Solution-recovery in ℓ_1-norm for non-square linear systems: deterministic conditions and open questions*

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Abstract

Consider an over-determined linear system $A^T x \approx b$ and an under-determined linear system $By = c$. Given $b = A^T\hat{x} + h$, under what conditions $\hat{x}$ will minimize the residual $A^T x - b$ in ℓ_1-norm (i.e., $\|h\|_1 = \min_x \|A^T x - b\|_1$)? On the other hand, given $c = Bh$, under what conditions h will be the minimum ℓ_1-norm solution of $By = c$? These two "solution-recovery" problems have been the focus of a number of recent works. Moreover, these two problems are equivalent under appropriate conditions on the data sets (A, b) and (B, c). In this paper, we give deterministic conditions for these solution-recoverary problems and raise a few open questuions. Some of the results in this paper are already known or partially known, but our derivations are different and thus may provide different perspectives.

1 Introduction

Let us consider the ℓ_1-norm approximation of an over-determined linear system:

$$(O1): \quad \min_{x \in \Re^p} \|A^T x - b\|_1 \tag{1}$$

where $A \in \Re^{p \times n}$ with $p < n$ and $b \in \Re^n$, and the minimum ℓ_1-norm solution to an under-determined linear system:

$$(U1): \quad \min_{y \in \Re^n} \{\|y\|_1 : By = c\} \tag{2}$$

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where $B \in \Re^{q \times n}$ with $q < n$ and $c \in \Re^q$. To avoid trivial cases, we always assume that b and c are nonzero vectors.

1.1 Solution Recovery Problems

We are concerned with the so-called *solution recovery* problems associated with $(O1)$ and $(U1)$. For problem $(O1)$ with a given right-hand side

$$b = A^T \hat{x} + h, \tag{3}$$

the question is under what conditions one can recover the solution $\hat{x}$ by minimizing the residual $A^T x - b$ in ℓ_1-norm; i.e., under what conditions

$$\hat{x} = \arg\min_x \; \|A^T x - b\|_1. \tag{4}$$

We note that (4) implies $\|h\|_1 = \min_x \|A^T x - b\|_1$. This solution recovery problem will also be called an *error-correction* problem, where the vector h represents errors.

For problem $(U1)$ with a given right-hand side

$$c = Bh, \tag{5}$$

the question is under what conditions one can recover the vector h by seeking the minimum ℓ_1-norm solution to the equation $By = c$; i.e., under what conditions

$$h = \arg\min_y \{\|y\|_1 : By = c\}. \tag{6}$$

These solution-recovery problems have recently been studied by a number of authors (for example, see [3, 4, 5, 6, 7, 8, 11, 12]), and many intriguing results have been obtained.

1.2 Conic Programing Representations

A conic program in $\Re^n$ is to minimize a linear function over the intersection of an affine space with a cone; more specifically,

$$\min_z \{f^T z : Cz = d, z \in \mathcal{K}\}, \tag{7}$$

where $f \in \Re^n$, $C \in \Re^{m \times n}$ $(m < n)$, $d \in \Re^m$, and $\mathcal{K}$ is a pointed, closed convex cone in $\Re^n$. Such a conic program always has a dual program,

$$\max_u \{d^T u : f - C^T u \in \mathcal{K}^*\}, \tag{8}$$

where $\mathcal{K}^*$ is the dual cone of $\mathcal{K}$ defined as

$$\mathcal{K}^* := \{x \in \Re^n : x^T z \geq 0, \ \forall z \in \mathcal{K}\}.$$

In particular, the following two cones,

$$\mathcal{K}_1 := \{(y, \xi) \in \Re^n \times \Re : \|y\|_1 \leq \xi\}, \tag{9}$$

$$\mathcal{K}_\infty := \{(v, \eta) \in \Re^n \times \Re : \|v\|_\infty \leq \eta\}, \tag{10}$$

are the dual cone of each other.

