Heat asymptotics for Lévy processes.

Rodrigo Bañuelos

Department of Mathematics
Purdue University
West Lafayette, IN 47907

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D open connected finite volume, Δ_D Dirichlet Laplacian.

$$
\begin{aligned}
Z_D(t) = trace(e^{t\Delta_D}) &= \sum_{j=0}^{\infty} e^{-t\lambda_j} = \int_D p_t^D x, x) dx \\
&= \frac{1}{(4\pi t)^{d/2}} \int_D P_x\{\tau_D > t \big| X_t = x\} dx,
\end{aligned}
$$

τ_D exit time from D of Brownian motion. In fact,

$$
\begin{aligned}
p_t^D(x,y) &= \frac{1}{(4\pi t)^{d/2}} e^{\frac{-|x-y|^2}{4t}} P_x\{\tau_D > t | B_t = y\} \\
&= p_t(x-y) - \mathbf{E}^x \left(\tau_D < t, p_{(t-\tau_D)}(X(\tau_D), y)\right) \\
&= p_t(x-y) - r_t^D(x,y).
\end{aligned}
$$

The function $r_t^D(x,y)$ is called a killing measure.

Theorem (M. Kac '51 (?))

For any $D \subset \mathbb{R}^d$ of finite volume

$$\lim_{t \downarrow 0} t^{d/2} Z_D(t) = \frac{|D|}{(4\pi)^{d/2}} = p_1(0)|D|$$

Corollary

Then (Karamata tauberian theorem)

$$\lim_{t \to 0} t^{\gamma} \int_0^{\infty} e^{-t\lambda} d\mu(\lambda) = A \Rightarrow \lim_{a \to \infty} a^{-\gamma} \mu[0, a) = \frac{A}{\Gamma(\gamma + 1)}$$

gives Weyl's asymptotics:

$$\lim_{\lambda \to \infty} \lambda^{-d/2} N(\lambda) = \frac{p_1(0)|D|}{\Gamma(d/2 + 1)}$$

$N(\lambda)$ be the number of eigenvalues $\{\lambda_j\}$ which not exceeding λ

Theorem (Minakshiusundaram '53–heat invariance)

$D \subset \mathbb{R}^d$ bounded "smooth". Then

$$Z_D(t) - \frac{1}{(4\pi t)^{d/2}} \sum_{j=0}^{m} c_j t^{j/2} = O(t^{(m-d+1)/2}), \qquad t \downarrow 0$$

$$c_1 = |D|, \qquad c_2 = -\frac{\sqrt{\pi}}{2}|\partial D|.$$

Theorem (McKean '67)

$D \subset \mathbb{R}^2$ with r holes. Then

$$\lim_{t\downarrow 0} \left\{ Z_D(t) - \frac{|D|}{4\pi t} + \frac{|\partial D|}{4(4\pi t)^{1/2}} \right\} = \frac{(1-r)}{6}$$

Theorem (C^1-domains: Brossard-Carmona '86. Lipschitz domains: R. Brown '93.)

$$Z_D(t) = (4\pi t)^{-d/2} \left(|D| - \frac{\sqrt{\pi t}}{2}|\partial D| + o(t^{1/2}) \right), \quad t \downarrow 0$$

For all Smooth Bounded Convex domains:

$$\frac{|D|}{(4\pi t)^{d/2}} - \frac{e^{d/2}|\partial D|}{(4\pi t)^{(d-1)2}} \leq Z_D(t) \leq \frac{|D|}{(4\pi t)^{d/2}}, \qquad t > 0$$

For Smooth Bounded Convex with mean curvature bounded by $\frac{1}{R}$

$$\left| Z_D(t) - \frac{|D|}{(4\pi t)^{d/2}} + \frac{|\partial D|}{4(4\pi t)^{(d-1)2}} \right| \leq \frac{|\partial D|}{t^{(d-2)/2}} \left\{ C_1 + C_2 \log\left(1 + \frac{R^2}{t}\right) \right\}$$

Theorem (van den Berg '87–sharp in t and "degree of smoothness")

