Exact Solutions in Linear and Integer Programming

Francois de Lucy

"The development of linear programming is -- in my opinion -- the most important contribution of the mathematics of the 20th century to the solution of practical problems arising in industry and commerce."

Martin Groetschel, 2006

Top Ten Algorithms of the Century

The Best of the 20th Century: Editors Name Top 10 Algorithms

By Barry A. Cipra

Algos is the Greek word for pain. *Algor* is Latin, to be cold. Neither is the root for *algorithm*, which stems instead from al-Khwarizmi, the name of the ninth-century Arab scholar whose book *al-jabr wa'l muqabalah* devolved into today's high school algebra textbooks. Al-Khwarizmi stressed the importance of methodical procedures for solving problems. Were he around today, he'd no doubt be impressed by the advances in his eponymous approach.

Some of the very best algorithms of the computer age are highlighted in the January/February 2000 issue of *Computing in Science & Engineering*, a joint publication of the American Institute of Physics and the IEEE Computer Society. Guest editors Jack Don-garra of the University of Tennessee and Oak Ridge National Laboratory and Fran-cis Sullivan of the Center for Comput-ing Sciences at the Institute for Defense Analyses put togeth-er a list they call the "Top Ten Algorithms of the Century."

"We tried to assemble the 10 al-gorithms with the greatest influence on the development and practice of science and engineering in the 20th century," Dongarra and Sullivan write. As with any top-10 list, their selections—and non-selections—are bound to be controversial, they acknowledge. When it comes to picking the algorithmic best, there seems to be no best algorithm.

Without further ado, here's the CiSE top-10 list, in chronological order. (Dates and names associated with the algorithms should be read as first-order approximations. Most algorithms take shape over time, with many contributors.)

1946: John von Neumann, Stan Ulam, and Nick Metropolis, all at the Los Alamos Scientific Laboratory, cook up the Metropolis algorithm, also known as the **Monte Carlo method**.

The Metropolis algorithm aims to obtain approximate solutions to numerical problems with unmanageably many degrees of freedom and to combinatorial problems of factorial size, by mimicking a random process. Given the digital computer's reputation for deterministic calculation, it's fitting that one of its earliest applications was the generation of random numbers.

In terms of widespread use, George Dantzig's simplex method is among the most successful algorithms of all time.

1947: George Dantzig, at the RAND Corporation, creates the **simplex method for linear programming**.

In terms of widespread application, Dantzig's algorithm is one of the most successful of all time: Linear programming dominates the world of industry, where economic survival depends on the ability to optimize within budgetary and other constraints. (Of course, the "real" problems of industry are often nonlinear; the use of linear programming is sometimes dictated by the computational budget.) The simplex method is an elegant way of arriving at optimal answers. Although theoretically susceptible to exponential delays, the algorithm in practice is highly efficient—which in itself says something interesting about the nature of computation.

1950: Magnus Hestenes, Eduard Stiefel, and Cornelius Lanczos, all from the Institute for Numerical Analysis at the National Bureau of Standards, initiate the development of **Krylov subspace iteration methods**.

These algorithms address the seemingly simple task of solving equations of the form $Ax = b$. The catch,

Top Ten Algorithms of the Century

In terms of widespread use, George Dantzig's simplex method is among the most successful algorithms of all time.