If both (7) and (8) have strictly feasible points, then they achieve optimality if and only if they both attain a same (finite) objective value.

It is not difficult to verify that problems $(O1)$ and $(U1)$ are equivalent to, respectively, the following two conic programs

$$\min_{x,y,\xi} \{\xi \in \Re : A^T x + y = b, \ (y, \xi) \in \mathcal{K}_1\}, \tag{11}$$

$$\min_{y,\xi} \{\xi \in \Re : By = c, \ (y, \xi) \in \mathcal{K}_1\}, \tag{12}$$

whose dual conic programs are, respectively,

$$\max_{v} \{b^T v : Av = 0, \ (v, 1) \in \mathcal{K}_\infty\}, \tag{13}$$

$$\max_{w} \{c^T w : (B^T w, 1) \in \mathcal{K}_\infty\}. \tag{14}$$

For any $x \in \Re^p$ and any dual feasible point v of (13), weak duality holds

$$b^T v = (b - A^T x)^T v \leq \|A^T x - b\|_1 \, \|v\|_\infty \leq \|A^T x - b\|_1.$$

Since both the primal and dual programs (11) and (13) are strictly feasible, optimality is attained at an $x \in \Re^p$ and a dual feasible v if and only if the strong duality $b^T v = \|A^T x - b\|_1$ holds. For an elementary introduction to the topic of convex conic programming, see [1], for example.

1.3 Equivalence

In the context of solution recovery, Candés and Tao (see [5], for example) have already observed the equivalence of problems $(O1)$ and $(U1)$ under the conditions $A^T B = 0$ and $c = Bb$. Here we give a slightly more general statement and a proof for it. This equivalence will allow us to concentrate on studying just one of the two problems.

Proposition 1 (Equivalence). *Let both $A \in \Re^{p\times n}$ and $B \in \Re^{q\times n}$ be of full-rank with $p+q=n$. Then (11) and (12) are identical if and only if*

$$AB^T = 0 \quad and \quad c = Bb. \tag{15}$$

Moreover, under this equivalence, if x^* solves (O1), then $b - A^T x^*$ solves (U1); and if y^* solves (U1), then $(AA^T)^{-1}A(b-y^*)$ solves (O1).

Proof. We observe that the conic programs (11) and (12) differ only in their defining affine spaces. Obviously, the two programs become identical if and only if the two involved affine spaces are the same; namely, $Y_1 = Y_2$ where

$$Y_1 := \{b - A^T x \in \Re^n : x \in \Re^p\}, \quad Y_2 := \{y \in \Re^n : By = c\}.$$

For arbitrary $y = b - A^T x \in Y_1$ to satisfy $By = c$, there must hold $(BA^T)x \equiv Bb - c$ for all $x \in \Re^p$, which is possible if and only if (15) holds. Therefore, $Y_1 \subset Y_2$ if and only if (15) holds. On the other hand, points in Y_2, satisfying $By = c$, have the form

$$y = B^T(BB^T)^{-1}c + (I - B^T(BB^T)^{-1}B)z$$

for arbitrary $z \in \Re^n$. Therefore, for $y \in Y_2$

$$b - y = B^T(BB^T)^{-1}(Bb - c) + (I - B^T(BB^T)^{-1}B)(b - z).$$

It is now easy to see that (15) is necessary and sufficient for $b - y$ to be in the range of A^T, or in other words for $y \in Y_1$. This proves that $Y_2 \subset Y_1$ if and only if (15) holds.

The rest of the proposition can be verified by substitutions. □

From this point on, we will always assume that (15) holds so that $(O1)$ and $(U1)$ are equivalent. Under this equivalence, without loss of generality we will need only to study $(O1)$, knowing that every result for $(O1)$ has an equivalent counterpart for $(U1)$.