If ∂D satisfies uniform inner and outer ball condition with radius R

$$\left| Z_D(t) - (4\pi t)^{-d/2} \left(|D| - \frac{\sqrt{\pi t}}{2} |\partial D| \right) \right| \leq \frac{d^4}{\pi^{d/2}} \frac{|D| t}{t^{d/2} R^2}, \quad t > 0.$$

Problem: **Investigate similar properties for “other" Lévy processes, and especially those subordinate to Brownian motion whose generators are simple transformations of the Laplacian**

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Definition

A Lévy Process is a stochastic process $X = (X_t), t \geq 0$ with

- X has independent and stationary increments
- $X_0 = 0$ (with probability 1)
- X is *stochastically continuous*: For all $\varepsilon > 0$,

$$\lim_{t \to s} P\{|X_t - X_s| > \varepsilon\} = 0$$

Note: Not the same as a.s. continuous paths. However, it gives “cadlag" paths: Right continuous with left limits.

- **Stationary increments:** $0 < s < t < \infty$, $A \in \mathbb{R}^d$ Borel

$$P\{X_t - X_s \in A\} = P\{X_{t-s} \in A\}$$

- **Independent increments:** For any given sequence of ordered times

$$0 < t_1 < t_2 < \cdots < t_m < \infty,$$

the random variables

$$X_{t_1} - X_0,\ X_{t_2} - X_{t_1}, \ldots, X_{t_m} - X_{t_{m-1}}$$

are independent.

The characteristic function of X_t is

$$\varphi_t(\xi) = E\left(e^{i\xi \cdot X_t}\right) = \int_{\mathbb{R}^d} e^{i\xi \cdot x} p_t(dx) = (2\pi)^{d/2} \widehat{p_t}(\xi)$$

where p_t is the distribution of X_t. Notation (same with measures)

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ix\cdot\xi} f(x)dx, \quad f(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(\xi)d\xi$$

The Lévy–Khintchine Formula

The characteristic function has the form $\varphi_t(\xi) = e^{t\rho(\xi)}$, where

$$\rho(\xi) = ib \cdot \xi - \frac{1}{2}\xi \cdot A\xi + \int_{\mathbb{R}^d} \left(e^{i\xi\cdot x} - 1 - i\xi \cdot x 1_{\{|x|<1\}}(x)\right)\nu(dx)$$

for some $b \in \mathbb{R}^d$, a non–negative definite symmetric $n \times n$ matrix A and a Borel measure ν on $\mathbb{R}^d$ with $\nu\{0\} = 0$ and

$$\int_{\mathbb{R}^d} \min\left(|x|^2, 1\right)\nu(dx) < \infty$$

$\rho(\xi)$ is called the **symbol** of the process or the **characteristic exponent.** The triple (b, A, ν) is called the **characteristics of the process.**

Converse also true. Given such a triple we can construct a Lévy process.

Example (The rotationally invariant stable processes:)

These are self–similar processes, denoted by X_t^α, in $\mathbb{R}^d$ with symbol

$$\rho(\xi) = -|\xi|^\alpha, \qquad 0 < \alpha \leq 2.$$

$\alpha = 2$ is **Brownian motion.** $\alpha = 1$ is the **Cauchy processes.**

Example (Relativistic Brownian motion)

According to quantum mechanics, a particle of mass m moving with momentum p has kinetic energy

$$E(p) = \sqrt{m^2c^4 + c^2|p|^2} - mc^2$$

where c is speed of light. Then $\rho(p) = -E(p)$ is the symbol of a Lévy process, called *"relativistic Brownian motion."*

In fact, these are Lévy processes of the form $X_t = B_{T_t}$ where B_t is Brownian motion and T_t is a "subordinator" independent of B_t.

Example (Subordinators)

A subordinator is a one-dimensional Lévy process $\{T_t\}$ such that

(i) $T_t \geq 0$ a.s. for each $t > 0$

(ii) $T_{t_1} \leq T_{t_2}$ a.s. whenever $t_1 \leq t_2$

Theorem (Laplace transforms)

$$E(e^{-\lambda T_t}) = e^{-t\psi(\lambda)},\ \lambda > 0,$$

$$\psi(\lambda) = b\lambda + \int_0^\infty \left(1 - e^{-\lambda s}\right) \nu(ds)$$

$b \geq 0$ and the Lévy measure satisfies $\nu(-\infty, 0) = 0$ and $\int_0^\infty \min(s,1)\nu(ds) < \infty$. ψ is called the Laplace exponent of the subordinator.