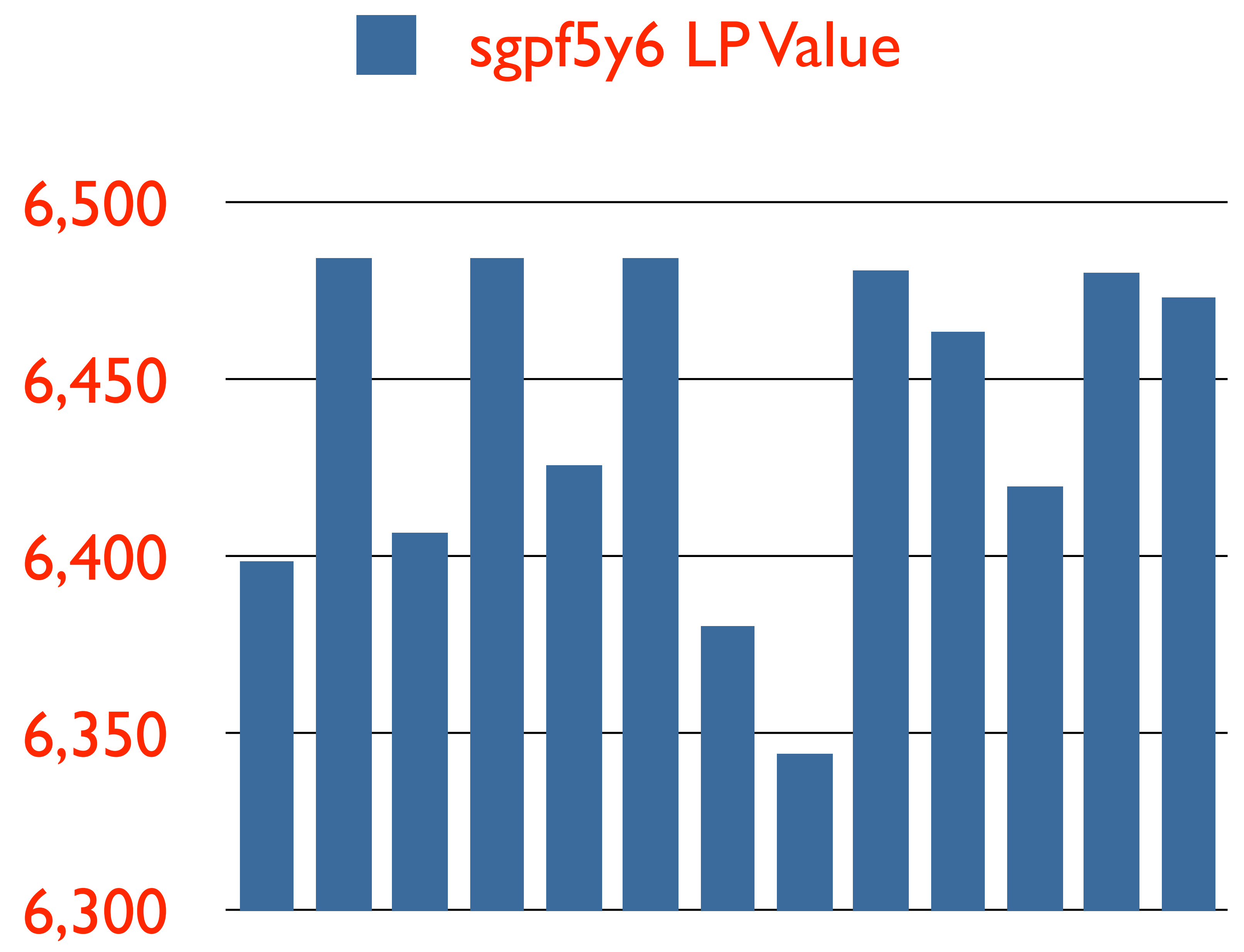
Professor Richard Tapia

Professor Richard Tapia

"Cook, second place sucks."

Example: sgpf5y6 (Mittelmann LP test set)

Cplex 7.1 Primal	6398.71
Cplex 7.1 Dual	6484.44
Cplex 9.0 Primal	6406.78
Cplex 9.0 Dual	6484.47
Cplex 11.0 Primal	6425.87
Cplex 11.0 Dual	6484.46
XPress-15 Primal	6380.45
XPress-15 Dual	6344.30
CPL-1.02.01	6480.95
GLPK-4.7	6463.66
QSopt Primal	6419.94
QSopt Dual	6480.33
Soplex 1.2.2	6473.33


sgpf5y6 LP Value
6,500
6,450
6,400
6,350
6,300

Volume 1 · Number 1 · March 2009

Mathematical Programming Computation

A Publication of the Mathematical Programming Society®

Springer

Nature, Vol. 395, 1 October 1998

Kepler's conjecture confirmed

Neil J. A. Sloane

One of the oldest unsolved problems in mathematics appears to have been settled. On 9 August, Thomas C. Hales announced that he had proved Kepler's assertion of 1611 that no packing of spheres can be denser than a face-centred-cubic lattice.

In face-centred-cubic packing (Fig. 1), seen in the piles of oranges in any grocer's shop, the spheres occupy 0.7405 of the total space available. Ambrose Rogers remarked[1] in 1958 that "many mathematicians believe and every physicist knows" that no denser packing is possible. So why has it taken 387 years for a proof to be found?

There are three reasons. First, technical difficulties come from the fact that the density of a packing is defined as the *limit* of the fraction of space occupied by the balls as the number of balls goes to infinity. This means that (say) a million balls can be thrown away without changing the density.

Second, even if one considers only packings without any obvious gaps, there are still

MATHEW B. BRADY/LIBRARY OF CONGRESS

Figure 1 **Cannonballs stacked in a face-centred-cubic lattice (Arlington, Virginia, about 1863). There is no denser way to do it.**

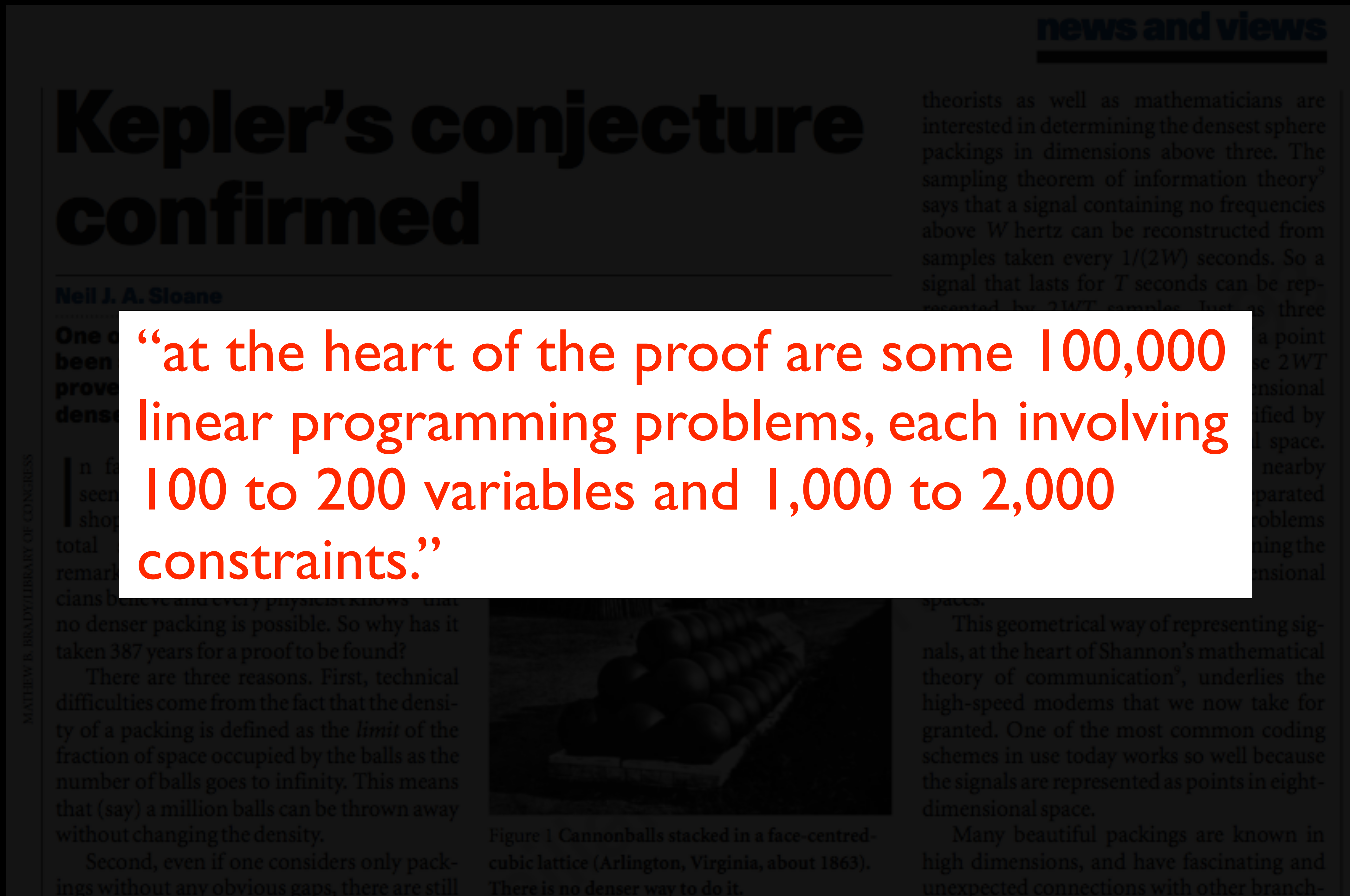
theorists as well as mathematicians are interested in determining the densest sphere packings in dimensions above three. The sampling theorem of information theory[9] says that a signal containing no frequencies above W hertz can be reconstructed from samples taken every $1/(2W)$ seconds. So a signal that lasts for T seconds can be represented by $2WT$ samples. Just as three numbers specify the coordinates of a point in three-dimensional space, so these $2WT$ samples specify a point in $2WT$-dimensional space. The whole waveform is specified by a single point in $2WT$-dimensional space. Similar signals are represented by nearby points, dissimilar signals by well-separated points. So one of the fundamental problems in communication theory is determining the densest packing of balls in high-dimensional spaces.

