SUPPLEMENTARY MATERIALS: SHAPE OPTIMIZATION OF SHELL STRUCTURE ACOUSTICS *

HARBIR ANTIL†, SEAN HARDESTY‡, AND MATTHIAS HEINKENSCHLOSS§

SM1. Introduction. This supplement contains details that were omitted from the main paper [SM2] because of page limitations.

SM2. The Model for the Shell Structure Acoustic Interaction. To avoid switching between the main paper and this supplement, we have included the entire expanded version of Section 2, not only the details and proofs missing from Section 2 of the main paper.

SM2.1. Naghdi Shells. When an elastic body is much smaller in one dimension than in the other two, it can be modeled using shell equations. The Naghdi shell model is derived from standard linear elasticity, reducing the original problem (1a-c) from three dimensions to two. The classical derivation makes use of a kinematic assumption and a mechanical assumption, which are not strictly necessary: Delfour has developed a version of the Naghdi model based on "intrinsic" differential calculus, which avoids recourse to these assumptions [SM13, SM14]. We follow the more classical approach proposed by Blouza and Le Dret [SM5, SM7] because it is well-known and closely tied to the implementation of the long-used MITC finite element methods; see [SM9, §6.3], [SM8, SM9].

In the derivation of Naghdi's model, we assume u to be real-valued. It will become complex-valued in §SM2.3, when we couple it with the Helmholtz equation. We consider "thin" domains, which are described by a so-called middle surface chart $\phi : \Omega_0 \subset \mathbb{R}^2 \to \mathbb{R}^3$ defined on a reference domain $\Omega_0 \subset \mathbb{R}^2$ and a thickness function $t : \Omega_0 \to \mathbb{R}^+$. We make the following assumptions.

(A1) The reference domain $\Omega_0 \subset \mathbb{R}^2$ is bounded and satisfies the strong local Lipschitz condition.

(A2) The chart function ϕ belongs to the set

$$\mathcal{C} = \Big\{ \phi \in W^2_\infty(\Omega_0)^3 : \phi \text{ is one-to-one and } \partial_\alpha \phi(x_1, x_2), \alpha = 1, 2, \text{are linearly independent for all } (x_1, x_2) \in \Omega_0 \Big\}. \tag{SM1}$$

We define the covariant basis vectors

$$a_\alpha(x_1, x_2) = \partial_\alpha \phi(x_1, x_2), \ \alpha = 1, 2, \tag{SM2a}$$

which span the plane tangent to the middle surface, and we define the unit normal to the middle surface

$$a_3(x_1, x_2) = \frac{a_1(x_1, x_2) \times a_2(x_1, x_2)}{|a_1(x_1, x_2) \times a_2(x_1, x_2)|}. \tag{SM2b}$$

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†Department of Mathematical Sciences, George Mason University, Fairfax, VA 22030. E-mail. hantil@gmu.edu

‡Z-Terra Inc., 17171 Park Row, Ste. 247, Houston, TX 77084. E-mail. hardesty@z-terra.com

§Department of Computational and Applied Mathematics, MS-134, Rice University, 6100 Main Street, Houston, TX 77005-1892. E-mail. heinken@rice.edu

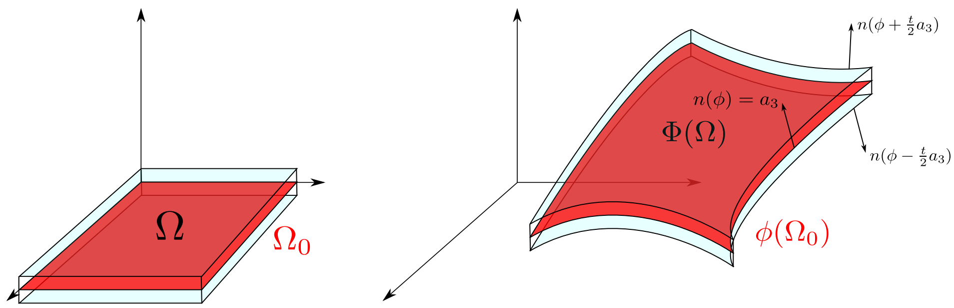

FIG. SM1. *Naghdi shell geometry: The physical domain $\widetilde{\Omega} = \Phi(\Omega)$ is constructed via the thickness function t and the chart function ϕ through the mapping $\Phi(x_1, x_2, x_3) = \phi(x_1, x_2) + x_3 a_3(x_1, x_2)$. of the reference domain $\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_2) \in \Omega_0$ and $|x_3| < t(x_1, x_2)/2\}$.*

The dual contravariant basis vectors a^i are defined via

$$a^i(x_1, x_2) \cdot a_j(x_1, x_2) = \delta^i_j, \; i, j = 1, 2, 3$$

where δ^i_j is the Kronecker delta. The covariant and contravariant components of the metric tensor are respectively

$$a_{ij} = a_i \cdot a_j, \; (a^{ij}) = (a_{ij})^{-1}.$$

Furthermore we define the change of metric factor

$$\sqrt{a(x_1, x_2)} = \sqrt{a_{11}(x_1, x_2) a_{22}(x_1, x_2) - a_{12}(x_1, x_2)^2}. \tag{SM2c}$$

See [SM10, SM9, SM16] for more details on shell geometries.

Our domain is the image of the reference domain

$$\Omega = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_2) \in \Omega_0 \text{ and } |x_3| < t(x_1, x_2)/2\right\}, \tag{SM3}$$

under the mapping

$$\Phi(x_1, x_2, x_3) = \phi(x_1, x_2) + x_3 a_3(x_1, x_2), \tag{SM4}$$

(see Figure SM1), and is given by

$$\widetilde{\Omega} = \Phi(\Omega).$$

For the remainder of this subsection we denote points in $\widetilde{\Omega}$ by $\widetilde{x}$ and points in Ω_0 by x. Greek subscripts and superscripts take values in $\{1, 2\}$, while Latin subscripts and superscripts take values in $\{1, 2, 3\}$. Throughout the paper we use the Einstein summation convention.

PROPOSITION SM2.1. *If* (A1) *and* (A2) *hold, then the map $\Phi : \Omega_0 \times \mathbb{R} \to \mathbb{R}^3$ defined in* (SM4) *is a local W^1_∞-diffeomorphism.*

Proof. Notice that by its definition, $a_3 \in W^1_\infty(\Omega)$. Therefore, the requirements of [SM1, Theorem 3.7] are fulfilled, and thus Φ is a local W^1_∞-diffeomorphism.