A linear program corresponding to the dual conic program (13) is

$$\max_v \{b^T v : Av = 0, \ -1 \le v \le 1\}. \tag{16}$$

It is worth noting that the feasibility set of the dual program (16) is the intersection of the unit cube in $\Re^n$,

$$\mathcal{C}_n = \{v \in \Re^n : \|v\|_\infty \le 1\}, \tag{17}$$

with the null space of A (or the column-space of B^T). This set (intersection) is invariant with respect to actual choices of bases for the involved subspace.

1.4 General Tso's Problem

We now continue with a puzzle. Once upon a time, General Tso [2] received two coded messages $b_i \in \Re^n$, $i = 1, 2$, from an agent behind the enemy line. Both messages were encoded from the same original message $\hat{x} \in \Re^p$ by multiplying $\hat{x}$ with a nonsingular encoding matrix $E \in \Re^{n \times n}$. However, the received messages were corrupted by unknown vectors $h_i \in \Re^n$, $i = 1, 2$, respectively; that is, $b_i = E\hat{x} + h_i$, $i = 1, 2$. General Tso knew, other than cooking chickens, what the encoding matrix E was and the fact that no message component was corrupted twice (i.e., h_1 and h_2 did not have non-zero entries at the same position). The General tried to minimize $\|Ex - b_1\|_1 + \|Ex - b_2\|_1$, corresponding to problem $(O1)$ with $A = [E^T \; E^T]$ and $b^T = (b_1^T \; b_2^T)$, and obtained a solution x^*. Now the question is, was he able to exactly decode the true message $\hat{x}$; in other words, was $x^* = \hat{x}$? This question will be answered later.

It is interesting to note that the above question is asked regardless of the actual values of E and h_i's? Therefore, an affirmative answer would imply that, at least in this case, whether or not the exact solution $\hat{x}$ can be recovered is invariant with respect to the actual values of E and h_i's.

1.5 Notations

For any $h \in \Re^n$, we partition the index set $\{1, 2, \cdots, n\}$ into two disjoint subsets: the support set $S(h)$ and its complement – the zero set $Z(h)$; more precisely,

$$S(h) = \{i : h_i \neq 0, \; 1 \leq i \leq n\}, \quad Z(h) = \{1, \cdots, n\} \setminus S(h). \tag{18}$$

We will sometimes omit the dependence of a partition (S, Z) on h when either it is clear from the context, or (S, Z) is not associated with any particular h.

For any index set J, $|J|$ is the cardinality of J. For any matrix $A \in \Re^{p \times n}$ and any index subset $J \subset \{1, 2, \cdots, n\}$, $A_J \in \Re^{p \times |J|}$ denotes the sub-matrix of A consisting of those columns of A whose indices are in J. For a vector $h \in \Re^n$, however, h_J denotes the sub-vector of h with those components whose indices are in J.

We call a vector a *binary vector* if all its components take value either -1 or $+1$ (not zero or one). We use $\mathcal{B}^k$ to denote the set of binary vectors in $\Re^k$.

We use $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ to denote the maximum and minimum, respectively, eigenvalues of matrices; and similarly, $\sigma_{\max}(\cdot)$ and $\sigma_{\min}(\cdot)$ for singular values.

2 Necessary and Sufficient Conditions

Lemma 1. *Let $h \in \Re^n$ be given and $(S, Z) := (S(h), Z(h))$. Then $\hat{x}$ solves (O1) if and only if there exists $v_Z \in \Re^{|Z|}$ that satisfies*

$$A_Z v_Z = -A_S \operatorname{sign}(h_S), \quad \|v_Z\|_\infty \leq 1; \tag{19}$$

or in geometric terms, if and only if the null space of A intersects with the $(n - |S|)$-dimensional face of the unit cube $\mathcal{C}_n$ defined by $v_S = \operatorname{sign}(h_S)$.