This geometrical way of representing signals, at the heart of Shannon's mathematical theory of communication[9], underlies the high-speed modems that we now take for granted. One of the most common coding schemes in use today works so well because the signals are represented as points in eight-dimensional space.

Many beautiful packings are known in high dimensions, and have fascinating and unexpected connections with other branch-

Thomas C. Hales

Nature, Vol. 395, 1 October 1998



news and views

Kepler's conjecture confirmed

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One c… been… prove… dens…

In fa… seem… shop… total… remark… cians b… that no denser packing is possible. So why has it taken 387 years for a proof to be found?

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Figure 1 Cannonballs stacked in a face-centred-cubic lattice (Arlington, Virginia, about 1863). There is no denser way to do it.

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"at the heart of the proof are some 100,000 linear programming problems, each involving 100 to 200 variables and 1,000 to 2,000 constraints."

Nature, Vol. 424, 3 July 2003

news feature

Does the proof stack up?

Think peer review takes too long? One mathematician has waited four years to have his paper refereed, only to hear that the exhausted reviewers can't be certain whether his proof is correct. George Szpiro investigates.

P. TURNLEY/S. SHERBELL/R. BICKEL/CORBIS; H. SITTON/GETTY

Grocers the world over know the most efficient way to stack spheres — but a mathematical proof for the method has brought reviewers to their knees.

Annals of Mathematics, **162** (2005), 1065–1185

A proof of the Kepler conjecture

By Thomas C. Hales*

To the memory of László Fejes Tóth

"Floating-point arithmetic was used freely in obtaining these bounds. The linear programming package CPLEX was used (see www.cplex.com). However, the results, once obtained, could be checked rigorously as follows."

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To encourage the submission of excellent short papers to the Annals, the editors announce that Annals papers under 20 printed pages in length will be published on an accelerated schedule. We will also make efforts to expedite the refereeing of excellent short papers.

Statement by the Editors on Computer-Assisted Proofs

Computer-assisted proofs of exceptionally important mathematical theorems will be considered by the *Annals*.

The human part of the proof, which reduces the original mathematical problem to one tractable by the computer, will be refereed for correctness in the traditional manner. The computer part may not be checked line-by-line, but will be examined for the methods by which the authors have eliminated or minimized possible sources of error: (e.g., round-off error eliminated by interval artihmetic, programming error minimized by transparent surveyable code and consistency checks, computer error minimized by redundant calculations, etc. [Surveyable means that an interested person can readily check that the code is essentially operating as claimed]).

We will print the human part of the paper in an issue of the *Annals*. The authors will provide the computer code, documentation necessary to understand it, and the computer output, all of which will be maintained on the *Annals of Mathematics* website online.

Last Updated: February 20, 2006

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“e.g, round-off error eliminated by interval arithmetic”

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The Flyspeck Project

Introduction

The purpose of the flyspeck project is to produce a formal proof of the Kepler Conjecture. The name `flyspeck' comes from matching the pattern `/f.*p.*k/` against an English dictionary. FPK in turn is an acronym for "The Formal Proof of Kepler."

Internal Links

- Flyspeck Wiki
- Formalizing the Text This is a proposal about how to formalize the written text of the proof.
- IMO-demo A demo on using HOL Light, including Harrison's IMO problem video.