Alternatively, from the definition of Φ we obtain

$$\nabla\Phi = \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} + x_3 \begin{pmatrix} \partial_1 a_3 & \partial_2 a_3 & 0 \end{pmatrix}.$$

For every $\epsilon > 0$, by (A2), we have that $\nabla\Phi \in L^\infty(\Omega_0 \times (-\epsilon, \epsilon))$. In addition, $\det \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} > 0$, which implies that $\det \nabla\Phi > 0$ on $\Omega_0 \times (-\epsilon, \epsilon)$ when ϵ is small enough, i.e., $\nabla\Phi$ is invertible.

Therefore the inverse function theorem can be applied, which shows that *locally* Φ is a W^1_∞-diffeomorphism. ◻

REMARK SM2.2. *Notice that it is possible to reduce the regularity requirement on a_1, a_2 to $W^1_\infty(\Omega)$, instead of $W^2_\infty(\Omega)$ and still arrive at Proposition SM2.1, see [SM1].*

Ciarlet in [SM10] uses a similar diffeomorphism property of Φ as in Proposition SM2.1 to show that the shell model is a limiting case of 3d elasticity. In our work, we will use Proposition SM2.1 in §SM2.2 in order to write the boundary integral equations for the Helmholtz screen problem.

We represent displacements on the reference domain Ω. For $\widetilde{x} = \Phi(x)$ we define

$$u(x) = \widetilde{u}(\Phi(x)) = \widetilde{u}(\widetilde{x}). \tag{SM5}$$

The Naghdi model of Blouza and Le Dret [SM5, SM7] is derived from 3d elasticity on Ω by use of the Reissner-Mindlin kinematic assumption, i.e., by assuming the following form for the 3d elastic displacement:

$$u(x_1, x_2, x_3) = z(x_1, x_2) + x_3\theta(x_1, x_2). \tag{SM6}$$

The displacement $u : \Omega \to \mathbb{R}^3$ is composed of the displacement $z : \Omega_0 \to \mathbb{R}^3$ of the middle surface plus a first-order rotation $\theta : \Omega_0 \to \mathbb{R}^3$, with $\theta \cdot a_3 = 0$. This means that material lines normal to the undeformed middle surface can translate and make a first-order rotation.

To derive the Naghdi shell equations, the functions z and θ can be represented in the locally-varying basis $\{a_1, a_2, a_3\}$ or the through their Cartesian components. To distinguish between between vectors and their components we temporarily use vector symbols. That is we write

$$\vec{u}(x_1, x_2, x_3) = \vec{z}(x_1, x_2) + x_3\vec{\theta}(x_1, x_2)$$

instead of (SM6). To derive the Naghdi shell equations, the locally-varying basis $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ is used to represent the shell geometry and the constitutive tensors. In the classical formulation (see, e.g., [SM9, §4.2.2]), it is also used to represent the vectors $\vec{z}$ and $\vec{\theta}$: they are identified with their covariant components $z = (z_1, z_2, z_3)$ and $\theta = (\theta_1, \theta_2)$ via

$$\vec{z}(x_1, x_2) = z_i(x_1, x_2)\vec{a}^i(x_1, x_2),\ \vec{\theta}(x_1, x_2) = \theta_\alpha(x_1, x_2)\vec{a}^\alpha(x_1, x_2).$$

Blouza and Le Dret [SM5, SM7] instead represent the vectors appearing in (SM6) through their Cartesian components; in this vein, see also the more specialized papers by Le Dret [SM15], Sprekels and Tiba [SM25], and Bletzinger et al. [SM4]. The presentation in [SM5, SM7] is simpler than the classical one and does not require differential geometry concepts such as covariant derivatives and the second fundamental form.

We adopt a hybrid approach in that we identify $\vec{\theta}$ with its covariant components and $\vec{z}$ with its Cartesian components, i.e., we drop the vector symbols and simply write $\theta = (\theta_1, \theta_2)$ to indicate the covariant components of $\vec{\theta}$, and $z = (z_1, z_2, z_3)$ to indicate the Cartesian components of $\vec{z}$. Thus, the kinematic assumption appears

$$u(x_1, x_2, x_3) = z(x_1, x_2) + x_3\theta_\alpha(x_1, x_2)a^\alpha(x_1, x_2). \tag{SM7}$$

This choice will allow us to formulate the problem in a function space that does not depend on ϕ as does that of Blouza and Le Dret through the tangency condition $\vec{\theta}\cdot\ \vec{a}_3 = 0$, and in fact is more closely tied to practical finite element implementation than either the classical approach or that of Blouza and Le Dret [SM5, SM7]; see [SM9, §6.3].

If we use the geometric assumptions on $\widetilde{\Omega} = \Phi(\Omega)$ stated above, (SM5), as well as the kinematic assumption (SM7) for u and a corresponding assumption $v(x_1, x_2, x_3) = y(x_1, x_2) + x_3\eta_\alpha(x_1, x_2)a^\alpha(x_1, x_2)$, for the test function, then the 3d elastic bilinear form

$$\int_{\widetilde{\Omega}} (H : e(\widetilde{u})(\widetilde{x})) : e(\widetilde{v})(\widetilde{x})\, d\widetilde{x}$$

leads to

$$\begin{aligned} K(\theta, z; \eta, y) := \int_{\Omega_0} \Big(C^{\alpha\beta\lambda\mu}\Big[t\gamma_{\alpha\beta}(z)\gamma_{\lambda\mu}(y) + \frac{t^3}{12}\chi_{\alpha\beta}(\theta, z)\chi_{\lambda\mu}(\eta, y)\Big] \\ + tD^{\lambda\mu}\zeta_\lambda(\theta, z)\zeta_\mu(\eta, y)\Big)\sqrt{a(x)}\, dx \end{aligned} \tag{SM8}$$

with $\gamma_{\alpha\beta}, \chi_{\alpha\beta}, \zeta_\alpha, C^{\alpha\beta\lambda\mu}, D^{\lambda\mu}$ defined next.