Proof. By strong duality, $\hat{x}$ solves (1) if and only if there exists a dual feasible solution $v \in \Re^n$ of (13), satisfying $Av = 0$ and $\|v\|_\infty \leq 1$, such that the duality gap is closed at $\hat{x}$ and $v = (v_Z, v_S)$. Namely,

$$b^T v \equiv h^T v = h_S^T v_S + h_Z^T v_Z = h_S^T v_S = \|h\|_1 \equiv \|A^T \hat{x} - b\|_1.$$

The equality $h_S^T v_S = \|h\|_1$ holds with $\|v_S\|_\infty \leq 1$ if and only if $v_S = \operatorname{sign}(h_S)$; and $v = (v_Z, \operatorname{sign}(h_S))$ is dual feasible if and only if v_Z satisfies the conditions in (19). Moreover, the geometric interpretation is obvious. □

Remark 1. *The result in Lemma 1 is unchange if A is replaced by RA for any nonsingular $p \times p$ matrix R since the equation (19) is invariant with respect to such a transformation. Hence, this result, as well as all the results in the rest of the paper, depends only on the subspace $\mathbb{A} \subset \Re^n$ spanned by the rows of A, but not on A itself. Furthermore, it depends on the support and the signs of the corruption vector h, but not on its magnitude.*

The results below follows from Lemma 1 in a straightforward way.

Theorem 1. *Let $\mathcal{T}$ consist of subsets of $\{1, 2, \cdots, n\}$. Then $\hat{x}$ solves (O1) for any h such that $S(h) \in \mathcal{T}$ and $|S(h)| \leq k$ if and only if for any $S \in \mathcal{T}$ with $|S| = k$ and any binary vectors $u \in \mathcal{B}^k$ the system*

$$A_Z v = -A_S u, \quad \|v\|_\infty \leq 1 \tag{20}$$

has a solution $v \in \Re^{n-k}$, where Z is the complement of S in $\{1, 2, \cdots, n\}$.

Moreover, if $\mathcal{T}$ includes all subsets of $\{1, 2, \cdots, n\}$, then $\hat{x}$ solves (O1) for any h with $|S(h)| \leq k$ if and only if the null space of A intersects with all $(n-k)$-dimensional faces of the unit cube $\mathcal{C}_n$ in $\Re^n$.

Remark 2. *After obtaining the above geometric condition, we learned that Rudelson and Veshynin [11] had recently derived the same result by a different approach based on convex geometry.*

Next we introduce numbers that completely characterize the solution recoverability of $(O1)$ for given sparsity levels of corruption vectors h.

Definition 1. *Let $\mathbb{A} \subset \Re^n$ be a p-dimensional subspace spanned by the rows of $A \in \Re^{p \times n}$. For $k \in \{1, 2, \cdots, n-1\}$, define*

$$\zeta_{\mathbb{A}}(k) := \max_{|S|=k} \max_{u \in \mathcal{B}^k} \min_{v \in \Re^{n-k}} \{\|v\|_\infty : A_Z v = -A_S u\}, \tag{21}$$

where the left-most maximum is taken over (S, Z) partitions of $\{1, 2, \cdots, n\}$.

For any given subspace $\mathbb{A}$, the numbers $\zeta_{\mathbb{A}}(k)$ are independent of the choice of basis and non-decreasing as k increases. They are computable in theory, but not so easily in practice. The following result is a direct consequence of Theorem 1.

Theorem 2. *The vector $\hat{x}$ solves $(O1)$ for any h with $|S(h)| \le k$ if and only if*

$$\zeta_{\mathbb{A}}(k) \le 1. \tag{22}$$

Now we apply the theory developed above to a few specific cases. The following result gives an affirmative answer to General Tso's question given at the beginning (when the ℓ_1-norm minimization problem involved has a unique solution).