External Links

- Flyspeck Google Group
- code for 1998 proof
- McLaughlin's revision of the kepler code
- QED Manifesto

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FlyspeckFactSheet

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HolLightDemo1

Groups: Flyspeck disucssion

Project owners: Join project

seanmcl, TCHales

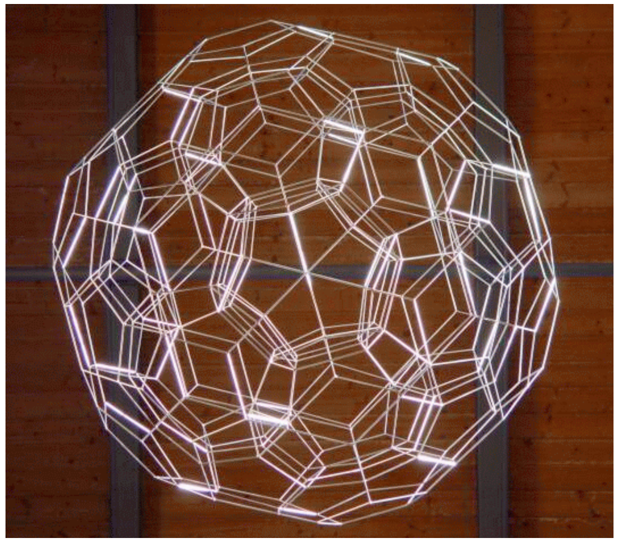
Project members:

florian.rabe, allegristas

Exact LP Code

D. Applegate, S. Dash, D. Espinoza

Applegate and Still (1995)
Kwappik et al. (2003)
Koch (2004)



Simplex Alg –> Basis
Solve Rational Linear System
Check Optimality Conditions

Implementation based on QSopt

- Increase precision on the fly (GNU–MP package)
- Full simplex code with steepest edge pricing
- Rational approximations of floating–point results to avoid rational linear solves (continued fractions)
- Callable library for modifying LP models
- Large scale instances

QSopt _ex vs QSopt
625 Test Instances from GAMS World

	Geom Mean	QSopt Seconds
Small	5.7	15.8
Medium	5.2	414.6
Large	1.8	3621.5

Use of QSopt_ex

Solve each Kepler LP
Store exact dual LP solution
Verify Kepler bounds via LP duality
Computations carried out in the ML language

Sean McLaughlin

Rational Number Reconstruction

Legendre

$$|\alpha - \frac{p}{q}| < \frac{1}{2q^2}, \text{then } \frac{p}{q} \text{ occurs as a convergent for } \alpha$$

Continued Fraction Expansion

Modular Reconstruction

$$nq \equiv p \mod M \text{ with } M > 2q^2, \text{ then we can construct } \frac{p}{q}$$

Extended Euclidean Algorithm

von zur Gathen and Gerhard (1999), Wang and Pan (2003, 2004)