Example ($\alpha/2$–Stable subordinator)

$\psi(\lambda) = \lambda^{\alpha/2}$, $0 < \alpha < 2$ gives the stable with $b = 0$ and

$$\nu(ds) = \frac{\alpha/2}{\Gamma(1 - \alpha/2)}\, s^{-1-\alpha/2}\, ds$$

Example (Relativistic stable subordinator:)

$0 < \alpha < 2$ and $m > 0$, $\Psi(\lambda) = (\lambda + m^{2/\alpha})^{\alpha/2} - m$.

$$\nu(ds) = \frac{\alpha/2}{\Gamma(1-\alpha/2)} e^{-m^{2/\alpha} s} s^{-1-\alpha/2} ds$$

Many others: "Gamma subordinators, Geometric stable subordinators, iterated geometric stable subordinators, Bessel subordinators,..."

Theorem

If X is an arbitrary Lévy process and T is a subordinator independent of X, then $Z_t = X_{T_t}$ is a Lévy process. For any Borel $A \subset \mathbb{R}^d$,

$$p_{Z_t}(A) = \int_0^\infty p_{X_s}(A) p_{T_t}(ds)$$

$$\mathbf{P}^x\left(X_t^\alpha \in A\right) = \int_A p_t^\alpha(x-y)dy, \qquad p_t^\alpha(x) = t^{-d/\alpha} p_1^\alpha\left(\frac{x}{t^{1/\alpha}}\right).$$

Heat Semigroup in D is the self-adjoint operator

$$T_t^D f(x) = E_x\left[f(X_t^\alpha); \tau_D > t\right] = \int_D p_t^{D,\alpha}(x,y) f(y) dy,$$

$$\begin{aligned} p_t^{D,\alpha}(x,y) &\le p_t^\alpha(x-y) \le p_1^\alpha(0) t^{-d/\alpha} \\ &= \left(\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-|\xi|^\alpha} d\xi\right) t^{-d/\alpha} = t^{-d/\alpha}\frac{\omega_d}{(2\pi)^d \alpha}\int_0^\infty e^{-s} s^{(\frac{n}{\alpha}-1)} ds \\ &= t^{-d/\alpha}\frac{\omega_d \Gamma(d/\alpha)}{(2\pi)^d \alpha}, \qquad \omega_d = \sigma(S_d) \end{aligned}$$

As before,

$$\begin{aligned} p_t^{D,\alpha}(x,y) &= p_t^\alpha(x-y) - \mathbf{E}^x\left(\tau_D < t, p_{t-\tau_D}^\alpha(X(\tau_D), y)\right) \\ &= p_t^\alpha(x-y) - r_t^{D,\alpha}(x,y). \end{aligned}$$

Symmetric Stable, $0 < \alpha < 2$

Two expressions for the free heat kernel: $g_{\alpha/2}(t,s) =$ density of T_t.

$$p_t^\alpha(x) = \int_0^\infty p_s^{(2)}(x)\, g_{\alpha/2}(t,s) ds = \int_0^\infty \frac{1}{(4\pi s)^{d/2}} e^{-|x|^2/4s} g_{\alpha/2}(t,s)\, ds$$

and

$$p_t^\alpha(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} e^{-t|\xi|^\alpha} d\xi$$

This leads to:

$$p_t^\alpha(x-y) \leq c\left(\frac{t}{|x-y|^{d+\alpha}} \wedge \frac{1}{t^{d/\alpha}}\right), \qquad x, y \in \mathbb{R}^d, \ \ t > 0$$

and

$$r_t^{D,\alpha}(x,x) \leq c(\frac{t}{\delta_D^{d+\alpha}(x)} \wedge \frac{1}{t^{d/\alpha}}), \qquad x \in D, \ \ t > 0$$