The tensors γ, χ, ζ, which correspond to membrane stretching, bending, and transverse shear, respectively, are given by

$$\begin{aligned} \gamma_{\alpha\beta}(z) &\stackrel{\text{def}}{=} \tfrac{1}{2}\big(a_\alpha(x)\cdot\partial_\beta z(x) + a_\beta(x)\cdot\partial_\alpha z(x)\big), \\ \chi_{\alpha\beta}(\theta, z) &\stackrel{\text{def}}{=} \tfrac{1}{2}\big(a_\alpha(x)\cdot\partial_\beta(\theta_\lambda(x)a^\lambda(x)) + a_\beta(x)\cdot\partial_\alpha(\theta_\lambda(x)a^\lambda(x)) \\ &\qquad + \partial_\alpha a_3(x)\cdot\partial_\beta z(x) + \partial_\beta a_3(x)\cdot\partial_\alpha z(x)\big), \\ \zeta_\alpha(\theta, z) &\stackrel{\text{def}}{=} \tfrac{1}{2}\big(\theta_\alpha(x) + a_3(x)\cdot\partial_\alpha z(x)\big). \end{aligned}$$

The form of these tensors is as in Blouza and Le Dret [SM5, SM7], in contrast to the classical formulation, which makes use of covariant derivatives and the second fundamental form. Using covariant derivatives here would simplify the appearance of the terms in χ involving θ, but would obscure the dependence on the chart function ϕ. We note that these tensors still make sense for charts that are W^2_∞: the classical formulation required a C^3 chart in order to formulate the standard problem of a shell clamped on an edge because the Koiter rigid-body lemma was not known until [SM6, Thm. 6] to hold for W^2_∞ charts. In our problem, derived from (1), no clamping is applied, and the rigid-body lemma is not required.

The Naghdi constitutive tensors $(C^{\alpha\beta\lambda\mu})$ and $(D^{\lambda\mu})$ are obtained from the constitutive tensor H in (1b). If the material is homogeneous and isotropic, i.e., if

$$H^{ijkl} = \frac{E}{2(1+\nu)}\left(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk} + \frac{2\nu}{1-2\nu}\delta^{ij}\delta^{kl}\right), \tag{SM10}$$

$i, j, k, l = 1, 2, 3$, where E is the Young modulus, ν is the Poisson ratio, and δ^{ij} is the Kronecker delta, then the components of the Naghdi constitutive tensor are given by

$$C^{\alpha\beta\lambda\mu} = \frac{E}{2(1+\nu)}\left(a^{\alpha\lambda}a^{\beta\mu} + a^{\alpha\mu}a^{\beta\lambda} + \frac{2\nu}{1-\nu}a^{\alpha\beta}a^{\lambda\mu}\right) \tag{SM11}$$

and $D^{\lambda\mu} = 2Ea^{\lambda\mu}/(1+\nu)$.[1] The appearance of $1-\nu$ in the second denominator of

[1] Blouza and Le Dret [SM7] use Lamé moduli instead of Young's modulus and Poisson ratio. We note that the formula (3.2) in [SM7] for the Naghdi constitutive tensor contains several typographical errors; the indices in [SM7, Eqn. (3.2)] should be the ones given in (SM11). See also [SM10, p. 147] and [SM9, Eqns. (4.34), (4.35)].

(SM11) instead of $1-2\nu$ as in (SM10) is due to the mechanical assumption of zero normal stress; see [SM9, §4.2.1].

Similarly, applying the geometry description and the Naghdi assumption (SM7) to the inertial term

$$\int_{\widetilde{\Omega}} \rho \widetilde{u}(\widetilde{x}) \cdot \widetilde{v}(\widetilde{x})\, d\widetilde{x},$$

which corresponds to the weak form of the left hand side in (1a), leads to the Naghdi inertial form

$$M(\theta, z; \eta, y) := \int_{\Omega_0} \rho \Big(tz(x) \cdot y(x) + \frac{t^3}{12} \theta_\alpha(x) a^{\alpha\beta} \eta_\beta(x) \Big) \sqrt{a(x)}\, dx. \tag{SM12}$$

The (weak form of the) equations (1a-c) are replaced by the following Naghdi shell equations. Let

$$\mathcal{S} = H^1(\Omega_0)^2 \times H^1(\Omega_0)^3. \tag{SM13}$$

We seek $(\theta, z) \in \mathcal{S}$ such that

$$K(\theta, z; \eta, y) - \omega^2 M(\theta, z; \eta, y) = \int_{\Omega_0} h(x) \cdot y(x) \sqrt{a(x)}\, dx \tag{SM14}$$

for all $(\eta, y) \in \mathcal{S}$. Here h is determined from the right hand side (1c).

We make the following assumptions on the Naghdi constitutive tensors.

(A3) There exists constants $0 < c_1 < c_2$ and $0 < c_3 < c_4$ such that for almost all $x \in \Omega_0$ and all symmetric tensors γ, χ and all vectors ζ, ξ,

$$C^{\alpha\beta\lambda\mu}(x)\gamma_{\alpha\beta}\gamma_{\lambda\mu} \geq c_1 \sum_{\alpha,\beta} \gamma_{\alpha\beta}^2, \qquad C^{\alpha\beta\lambda\mu}(x)\gamma_{\alpha\beta}\chi_{\lambda\mu} \leq c_2 \gamma_{\alpha\beta}\chi_{\alpha\beta},$$
$$D^{\lambda\mu}(x)\zeta_\lambda\zeta_\mu \geq c_3 \sum_{\lambda} \zeta_\lambda^2, \qquad D^{\lambda\mu}(x)\zeta_\lambda\xi_\mu \leq c_4 \zeta_\lambda\xi_\lambda.$$

Assumption (A3) holds if (A2) is valid and the Naghdi constitutive tensors $(C^{\alpha\beta\lambda\mu})$ and $(D^{\lambda\mu})$ are obtained from the tensor (SM10) for homogeneous isotropic materials with $E > 0$ and $\nu \in (0, 1/2)$.

Blouza and LeDret [SM7, Lemma 3.6] use assumptions (A2) and (A3) to prove that the bilinear form K in (SM8) is bounded and coercive on $\mathcal{S}_0 \times \mathcal{S}_0$ provided that the shell is clamped and $\mathcal{S}_0 \subset \mathcal{S}$ is the subspace that incorporates the clamping condition. To analyze the coupled problem we will need a slightly different result.