Solving Rational Linear Systems

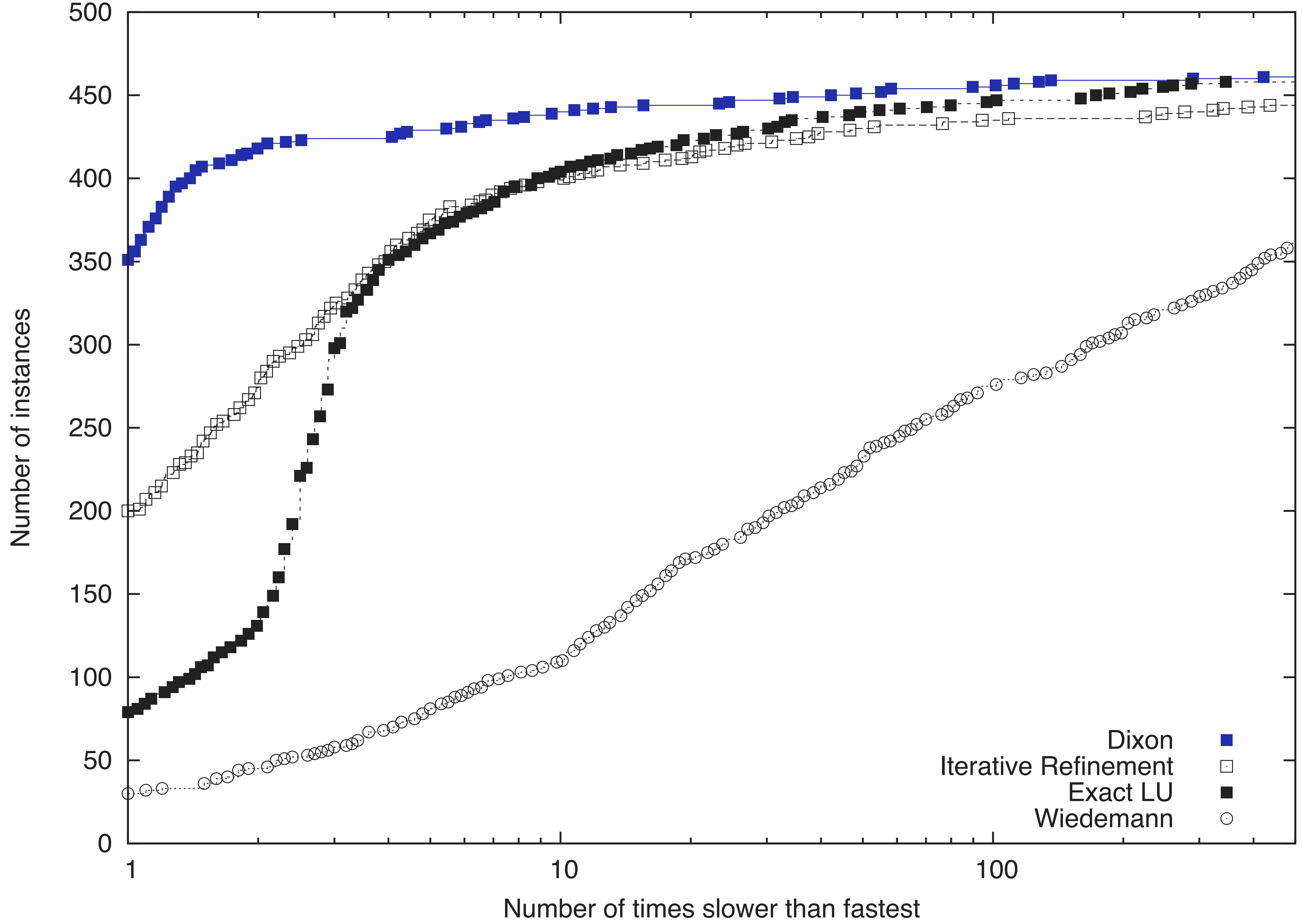
Dixon (1982): p-adaic lifting

Wiedemann (1986): solving over finite fields

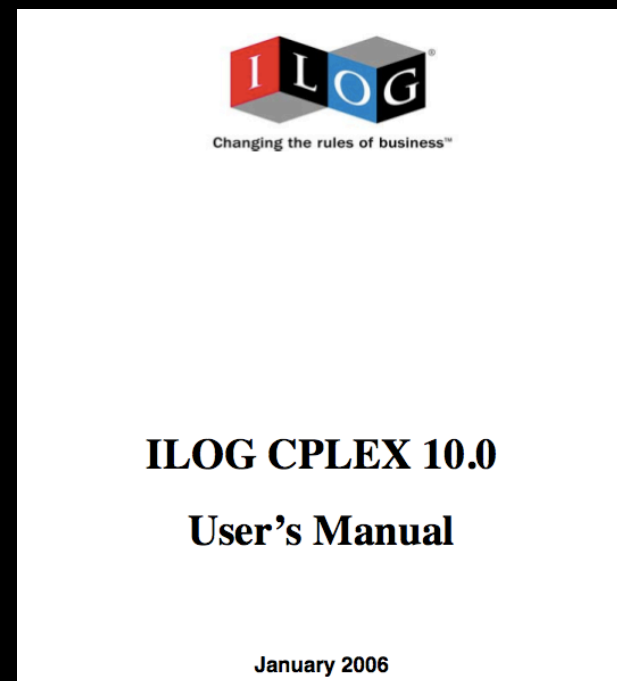
Computational Studies

LaMacchia and Odlyzko (1990), Eberly et al. (2006), Wan (2006)

LinBox C++ Software



Number of instances
Number of times slower than fastest
Dixon
Iterative Refinement
Exact LU
Wiedemann
0
50
100
150
200
250
300
350
400
450
500
1
10
100


Changing the rules of business™

ILOG CPLEX 10.0

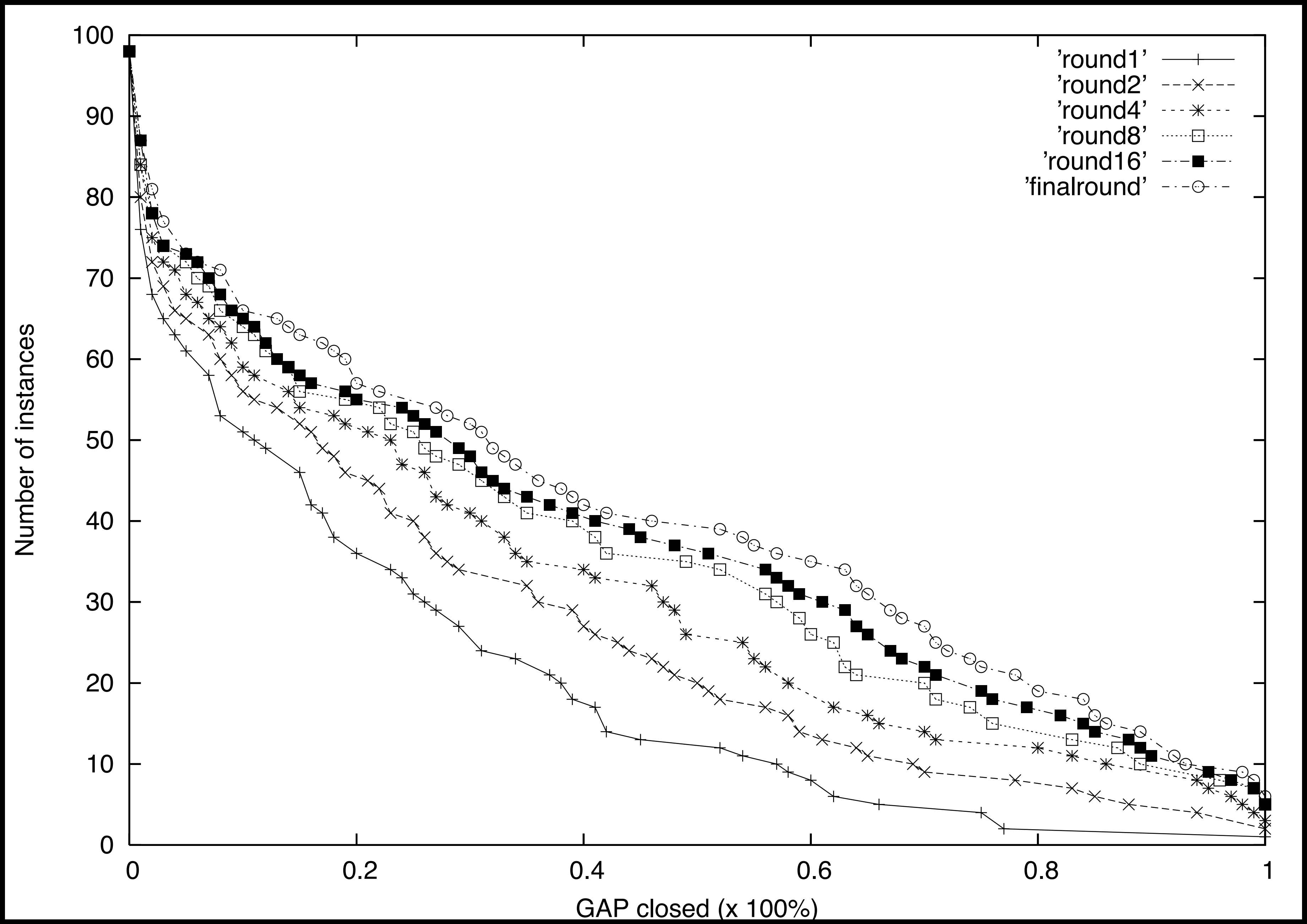
User's Manual

January 2006

"These default values indicate to CPLEX to stop when an integer feasible solution has been proved to be within 0.01% of optimality."