Relativistic Symmetric Stable, $0 < \alpha < 2$, $m > 0$

Two expressions for the "free density"

$$p_t^{\alpha,m}(x) = e^{mt} \int_0^\infty \frac{1}{(4\pi s)^{d/2}} e^{\frac{-|x|^2}{4s}} e^{(-m^{1/\beta} s)} g_{\alpha/2}(t,s) ds,$$

$$p_t^{\alpha,m}(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} e^{-t\{\left(m^{2/\alpha}+|\xi|^2\right)^{\alpha/2}-m\}} d\xi$$

$$p_t^{\alpha,m}(x-y) \le c(\alpha,d)\left\{\frac{m^{d/\alpha-d/2}}{t^{d/2}} + \frac{1}{t^{d/\alpha}}\right\}, \qquad x,y \in \mathbb{R}^d, \ \ t > 0$$

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$$p_t^{\alpha,m}(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} e^{-t\{\left(m^{2/\alpha}+|\xi|^2\right)^{\alpha/2}-m\}}\,d\xi$$

$$p_t^{\alpha,m}(x-y) \le c(\alpha,d)\left\{\frac{m^{d/\alpha-d/2}}{t^{d/2}} + \frac{1}{t^{d/\alpha}}\right\}, \qquad x,y \in \mathbb{R}^d, \ \ t>0$$

$$p_t^{\alpha,m}(x-y) \le c_1 e^{mt}\left\{\frac{t\,e^{-c_2|x-y|}}{|x-y|^{d+\alpha}} \wedge \frac{1}{t^{d/\alpha}}\right\}, \qquad x,y \in \mathbb{R}^d, \ \ t>0$$

$$r_t^{D,\alpha,m}(x,x) \le c_1 e^{mt}\left\{\frac{t\,e^{-c_2\delta_D(x)}}{\delta_D(x)^{d+\alpha}} \wedge \frac{1}{t^{d/\alpha}}\right\}, \qquad x \in D, \ \ t>0$$

$$\lim_{t\to 0} p_t^{\alpha,m}(0) e^{-mt} t^{d/\alpha} = C_1(\alpha,d) = \frac{\omega_d \Gamma(d/\alpha)}{(2\pi)^d \alpha}, \qquad \omega_d = \sigma(S_d)$$

Trace, stable and relativistic stable (we drop the α, and m)

$$\begin{aligned} Z_D(t) &= \int_D p_D(t,x,x)dx = \int_D p_t(x-x)dx - \int_D r_t^D(x,x)dx \\ &= p_t(0)|D| - \int_D r_t^D(x,x)dx \end{aligned}$$

Lemma (Both Stable and Relativistic Stable)

$$\lim_{t\to 0} t^{d/\alpha} \int_D r_t^D(x,x)dx = 0$$

Proof.

Recall $t^{d/\alpha} r_t^D(x,x) \leq C(\frac{t^{d/\alpha+1}}{\delta_D^{d+\alpha}(x)} \wedge 1)$. Set $D_t = \left\{x \in D : d_d(x) > t^{1/2\alpha}\right\}$. Then

$$t^{d/\alpha} \int_{D\setminus D_t} r_t^D(x,x)dx \leq C|D \setminus D_t|,$$

$$t^{d/\alpha} \int_{D_t} r_t^D(x,x)dx \leq Ct^{d/2\alpha+1/2}|D|, \qquad t \ll 1$$

Corollary (For any set of finite volume D)

$$\lim_{t\to 0} t^{d/\alpha} Z_D(t) = C_1(\alpha,d)|D| = \frac{\omega_d \Gamma(d/\alpha)}{(2\pi)^d \alpha}|D|, \qquad \textbf{Stable}$$

$$\lim_{t\to 0} t^{d/\alpha} e^{-mt} Z_D(t) = C_1(\alpha,d)|D| = \frac{\omega_d \Gamma(d/\alpha)}{(2\pi)^d \alpha}|D|, \qquad \textbf{Relativistic Stable}$$

Stable proved under assumption $vol_d(\partial D) = 0$ by Blumenthal-Getoor 1959.