We first make the following standard assumption for the thickness function t.

(A4) The thickness function t belongs to the set

$$\mathcal{T} = \left\{ t \in L^\infty(\Omega_0) : t(x) \geq t_{\min} > 0 \text{ for almost all } x \in \Omega_0 \right\}. \tag{SM15}$$

Lemma SM2.3. *If the assumptions (A1)–(A4) hold, then there exists a constant $k_0 > 0$ such that the bilinear form*

$$((\theta, z), (\eta, y)) \mapsto K(\theta, z; \eta, y) + k_0 \int_{\Omega_0} \theta(x) \cdot \eta(x) + z(x) \cdot y(x)\, dx$$

on $\mathcal{S} \times \mathcal{S}$ is bounded and coercive.

Proof. Boundedness follows in a straightforward manner from (A1)–(A4).

For coercivity, we need (A3) plus bounds on the L^2-norms of the various tensors. The proof uses ideas from [SM9, p. 104].

Given a vector $w = (w_1, w_2)$, define

$$e_{\alpha\beta}(w) = \tfrac{1}{2}(\partial_\alpha w_\beta + \partial_\beta w_\alpha).$$

Then, Korn's inequality [SM8, §VI.3.1] states that there exists $c > 0$ such that

$$\sum_{\alpha,\beta} \|e_{\alpha\beta}(w)\|^2_{L^2(\Omega_0)} + \|w\|^2_{L^2(\Omega_0)^2} \geq c\|w\|^2_{H^1(\Omega_0)^2}.$$

We consider the tensors γ, χ, δ separately.

- γ: define $w_\alpha = z \cdot a_\alpha$. Then,

$$\begin{aligned} e_{\alpha\beta}(w) =& \tfrac{1}{2}(\partial_\alpha(z \cdot a_\beta) + \partial_\beta(z \cdot a_\alpha)) \\ =& \underbrace{\tfrac{1}{2}(\partial_\alpha z \cdot a_\beta + \partial_\beta z \cdot a_\alpha)}_{=\gamma_{\alpha\beta}(z)} + \tfrac{1}{2} z \cdot (\partial_\alpha a_\beta + \partial_\beta a_\alpha). \end{aligned}$$

  Using Korn's inequality, the triangle inequality, and $\phi \in W^2_\infty(\Omega_0)^3$ (which implies that the quantity $\|\partial_\alpha a_\beta + \partial_\beta a_\alpha\|_{L^\infty(\Omega_0)}$ can be bounded by a constant) yields the bound

$$\text{(SM16)} \qquad \sum_{\alpha,\beta} \|\gamma_{\alpha\beta}(z)\|^2_{L^2(\Omega_0)} + \|z\|^2_{L^2(\Omega_0)^3} \geq c_\gamma \|(z \cdot a_1, z \cdot a_2)\|^2_{H^1(\Omega_0)^2}.$$

  Note that (SM16) controls only the in-plane part of $\|z\|_{H^1(\Omega_0)^3}$.

- χ: define $w_\alpha = a_\alpha \cdot (\theta_\lambda a^\lambda)$. Then,

$$\begin{aligned} e_{\alpha\beta}(w) =& \tfrac{1}{2}(\partial_\alpha w_\beta + \partial_\beta w_\alpha) \\ =& \underbrace{\tfrac{1}{2}(a_\beta \cdot \partial_\alpha(\theta_\lambda a^\lambda) + a_\alpha \cdot \partial_\beta(\theta_\lambda a^\lambda))}_{=\chi_{\alpha\beta}(\theta,z) - \frac{1}{2}(\partial_\alpha z \cdot \partial_\beta a_3 + \partial_\beta z \cdot \partial_\alpha a_3)} + \tfrac{1}{2}(\theta_\lambda a^\lambda) \cdot (\partial_\alpha a_\beta + \partial_\beta a_\alpha). \end{aligned}$$

  Arguing as with γ, we obtain the bound

$$\text{(SM17)} \quad \sum_{\alpha,\beta} \|\chi_{\alpha\beta}(\theta, z)\|^2_{L^2(\Omega_0)} + \|z\|^2_{H^1(\Omega_0)^3} + \|\theta\|^2_{L^2(\Omega_0)^2} \geq c_\chi \|\theta\|^2_{H^1(\Omega_0)^2}.$$

- ζ: it remains to control $\|z \cdot a_3\|_{H^1(\Omega_0)}$. Observe that

$$\partial_\alpha(z \cdot a_3) = \underbrace{\partial_\alpha z \cdot a_3}_{=2\zeta_\alpha(\theta,z) - \theta_\alpha} + z \cdot \partial_\alpha a_3.$$

  Thus, we obtain the bound

$$\text{(SM18)} \quad \sum_{\alpha} \|\zeta_\alpha(\theta, z)\|^2_{L^2(\Omega_0)} + \|\theta\|^2_{L^2(\Omega_0)^2} + \|z\|^2_{L^2(\Omega_0)^3} \geq c_\zeta \|z \cdot a_3\|^2_{H^1(\Omega_0)}.$$

Combining (SM16), (SM17), and (SM18), we arrive at

$$\begin{aligned} &\sum_{\alpha,\beta} \left(\|\gamma_{\alpha\beta}(z)\|^2_{L^2(\Omega_0)} + \|\chi_{\alpha\beta}(\theta, z)\|^2_{L^2(\Omega_0)} \right) + \\ &\quad \sum_{\alpha} \|\zeta_\alpha(\theta, z)\|_{L^2(\Omega_0)^2} + \|z\|^2_{L^2(\Omega_0)^3} + \|\theta\|^2_{L^2(\Omega_0)^2} \\ &\quad \geq C \left(\|z\|^2_{H^1(\Omega_0)^3} + \|\theta\|^2_{H^1(\Omega_0)^2} \right). \end{aligned}$$

∎

REMARK SM2.4. *In the next section, and for the rest of the paper, the field variables* (θ, z) *are assumed to be complex-valued, with the real part being the physical value. For a complex-valued forcing function* h*, the analogue to* (SM14) *is*

$$K(\theta, z; \overline{\eta}, \overline{y}) - \omega^2 M(\theta, z; \overline{\eta}, \overline{y}) = \int_{\Omega_0} h(x) \cdot \overline{y}(x) dx. \tag{SM19}$$

In particular h *depends on* φ *and is used to couple (SM19) with the approximation of the Helmholtz equation (1d-f).*

Since K (SM8) *and* M (SM12) *are real, symmetric bilinear forms, Lemma SM2.3 remains valid if complex-valued* (θ, z) *and* $(\overline{\eta}, \overline{y})$ *are used.*

In the next subsection we describe the approximation of the Helmholtz equation (1d-f), and then we state and analyze the coupled system, which contains (SM19).