Corollary 1. *Let $A \in \Re^{n \times 2n}$ such that $A = [E^T \; E^T]$ and $h \in \Re^{2n}$ be such that $h_i h_{n+i} = 0$, $i = 1, 2, \cdots, n$. Then $\hat{x}$ solves $(O1)$.*

Proof. We can assume, with reordering if necessary, that $S(h) \subset \{1, 2, \cdots, n\}$. Set $v_{S+n} = -\text{sign}(h_S)$ and the rest of v_Z-components to zeros. Then the equation (19) is satisfied because $E_Z^T v_Z = E_{S+n}^T v_{S+n} \equiv -E_S^T(\text{sign}(h_S))$. Moreover, $\|v_Z\|_\infty = 1$. So Lemma 1 applies. □

An application of Theorem 1 gives the following necessary and sufficient conditions for matrices consisting of repeated orthogonal matrices.

Corollary 2. *Let A be a $p \times 2rp$ matrix consisting of $2r$ copies of a $(p \times p)$ orthogonal matrix. Then $\hat{x}$ solves $(O1)$ for any h with $|S(h)| \le k$ if and only if $k \le r$.*

Proof. Let $A = [Q \; \cdots \; Q]$ where Q is orthogonal and repeats $2r$ times. The sufficiency part can be proven in a similar fashion as in Corollary 1. For the necessity part, suppose that $k = r + 1$. One can pick $u_i = 1$, $i = 1, \cdots, r+1$, all corresponding to q_1 — the first column of Q. As such, $A_S u = (r+1) q_1$. Now there are only $(r-1)$ q_1-columns left in A_Z, and the rest of columns in A_Z are all orthogonal to q_1. Hence, there can be no solution to the equation (20), and $\hat{x}$ cannot be a solution to $(O1)$. □

3 Sufficient Conditions

Unlike the necessary and sufficient conditions in the preceding section, we need to impose extra requirements in order to obtain sufficient conditions.

Definition 2. *Let $C \in \Re^{p\times n}$ with $p < n$ and $S \subset \{1, 2, \cdots, n\}$. We say that (i) C is uniformly rank-p if $\mathrm{rank}(C_S) = p$ whenever $|S| \geq p$; and (ii) C is uniformly full-rank if $\mathrm{rank}(C_S) = \min(p, |S|)$ for any S. (Clearly, C is uniformly full-rank implies that C is uniformly rank-p.)*

Remark 3. *The above properties are really properties for the range of C, not just for C itself, since any matrix of the form RC for some $p \times p$ nonsingular matrix R would share the same property with C. These properties typically hold for random subspaces with high probability.*

Theorem 3 (Kernel Condition). *Let $A \in \Re^{p\times n}$ be full-rank and $B \in \Re^{q\times n}$ be uniformly full-rank such that $BA^T = 0$ and $p + q = n$. Then $\hat{x}$ solves (O1) for any $h \in \Re^n$ with $|S(h)| \leq k \leq q$ if*

$$\max_{|S|=k} \|(B_S^T B_S)^{-1} B_S^T B_Z\|_1 \leq 1, \tag{23}$$

where the maximum is taken over all partitions (S, Z) with $|S| = k$.

Proof. Let (S, Z) be a partition with $|S| = k \leq q$. Hence, $B_S^T \in \Re^{k\times q}$ with $k \leq q$. First observe that for any binary vector $u \in \mathcal{B}^k$,

$$\min_v \{\|v\|_\infty : A_Z v + A_S u = 0\} \quad \Leftrightarrow \quad \min_w \{\|B_Z^T w\|_\infty : B_S^T\, w = u\}.$$