Accurate Gomory Cuts

Sanjeeb Dash, Ricardo Fukasawa, Marcos Goycoolea



Any hope for a solution?

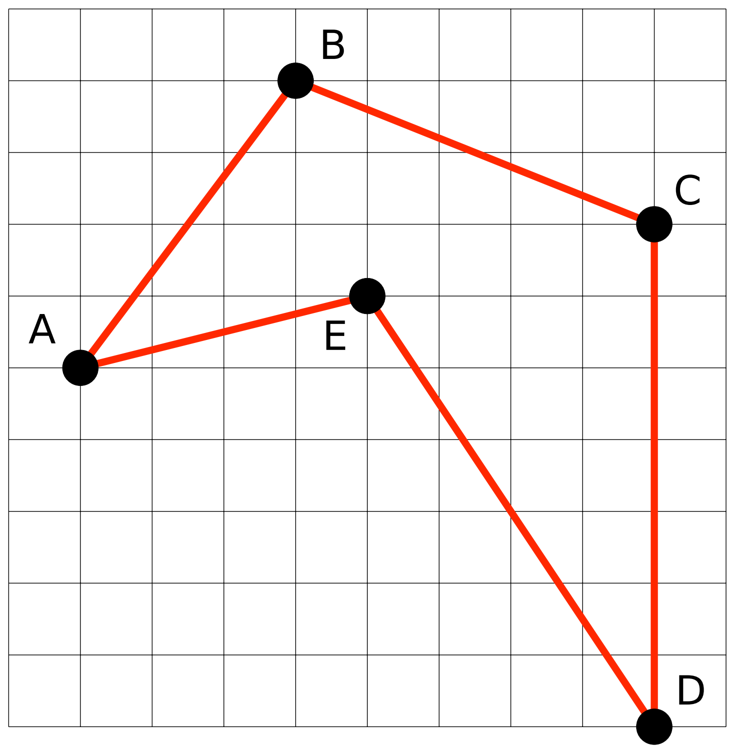
RSA-2048 (Formerly $200,000)

MIPs with 7,000 Variables

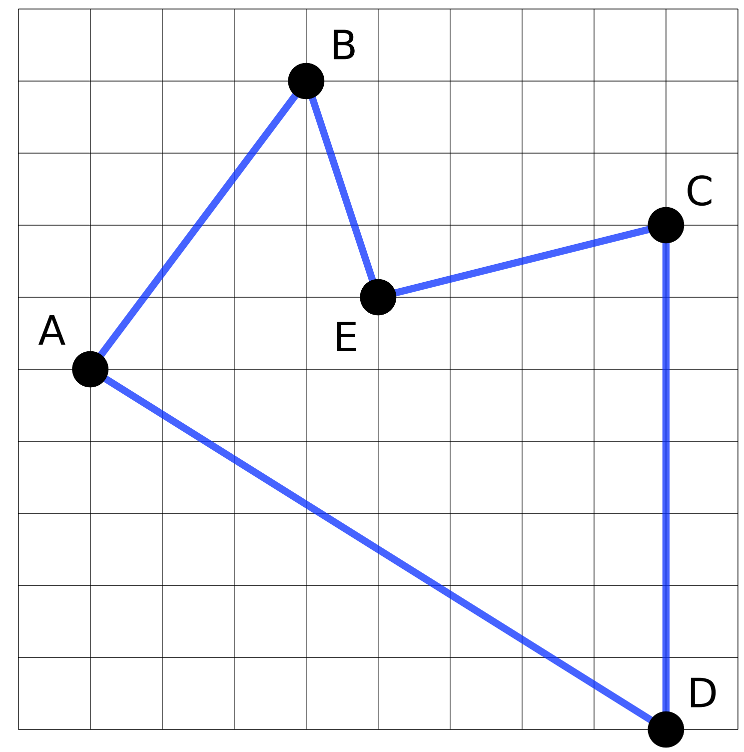
Dan Steffy: Chvatal/MIR Closures of Horner Systems

Optimizing Irrational Functions

Example: Euclidean TSP



28.71937



28.71936

Exact Geometric Computation: LEDA, CGAL, CORE
Yap (2003) -- survey paper

Sum of Square Roots Problem

Garey, Graham, Johnson (STOC 1976),
Ron Graham in early 1980s,
O'Rourke (American Math Monthly 1981)

$a_1, \ldots, a_n, b_1, \ldots, b_n$ integers (k-bit)

$$\left| \sum \sqrt{a_i} - \sum \sqrt{b_i} \right| \neq 0$$

How small can this value be?
Polynomial bound on number of bits?