SM2.2. Screen Boundary Integral Equations. We continue to use the geometry description of the shell introduced in the previous section. In this section we discuss the approximation of the Helmholtz equation (1d-f) by a boundary integral equation on the midsurface $\phi(\Omega_0)$ of the shell. This leads to the so-called "screen" problem studied by Stephan [SM26]. In this section we outline the approximations that lead to the screen problem, sketch the derivation of the corresponding boundary integral equations and summarize results that will be needed to state and analyze the coupled problem.

To arrive at the screen problem we consider the Helmholtz equation (1d-f) and make the following approximations; see Martinez [SM22].

(S1) The surfaces $\{\phi(x) \pm t(x)a_3(x)/2 : x \in \Omega_0\}$ of the shell (see Figure SM1) are approximated by the middle surface $\phi(\Omega_0)$.

(S2) The normals on the shell surfaces $\{\phi(x) \pm t(x)a_3(x)/2 : x \in \Omega_0\}$ are approximated using the midsurface normal $n(\phi(x)) = a_3(x)$.

We emphasize the physical nature of the approximations (S1,S2): if the thickness of the shell is small compared to acoustic wavelengths, it is reasonable to pose the problem as if the acoustic coupling actually happens on the middle surface. The thickness function t retains a strong influence over the way in which the shell moves, but has no direct effect on the acoustic coupling. Due to the smoothness assumption (A2) on the chart function ϕ, the midsurface normal $n(\phi(x)) = a_3(x)$ is well-defined. The Reissner-Mindlin kinematic assumption (SM7) assures that the normal velocity $u \cdot n$ is the same on both sides of the shell and is given by $z \cdot n$, and therefore the Helmholtz screen equation with Neumann data can be coupled with the Naghdi model in a kinematically consistent way.

With the approximations (S1,S2), the Helmholtz equation (1d-f) on the exterior of the shell $\widetilde{\Omega} = \Phi(\Omega)$ can be approximated by the following equation on the exterior of the midsurface $\phi(\Omega_0)$ of the shell. To simplify the notation, we denote the midsurface by

$$\Gamma_0 \stackrel{\text{def}}{=} \phi(\Omega_0). \tag{SM20}$$

The approximation of (1d-f) is given by

$$\Delta\varphi(\widetilde{x}) + \kappa^2\varphi(\widetilde{x}) = 0 \qquad \widetilde{x} \in \mathbb{R}^3 \setminus \Gamma_0 \tag{SM21a}$$

$$\partial_n\varphi(\widetilde{x}) = g(\widetilde{x}) \qquad \widetilde{x} \in \Gamma_0 \tag{SM21b}$$

$$\left|\nabla\varphi \cdot \widetilde{x}/|\widetilde{x}| - i\kappa\varphi\right| = O\left(1/|\widetilde{x}|^2\right) \qquad \text{as } |\widetilde{x}| \to \infty. \tag{SM21c}$$

The right hand side g in (SM21b) is derived from the right hand side in (1d). Let $\widetilde{x} = \phi(x)$ and let $x \in \Omega_0$ be a point on the midsurface Γ_0. If we use (SM5), (SM7) and (S1,S2), then the right hand side $-i\omega\widetilde{u}(\widetilde{x}) \cdot n(\widetilde{x})$ in (1d) corresponds to

$$g(\phi(x)) = -i\omega z(x) \cdot n(\phi(x)) = -i\omega z(x) \cdot a_3(x). \tag{SM22}$$

where $\phi(x)$, $x \in \Omega_0$, is a point on the midsurface Γ_0.

Next, we recall the uniqueness result for the screen problem (SM21); the existence result will be provided in Theorem SM2.7, following from the equivalence between the screen problem and a corresponding boundary integral equation, established in Lemma SM2.6.

LEMMA SM2.5. *Let (A1) and (A2) hold. If* $\varphi \in \{\varrho \in H^1_{loc}(\mathbb{R}^3 \setminus \Gamma_0) \,:\, (\Delta + \kappa^2)\varrho = 0$ *in* $\mathbb{R}^3 \setminus \Gamma_0$ *and* ϱ *satisfies (SM21c)*$\}$ *solves (SM21) with* $g = 0$*, then* $\varphi = 0$.

Proof. See [SM26, Lemma 2.1] and [SM16, Lemma 4.3.5]. ∎

We equivalently reformulate (SM21) as a boundary integral equation. The Green's function for the Helmholtz equation is

$$G(\widetilde{x}, \widetilde{x}') = \frac{\exp(i\kappa|\widetilde{x}' - \widetilde{x}|)}{4\pi|\widetilde{x}' - \widetilde{x}|}.$$

We extend the shell midsurface Γ_0 to a Lipschitz continuous surface $\Gamma \supset \Gamma_0$ such that Γ is the boundary of a bounded Lipschitz domain $G_1 \subset \mathbb{R}^3$. Thus,

$$\Gamma_0 \subset \Gamma = \partial G_1. \tag{SM23}$$

We denote by γ^- and γ^+ respectively the trace operator from the interior and exterior of G_1, and by ∂_n^- and ∂_n^+ respectively the interior and exterior normal derivative on ∂G_1. The boundary integral equation corresponding to (SM21) is derived from the boundary integral equation for the Helmholtz equation for G_1, see [SM26, p. 243]. Given boundary data from an exterior solution to the Helmholtz equation, the exterior representation formula yields zero when computed at interior points. Likewise, given boundary data from an interior solution to the Helmholtz equation, the interior representation formula yields zero when computed at exterior points; see [SM11, Thm. 3.1]. Thus, for points interior to G_1 we have