Corollary

Gives Weyl's asymptotics:

$$\lim_{\lambda\to\infty} \lambda^{-d/\alpha} N(\lambda) = \frac{C_1(\alpha,d)|D|}{\Gamma(d/2+1)}, \qquad \textbf{Stable}$$

and

$$\lim_{\lambda\to\infty} \lambda^{-d/\alpha} e^{m/\lambda} N(\lambda) = \frac{C_1(\alpha,d)|D|}{\Gamma(d/2+1)}, \qquad \textbf{Relativistic Stable}$$

$N(\lambda)$ be the number of eigenvalues $\{\lambda_j\}$ which not exceeding λ

From now on, only α-stable, $0 < \alpha < 2$

Theorem (R-smooth domains: B.–Kulczycki '08)

$$\left| Z_D(t) - \frac{C_1(\alpha,d)|D|}{t^{d/\alpha}} + \frac{C_2(\alpha,d)|\partial D| t^{1/\alpha}}{t^{d/\alpha}} \right| \le \frac{C_3 |D| t^{2/\alpha}}{R^2 t^{d/\alpha}}, \quad t > 0.$$

Theorem (Lipschitz domains: B.–Kulczycki–Siudeja (preprint))

$$t^{d/\alpha} Z_D(t) = C_1(\alpha,d)|D| - C_2(\alpha,d)|\partial D| t^{1/\alpha} + o\left(t^{1/\alpha}\right), \quad t \downarrow 0$$

$$C_1(\alpha,d) = p_1^\alpha(0) = \frac{\omega_d \Gamma(d/\alpha)}{(2\pi)^d \alpha},$$

$$C_2(\alpha,d) = \int_0^\infty r_1^H(q,0,\ldots,0),(q,0,\ldots,0))dq, \text{ where } H = \{x : x_1 > 0\}.$$

Lemma (A geometric property of *R*–smooth domains)

Let $D \subset \mathbb{R}^d$ be R-smooth. Set $D_q = \{x \in D : d_D(x) > q\}$. Then for any $0 < q \leq R/2$

(i)

$$2^{-d+1}|\partial D| \leq |\partial D_q| \leq 2^{d-1}|\partial D|,$$

(ii)

$$|\partial D| \leq \frac{2^d|D|}{R},$$

(iii)

$$\Big||\partial D_q| - |\partial D|\Big| \leq \frac{2^d dq|\partial D|}{R} \leq \frac{2^{2d} dq|D|}{R^2}.$$

Proposition ($t^{1/\alpha} > R/2$)

$$Z_D(t) \leq \frac{C_1|D|}{t^{d/\alpha}} \leq \frac{C_1|D|t^{2\alpha}}{R^2 t^{d/\alpha}}$$

and by (ii),

$$\frac{C_2|\partial D|t^{1/\alpha}}{t^{d/\alpha}} \leq \frac{2^d C_2|D|t^{1/\alpha}}{Rt^{d/\alpha}} \leq \frac{2^{d+1} C_2|D|t^{2/\alpha}}{R^2 t^{d/\alpha}}$$

This implies Theorem for $t^{1/\alpha} > R/2$.

$$Z_D(t) - \frac{C_1|D|}{t^{d/\alpha}} = -\int_D r_t^D(x,x)dx = -\int_{D_{R/2}} r_t^D(x,x)dx - \int_{D\setminus D_{R/2}} r_t^D(x,x)dx$$

As before, for $t^{1/\alpha} \leq R/2$,

$$\int_{D_{R/2}} r_t^D(x,x)dx \leq \frac{C|D|t^{2/\alpha}}{R^2 t^{d/\alpha}}$$

Lemma

For $x \in D \setminus D_{R/2}$, let $x_* \in \partial D$ with $d_D(x) = |x - x_*|$. Let $B_1(z_1, R)$ and $B_2(z_2, R)$ be the balls of radius R passing through x_* with $B_1 \subset D$ and $B_2 \subset D^c$. Let $H(x)$ be the half space containing B_1 perpendicular to $\overline{z_1 z_2}$. For $t^{1/\alpha} < R$,

$$\left| \int_{D \setminus D_{R/2}} r_t^D(x,x)dx - \int_{D \setminus D_{R/2}} r_t^{H(x)}(x,x)dx \right| \leq \frac{C|D|t^{2/\alpha}}{R^2 t^{d/\alpha}}$$