Every solution of the equation $B_S^T\, w = u$ is of the form $w = B_S(B_S^T B_S)^{-1}u + z$ for some $z \in \Re^q$ satisfying $B_S^T\, z = 0$. Therefore,

$$\begin{aligned}
& \max_{u\in\mathcal{B}^k} \min_v \{\|v\|_\infty : A_Z v = -A_S u\} \\
= \; & \max_{u\in\mathcal{B}^k} \min_w \{\|B_Z^T w\|_\infty : B_S^T\, w = u\} \\
= \; & \max_{u\in\mathcal{B}^k} \min_z \{\|B_Z^T (B_S(B_S^T B_S)^{-1}u + z)\|_\infty : B_S^T\, z = 0\} \\
\leq \; & \max_{u\in\mathcal{B}^k} \|B_Z^T B_S(B_S^T B_S)^{-1}u\|_\infty \qquad (24) \\
= \; & \|B_Z^T B_S(B_S^T B_S)^{-1}\|_\infty = \|(B_S^T B_S)^{-1} B_S^T B_Z\|_1.
\end{aligned}$$

Substituting the above inequality into the definition (21), we have

$$\zeta_{\mathbb{A}}(k) \leq \max_{|S|=k} \|(B_S^T B_S)^{-1} B_S^T B_Z\|_1.$$

Setting the right-hand side less than or equal to one and invoking Theorem 2, we obtain (23) and complete the proof. □

Theorem 4 (Range Condition). *Let $A \in \Re^{p\times n}$ be uniformly rank-p. Then $\hat{x}$ solves (O1) for any h with $|S(h)| \le k \le n-p$ if*

$$\max_{|Z|=n-k} \|A_S^T(A_Z A_Z^T)^{-1}A_Z\|_1 \le 1, \tag{25}$$

where the maximum is taken over all partitions (S,Z) with $|Z| = n-k$.

Proof. Let (S,Z) be a partition with $|S| = k \le n-p$. Hence, $A_Z \in \Re^{p\times(n-k)}$ with $p \le n-k$. Since all the solutions to the equation $A_Z v = -A_S u$ can be written into the form $v = -A_Z^T(A_Z A_Z^T)^{-1}A_S u + w$ where $A_Z w = 0$, we have

$$\begin{aligned} & \max_{u\in\mathcal{B}^k} \min_{v} \{\|v\|_\infty : A_Z v = -A_S u\} \\ = \;& \max_{u\in\mathcal{B}^k} \min_{w} \{\| -A_Z^T(A_Z A_Z^T)^{-1}A_S u + w\|_\infty : A_Z w = 0\} \\ \le \;& \max_{u\in\mathcal{B}^k} \|A_Z^T(A_Z A_Z^T)^{-1}A_S u\|_\infty \\ = \;& \|A_Z^T(A_Z A_Z^T)^{-1}A_S\|_\infty = \|A_S^T(A_Z A_Z^T)^{-1}A_Z\|_1. \end{aligned} \tag{26}$$

Now the theorem follows from the same arguments as in the proof of Theorem 3. □

Remark 4. *In each of the above proofs, only one relaxation (replacing an equality by an inequality) is made in, respectively, (24) and (26). In fact, no relaxation is made when $k = q$ (so B_S^T and A_Z become square and nonsingular), and consequently conditions (23) and (25) both become necessary and sufficient.*

To further relax the left-hand side of (23), we observe that for some index j,

$$\begin{aligned} & \|(B_S^T B_S)^{-1}B_S^T B_Z\|_1 = \|(B_S^T B_S)^{-1}B_S^T B_Z e_j\|_1 \\ \le \;& \sqrt{k}\, \|(B_S^T B_S)^{-1}B_S^T\|_2 \, \|B_Z e_j\|_2 \\ \le \;& \sqrt{k}\, \lambda_{\max}^{1/2}((B_S^T B_S)^{-1}) \max_{1\le i\le n} \|Be_i\|_2 \\ \le \;& \sqrt{k}\, \lambda_{\min}^{-1/2}(B_S^T B_S) \max_{1\le i\le n} \|Be_i\|_2. \end{aligned}$$

Consequently, we obtain the following condition after setting the squares of the above less than or equal to one, taking the maximum over S and rearranging.