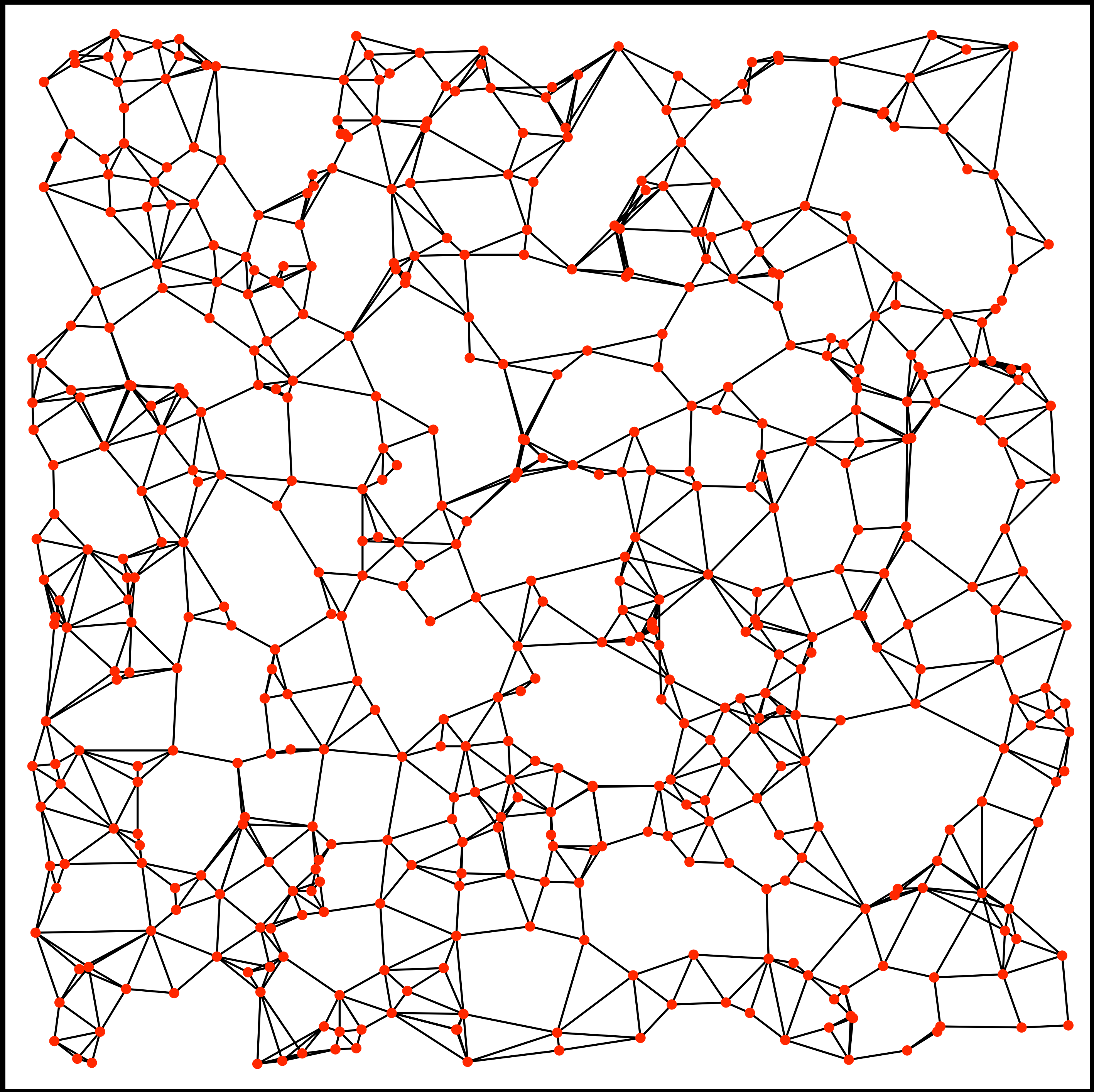
$\geq (2n - \frac{3}{2})k$ (Qian and Wang 2006), $\leq 4k2^{2n}$ (Burnikel et al. 2000)

Branch-and-Bound with Increasing Precision

Cannot Discard LPs with Integer Optimal Solutions

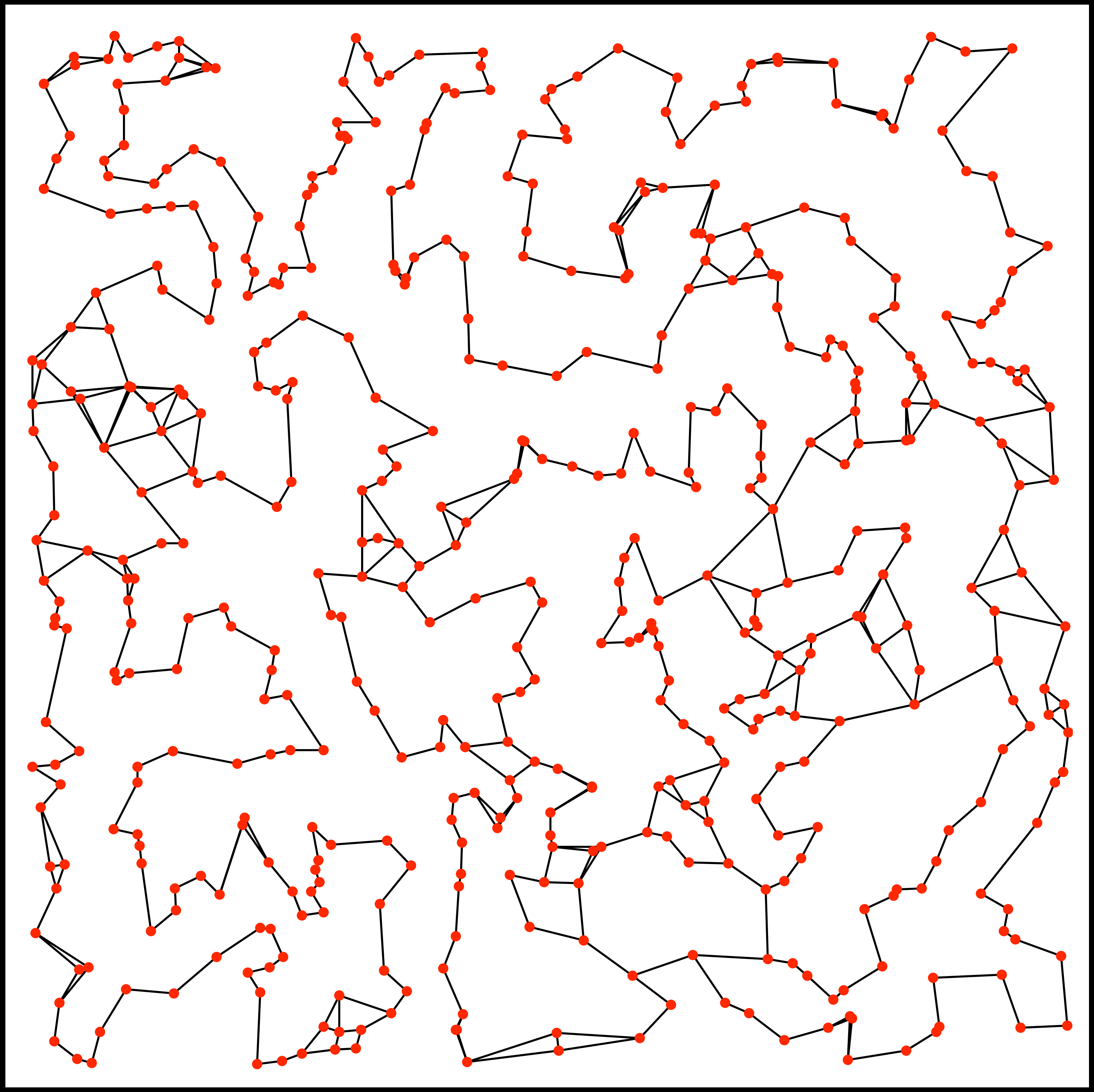
Must Explore Search Tree to reach all Indistinguishable Tours