Recall

$$H = \{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : x_1 > 0\}$$

Set

$$f_H(t,q) = r_t^H((q,0,\ldots,0),(q,0,\ldots,0)), \qquad q > 0$$

Then,

$$r_t^{H(x)}(x,x) = f_H(t, d_{H(x)}(x))$$

and

$$f_H(t,q) = t^{-d/\alpha} f_H(1, qt^{-1/\alpha}), \quad f_H(1,q) \leq c(q^{-d-\alpha} \wedge 1).$$

$$
\begin{aligned}
\int_{D\setminus D_{R/2}} r_t^{H(x)}(x,x)\,dx &= \int_0^{R/2} |\partial D_u| f_H(t,u)\,du \\
&= \frac{1}{t^{d/\alpha}} \int_0^{R/2} |\partial D_u| f_H(1, ut^{-1/\alpha})\,du \\
&= \frac{t^{1/\alpha}}{t^{d/\alpha}} \int_0^{R/(2t^{1/\alpha})} |\partial D_{t^{1/\alpha}q}| f_H(1,q)\,dq,
\end{aligned}
$$

For R-smooth regions,

$$
\begin{aligned}
\frac{t^{1/\alpha}}{t^{d/\alpha}} \int_0^{R/(2t^{1/\alpha})} \left| |\partial D_{t^{1/\alpha}q}| - |\partial D| \right| f_H(1,q) dq &\leq \frac{c|D|t^{2/\alpha}}{R^2 t^{d/\alpha}} \int_0^{R/(2t^{1/\alpha})} q f_H(1,q) dq \\
&\leq \frac{c|D|t^{2/\alpha}}{R^2 t^{d/\alpha}} \int_0^{\infty} q(q^{-d-\alpha} \wedge 1)\,dq \\
&\leq \frac{c|D|t^{2/\alpha}}{R^2 t^{d/\alpha}}.
\end{aligned}
$$

Remains to show:

$$\left|\frac{t^{1/\alpha}|\partial D|}{t^{d/\alpha}}\int_0^{R/(2t^{1/\alpha})} f_H(1,q)\,dq - \frac{t^{1/\alpha}|\partial D|}{t^{d/\alpha}}\int_0^{\infty} f_H(1,q)\,dq\right| \le \frac{c|D|t^{2/\alpha}}{R^2\,t^{d/\alpha}}.$$

or

$$\frac{t^{1/\alpha}|\partial D|}{t^{d/\alpha}}\left|\int_{R/(2t^{1/\alpha})}^{\infty} f_H(1,q)\,dq\right| \le \frac{c|D|t^{2/\alpha}}{R^2\,t^{d/\alpha}}.$$

Remains to show:

$$\left|\frac{t^{1/\alpha}|\partial D|}{t^{d/\alpha}}\int_0^{R/(2t^{1/\alpha})} f_H(1,q)\,dq - \frac{t^{1/\alpha}|\partial D|}{t^{d/\alpha}}\int_0^{\infty} f_H(1,q)\,dq\right| \le \frac{c|D|t^{2/\alpha}}{R^2 t^{d/\alpha}}.$$

or

$$\frac{t^{1/\alpha}|\partial D|}{t^{d/\alpha}}\left|\int_{R/(2t^{1/\alpha})}^{\infty} f_H(1,q)\,dq\right| \le \frac{c|D|t^{2/\alpha}}{R^2 t^{d/\alpha}}.$$

Recall: $R/(2t^{1/\alpha}) \ge 1$. Thus, for $q \ge R/(2t^{1/\alpha})$ we have

$$f_H(1,q) \le cq^{-d-\alpha} \le cq^{-2},$$

$$\Rightarrow \quad \int_{R/(2t^{1/\alpha})}^{\infty} f_H(1,q)\,dq \le c\int_{R/(2t^{1/\alpha})}^{\infty} \frac{dq}{q^2} \le \frac{ct^{1/\alpha}}{R}.$$

Again, use

$$|\partial D| \le \frac{2^d|D|}{R},$$

to conclude.

Happy Birthday Richard