$$\begin{aligned}
\varphi(\widetilde{x}') &= \int_\Gamma \partial_n^- \varphi(\widetilde{x}) G(\widetilde{x}, \widetilde{x}') - \partial_{n,\widetilde{x}} G(\widetilde{x}, \widetilde{x}')\gamma^- \varphi(\widetilde{x})\, d\widetilde{x} \\
0 &= \int_\Gamma \partial_{n,\widetilde{x}} G(\widetilde{x}, \widetilde{x}')\gamma^+ \varphi(\widetilde{x}) - \partial_n^+ \varphi(\widetilde{x}) G(\widetilde{x}, \widetilde{x}')\, d\widetilde{x},
\end{aligned}$$

and for points exterior to G_1,

$$\begin{aligned}
0 &= \int_\Gamma \partial_n^- \varphi(\widetilde{x}) G(\widetilde{x}, \widetilde{x}') - \partial_{n,\widetilde{x}} G(\widetilde{x}, \widetilde{x}')\gamma^- \varphi(\widetilde{x})\, d\widetilde{x} \\
\varphi(\widetilde{x}') &= \int_\Gamma \partial_{n,\widetilde{x}} G(\widetilde{x}, \widetilde{x}')\gamma^+ \varphi(\widetilde{x}) - \partial_n^+ \varphi(\widetilde{x}) G(\widetilde{x}, \widetilde{x}')\, d\widetilde{x}.
\end{aligned}$$

In either case, the two equations add to

$$\begin{aligned}
\varphi(\widetilde{x}') = \int_\Gamma & G(\widetilde{x}, \widetilde{x}') \underbrace{\left(\partial_n^- \varphi(\widetilde{x}) - \partial_n^+ \varphi(\widetilde{x})\right)}_{=0 \text{ on } \Gamma} \\
&+ \partial_{n,\widetilde{x}} G(\widetilde{x}, \widetilde{x}') \underbrace{\left(\gamma^+ \varphi(\widetilde{x}) - \gamma^- \varphi(\widetilde{x})\right)}_{=0 \text{ on } \Gamma \setminus \Gamma_0} d\widetilde{x}.
\end{aligned} \tag{SM24}$$

Let

$$[\varphi] \stackrel{\text{def}}{=} \gamma^+\varphi - \gamma^-\varphi \tag{SM25}$$

denote the jump of φ on the shell midsurface Γ_0. Equation (SM24) immediately yields the representation formula

$$\varphi(\widetilde{x}') = \int_{\Gamma_0} [\varphi](\widetilde{x})\, \partial_{n,\widetilde{x}} G(\widetilde{x}, \widetilde{x}')\, d\widetilde{x} \quad \text{for } \widetilde{x}' \notin \Gamma_0 \tag{SM26}$$

for the solution φ to the screen problem (SM21); see [SM26, p. 243]. Since we are given Neumann boundary data in (SM21), we take the normal derivative of (SM26) to derive an integral equation for the jump of φ across the shell midsurface Γ_0.

To derive the boundary integral operator corresponding to the screen problem, we first review the hypersingular operator corresponding to the Helmholtz equation for $\Gamma = \partial G_1$. The double-layer potential DL applied to $\varsigma \in L^1(\Gamma)$ is

$$\big(DL\varsigma\big)(\widetilde{x}') = \int_{\Gamma} \partial^+_{n,\widetilde{x}} G(\widetilde{x}, \widetilde{x}')\varsigma(\widetilde{x})\, d\widetilde{x},$$

see, e.g., [SM12], and the hypersingular integral operator is defined by

$$D \stackrel{\text{def}}{=} -\partial_n^{\pm} DL : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma).$$

The representation of $\langle D\varphi, \varrho\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)}$ can be found, e.g., in [SM23, Thm. 9.15] and is given by the right hand side in (SM28b) below with Γ_0 replaced by Γ.

The restriction of D to the surface patch $\Gamma_0 = \phi(\Omega_0)$ is denoted D_{Γ_0}. It is shown in [SM26, Lemma 2.2] that the proper space for the jump $[\varphi]$ is

$$\widetilde{H}^{1/2}(\Gamma_0) = \Big\{ u \in H^{1/2}(\Gamma) : \operatorname{supp}(u) \subset \overline{\Gamma}_0 \Big\}. \tag{SM27}$$

The physical significance of this is that the pressure jump at the edge is zero. We define

$$D_{\Gamma_0} : \widetilde{H}^{1/2}(\Gamma_0) \to H^{-1/2}(\Gamma_0) \tag{SM28a}$$

with

$$\begin{aligned} \langle D_{\Gamma_0}[\varphi], \varrho\rangle_{H^{-1/2}(\Gamma_0)\times \widetilde{H}^{1/2}(\Gamma_0)} &= \iint_{\Gamma_0} G(\widetilde{x}, \widetilde{x}')(n_{\widetilde{x}} \times \nabla[\varphi]) \cdot (n_{\widetilde{x}'} \times \nabla \varrho)\, d\widetilde{x} d\widetilde{x}' \\ &\quad - \kappa^2 \iint_{\Gamma_0} G(\widetilde{x}, \widetilde{x}')([\varphi] n_{\widetilde{x}}) \cdot (\varrho n_{\widetilde{x}'})\, d\widetilde{x} d\widetilde{x}'. \end{aligned} \tag{SM28b}$$

The next lemma establishes the equivalence between the differential equation (SM21) and the integral equation (SM29) below. As mentioned before, the idea behind (SM29) is taking the normal derivative of (SM26) and replacing the resulting left hand side $\partial_n\varphi$ on the shell midsurface Γ_0 be the boundary data g.

Lemma SM2.6. *Assume that (A1) and (A2) hold and let the Neumann data $g \in H^{-1/2}(\Gamma_0)$ be given. The function $\varphi \in H^1_{loc}(\mathbb{R}^3 \setminus \Gamma_0)$ solves (SM21) if and only if its jump $[\varphi] \in \widetilde{H}^{1/2}(\Gamma_0)$ solves the integral equation*

$$D_{\Gamma_0}[\varphi] = -g. \tag{SM29}$$

Proof. This result was shown for smooth surfaces in [SM26, Theorem 2.6] but can be easily extended to Lipschitz surfaces using the ideas from Costabel [SM12]. ∎

THEOREM SM2.7. *Assume that (A1) and (A2) hold and let the Neumann data $g \in H^{-1/2}(\Gamma_0)$ be given. The integral equation (SM29) has a unique solution $[\varphi] \in \widetilde{H}^{1/2}(\Gamma_0)$ that depends continuously on g, i.e., there exists a constant $c > 0$ such that*

$$\|[\varphi]\|_{\widetilde{H}^{1/2}(\Gamma_0)} \leq c\|g\|_{H^{-1/2}(\Gamma_0)}.$$

Proof. This result was shown for smooth surfaces in [SM26, Theorem 2.7] but can be easily extended to Lipschitz surfaces using the ideas from Costabel [SM12]. ∎

Coupling of the Naghdi equations (SM14) with the boundary integral equation (SM29) is discussed next, in §SM2.3. Later, in §3, we compute shape derivatives of the coupled equations (SM34). Because the Naghdi chart function ϕ determines the geometry for both the shell and boundary integral equations, this requires us to formulate (SM29) on the reference domain Ω_0.