Lemma 2. *Let $B \in \Re^{q \times n}$ with $p + q = n$ be uniformly full-rank such that $BA^T = 0$. Then $\hat{x}$ solves $(O1)$ for any h with $|S(h)| \le k \le q$ if*

$$\frac{1}{k} \min_{|S|=k} \lambda_{\min}(B_S^T B_S) \;\ge\; \max_{1 \le i \le n} \|Be_i\|_2^2, \tag{27}$$

where the minimum is taken over all partitions (S, Z) with $|S| = k$.

We note that so far we have been careful in saying that "$\hat{x}$ solves $(O1)$" instead of "$\hat{x}$ is recovered by solving $(O1)$". However, the two statements are equivalent whenever the solution is unique which can be expected under appropriate conditions.

4 Open Questions

It is worth reiterating that the solution recoverablity by solving $(O1)$ (or $(U1)$ for that matter) depends on (i) the range or kernel of A^T but not specifically on the matrix A itself; and (ii) the support of error vector h but not the values of h.

We need to introduce a couple of more notations before we raise some open questions.

Definition 3. *For a given integer $n > 0$, let $\mathcal{F}_{n-k}$ be the set of $(n-k)$-dimensional faces (or in short $(n-k)$-faces) of the unit cube $\mathcal{C}_n \subset \Re^n$. Clearly,*

$$|\mathcal{F}_{n-k}| = 2^k \begin{pmatrix} n \\ k \end{pmatrix}.$$

Let $\mathcal{S}_n(q)$ be the set of all q-dimensional subspaces of $\Re^n$. For any $S \in \mathcal{S}_n(q)$, let $\omega(S, q, k)$ be the number of members of $\mathcal{F}_{n-k}$ that S intersects with. Moreover, let

$$\Omega_n(q, k) = \max_{S \in \mathcal{S}_n(q)} \omega(S, q, k).$$

Given $p < n$ and $k < n$, it is not difficult to see that the ratio $\Omega_n(n-p, k)/|\mathcal{F}_{n-k}|$ is the best-scenoria probability of recovering $\hat{x}$ from $(O1)$ (i.e., $\min_x \|A^T x - b\|_1$) where $A \in \Re^{p \times n}$, $b = A^T \hat{x} + h$, and h has k non-zeros that may occur equally likely at any set of k indices. An analogous interpretation can be made for the under-determined problem $(U1)$.

Recently, a great deal of progress (see for example, [3, 4, 5, 6, 7, 8, 11, 12]) has been made for the solution recovery problems when the matrix A or B is random (in particular, when the entries are i.i.d standard Gaussian or Bernoulli with the equal probability).

Remark 5. *In a nutshell, it has been established that when the row-column ratio of the matrix A or B is fixed, then asymptotically (as $n \to \infty$) solutions can be recovered at a high and increasing probability, as long as the number of non-zeros in the error vector h does not exceed a small fraction of its length n.*

In the following, we raise some questions of deterministic and finite nature, rather than probabilistic and asymptotic. These questions appear to be unsolved at this point.

Open Questions

1. In general, for $q < n$ and $k < n$

$$\Omega_n(q, k) = ?$$

   Short of it, can one find tight bounds on $\Omega_n(q, k)$?

2. Is it possible to construct a deterministic subspace $S \in \mathcal{S}_n(q)$ so that

$$\omega(S, q, k) = \Omega_n(q, k)?$$

   Short of it, can one construct a deterministic subspace $S \in \mathcal{S}_n(q)$ so that $\omega(S, q, k)$ is provably close to $\Omega_n(q, k)$?

A partial answer to the first question can be derived from recent results in [9] and [10] by examining the neighborliness of centrally symmetric polytopes. In particular, using Theorem 1.1 in [10], one can derive that

$$\Omega_{2q}(q, k) = |\mathcal{F}_{2q-k}|, \quad \forall\, k \le \frac{q}{400}. \tag{28}$$

This deterministic bound on k is, to the best of our knowledge, tighter than the best available probabilistic bound of today.

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