500 Random Points in 1,000 by 1,000 Grid



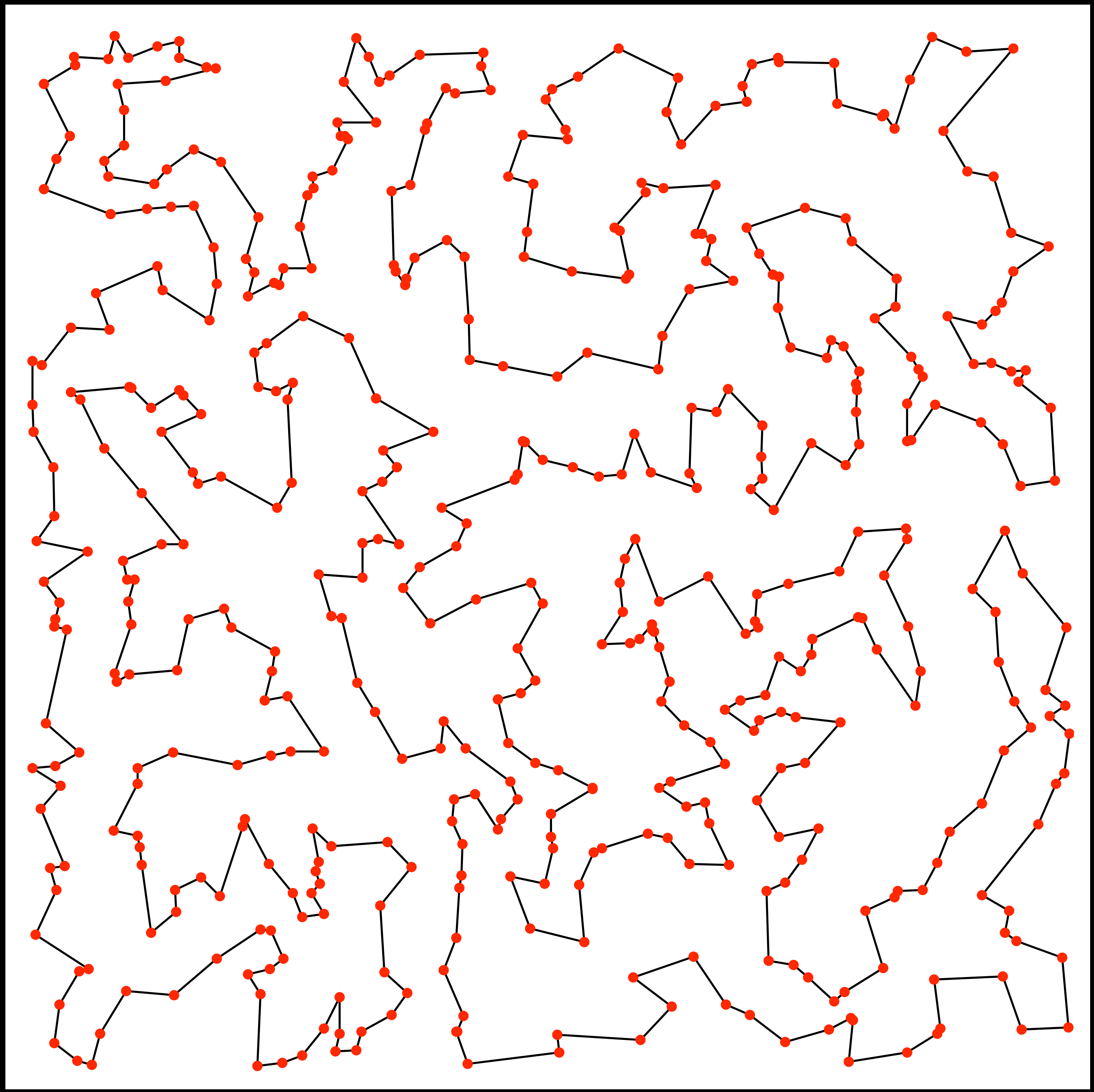
1,188 edges after duality elimination

500 Random Points in 1,000 by 1,000 Grid



615 edges after depth-1 branching

500 Random Points in 1,000 by 1,000 Grid



Optimal Tour

48 States in 10 Days

Maura Gatensby:
“There is more poetry in walking in the footsteps of giants.”

ON STAGE

PRESENTS

TONE DEF

November 6, 2007