Let $\Omega_0 \subset \mathbb{R}^2$, with the chart function $\phi \in \mathcal{C} \subset W^2_\infty(\Omega_0)^3$. The weak form to be evaluated for the hypersingular operator D is (SM28). In order to transform this integral over $\Gamma_0 = \phi(\Omega_0)$ into an integral over Ω_0, the quantities a_3 and $\sqrt{a}$, defined in (SM2) are needed.

Through transformation of the integral over $\Gamma_0 = \phi(\Omega_0)$ into an integral over Ω_0, the product $\langle D_{\Gamma_0}[\varphi], \varrho\rangle_{H^{-1/2}(\phi(\Omega_0))\times\widetilde{H}^{1/2}(\phi(\Omega_0))}$ becomes

(SM30)

$$\begin{aligned}
&\langle D(\phi)\psi, \varrho\rangle_{\Omega_0} \\
&= \iint_{\Omega_0} G(\phi(x), \phi(x'))\,(a_3(x') \times \nabla\Phi^{-T}(x')\widetilde{\nabla}\psi(x')) \cdot (a_3(x) \times \nabla\Phi^{-T}(x)\widetilde{\nabla}\varrho(x)) \\
&\qquad \sqrt{a(x')a(x)}\,dx'dx \\
&\quad - \kappa^2 \iint_{\Omega_0} G(\phi(x), \phi(x'))\,(a_3(x')\psi(x')) \cdot (a_3(x)\varrho(x))\sqrt{a(x')a(x)}\,dx'dx,
\end{aligned}$$

where Φ is defined in (SM4) and the derivative $\widetilde{\nabla}$ is defined via

$$\widetilde{\nabla}\rho(x) = \begin{pmatrix}\partial_{x_1}\rho(x) & \partial_{x_2}\rho(x) & 0\end{pmatrix}^T.$$

In (SM30) the notation $D(\phi)$ is used to emphasize that for the screen $\phi(\Omega_0)$ the hypersingular operator depends on the chart function ϕ. We use $\psi \in \widetilde{H}^{1/2}(\Omega_0)$ for the jump in the velocity potential defined in the reference domain.

COROLLARY SM2.8. *Under the assumptions of Theorem SM2.7 there exists a unique solution $\psi \in \widetilde{H}^{1/2}(\Omega_0)$ to*

(SM31)

$$\langle D(\phi)\psi, \varrho\rangle_{H^{-1/2}(\Omega_0)\times\widetilde{H}^{1/2}(\Omega_0)} = -\langle g \circ \phi, \varrho\rangle_{H^{-1/2}(\Omega_0),\widetilde{H}^{1/2}(\Omega_0)} \quad \forall \varrho \in \widetilde{H}^{1/2}(\Omega_0).$$

where $D(\phi)$ is as given in (SM30). In addition, the solution depends continuously on the data g, i.e., there exists a constant $c > 0$ such that

$$\|\psi\|_{\widetilde{H}^{1/2}(\Omega_0)} \leq c\|g \circ \phi\|_{H^{-1/2}(\Omega_0)}.$$

Proof. We recall that

$$\langle D_{\Gamma_0}[\varphi], \varrho\rangle_{H^{-1/2}(\Gamma_0)\times \widetilde{H}^{1/2}(\Gamma_0)} = \langle D(\phi)\psi, \varrho\circ\phi\rangle_{H^{-1/2}(\Omega_0)\times\widetilde{H}^{1/2}(\Omega_0)},$$

where $\psi = [\varphi]\circ\phi$, D_{Γ_0} and $D(\phi)$ are defined in (SM28b) and (SM30) respectively. The assertions then follow by invoking Proposition SM2.1 and Theorem SM2.7. ∎

SM2.3. The Coupled Shell and Boundary Integral Equations. In this section we couple the models from §SM2.1 and §SM2.2 and prove existence and uniqueness of the solution. Recall that $\phi \in \mathcal{C} \subset W^2_\infty(\Omega_0)^3$ is the chart function representing the midsurface

$$\Gamma_0 = \phi(\Omega_0)$$

of the shell. For the rest of the paper, all function spaces are taken to be over $\mathbb{C}$, including $\mathcal{S}$, defined in (SM13). The physical mid-surface displacement z, rotation angle θ, and velocity potential φ are simply the real parts of these complex-valued functions.

In the following, x denotes a point in Ω_0 and $\widetilde{x}$ denotes a point in $\mathbb{R}^3\setminus\Gamma_0$ or on Γ_0. The functions $\theta, z, \eta, y, f, \rho$ are defined on the reference domain Ω_0, while the velocity potential φ and the normal n are defined respectively on $\mathbb{R}^3\setminus\Gamma_0$ and on Γ_0. Therefore, the argument of n is $\phi(x)$ if, for example, we integrate over Ω_0. For $\widetilde{x}\in\Gamma_0=\phi(\Omega_0)$ we often write $z(x)\cdot n(\widetilde{x})$. In this case the argument $x\in\Omega_0$ of z is the point in the reference domain Ω_0 such that $\widetilde{x}=\phi(x)$.

First, we combine the Naghdi shell equations from §SM2.1 with the screen problem (SM21) to approximate (1). We seek $(\theta, z)\in\mathcal{S}$, where $\mathcal{S}$ is defined in (SM13), and φ in $H^1_{\text{loc}}(\mathbb{R}^3\setminus\phi(\Omega_0))$ such that

$$\begin{aligned} &K(\theta, z; \overline{\eta}, \overline{y}) - \omega^2 M(\theta, z; \overline{\eta}, \overline{y}) \\ &\quad = \int_{\Omega_0} \big(f(x)\cdot\overline{y}(x) - i\omega\rho_0[\varphi](\phi(x))\,\overline{y}(x)\cdot n(\phi(x))\big)\sqrt{a(x)}\,dx \end{aligned} \tag{SM32a}$$

for all $(\eta, y)\in\mathcal{S}$, and

$$\partial_n\varphi(\widetilde{x}) = -\,i\omega z(x)\cdot n(\widetilde{x}) \qquad \widetilde{x}\in\phi(\Omega_0) \tag{SM32b}$$

$$\triangle\varphi(\widetilde{x}) + \kappa^2\varphi(\widetilde{x}) = 0 \qquad \widetilde{x}\in\mathbb{R}^3\setminus\phi(\Omega_0) \tag{SM32c}$$

$$\big|\nabla\varphi(\widetilde{x})\cdot\widetilde{x}/|\widetilde{x}| - i\kappa\varphi(\widetilde{x})\big| = O\left(1/|\widetilde{x}|^2\right) \qquad \text{as } |\widetilde{x}|\to\infty. \tag{SM32d}$$

In (SM32a), the shell is driven by the given function f, specifying a time-harmonic driving force applied to the shell at angular frequency $\omega = c\kappa$, and by the jump $-i\omega\rho_0[\varphi]$ in the air pressure across the shell midsurface (see (SM25)), where ρ_0 is the density of the air.

Existence and uniqueness of solutions to the related problem of 3d elasticity (instead of the Naghdi shell equations) coupled to standard Helmholtz integral equations (instead of the screen integral equation) is addressed in the papers by Bielak, MacCamy, Zeng [SM3] and Luke and Martin [SM21]. Uniqueness can fail in general if the elastic body has a "Jones mode," a free eigenmode that exhibits no surface motion in the normal direction, and thus does not drive the acoustics. This is known to be possible for spheres and axisymmetric structures [SM21], but almost never happens for general shapes, as shown by Hargé [SM17], since any sufficiently smooth boundary can be approximated arbitrarily well by shapes that have no Jones modes. Existence and uniqueness results were later proven for other fluid-structure interaction problems. For example, Jentsch, Natroshvili [SM19, SM20] consider an anisotropic inviscid fluid and an anisotropic thermoelastic body. However, it appears that existence and uniqueness of solutions to (SM32) has not been addressed in the literature. In the following we extend the existence and uniqueness results in [SM3, SM21] to the problem (SM32).

Jones modes in the context of Nadghi shells are defined as follows.

DEFINITION SM2.9. *The pair* $(\theta, z) \in \mathcal{S}$*,* $(\theta, z) \neq 0$ *is called a* Jones mode *at frequency* ω *if*

$$\begin{aligned} K(\theta, z; \overline{\eta}, \overline{y}) - \omega^2 M(\theta, z; \overline{\eta}, \overline{y}) &= 0 && \text{for all } (\eta, y) \in \mathcal{S} \\ z(x) \cdot n(\phi(x)) &= 0, && x \in \Omega_0. \end{aligned}$$

The following lemma characterizes conditions required for uniqueness to solutions of (SM32).

LEMMA SM2.10. *Provided that there do not exist any Jones modes at frequency* ω*, (SM32) has at most one solution.*

Proof. It suffices to show that if (θ, z, φ) solves (SM32) with $f = 0$, then $\varphi = 0$ in $\mathbb{R}^3 \setminus \phi(\Omega_0)$. Given that this is true, $\varphi = \partial_n \varphi = 0$ on $\phi(\Omega_0)$, and therefore either $(\theta, z) = 0$, or (θ, z) is a Jones mode.

So let $f = 0$. Combining (SM32a,b) and taking $(\eta, y) = (\theta, z)$,

$$K(\theta, z; \overline{\theta}, \overline{z}) - \omega^2 M(\theta, z; \overline{\theta}, \overline{z}) = \rho_0 \omega \int_{\Omega_0} [\varphi](\phi(x))\, \partial_n \overline{\varphi(\phi(x))} \sqrt{a(x)}\, dx.$$

By symmetry of the shell bilinear form K (SM8), the imaginary part of the left-hand side equals zero. Therefore,

$$0 = \operatorname{Im}\left(\int_{\Omega_0} [\varphi](\phi(x))\, \partial_n \overline{\varphi(\phi(x))} \sqrt{a(x)}\, dx \right) = \operatorname{Im}\left(\int_{\Gamma_0} [\varphi](\widetilde{x})\, \partial_n \overline{\varphi(\widetilde{x})}\, d\widetilde{x} \right), \tag{SM33}$$

where we have used $\Gamma_0 = \phi(\Omega_0)$. If Γ_0 were a closed surface, (SM33) would be exactly the condition needed to use a theorem of Rellich [SM27, Theorem 4.2] to conclude that $\varphi = 0$ everywhere. To apply it here, we extend as in §SM2.2 Γ_0 to a Lipschitz surface Γ enclosing a bounded domain. The proof of lemma SM2.5 uses the same geometric setting and arguments about transmission conditions, and we refer to [SM16, Lemmas 5.3.1,4.3.5] for the details. ∎

With this uniqueness result, the Helmholtz screen problem in (SM32) can be replaced with the equivalent boundary integral equations (SM29), where g is given by (SM22).

The coupled shell structure acoustic problem is given as follows: find $(\theta, z) \in \mathcal{S}$ and $[\varphi] \in \widetilde{H}^{1/2}(\Gamma_0)$ such that

$$K(\theta, z; \overline{\eta}, \overline{y}) - \omega^2 M(\theta, z; \overline{\eta}, \overline{y}) = \int_{\Omega_0} \left(f \cdot \overline{y} - i\omega\rho_0 [\varphi](\phi(x))\, \overline{y} \cdot n \right) \sqrt{a}\, dx \tag{SM34a}$$

$$\langle D_{\Gamma_0}[\varphi], \overline{\varrho} \rangle_{H^{-1/2}(\Gamma_0) \times \widetilde{H}^{1/2}(\Gamma_0))} = \int_{\Gamma_0} i\omega z(x) \cdot n(\widetilde{x})\, \overline{\varrho}(\widetilde{x})\, d\widetilde{x} \tag{SM34b}$$

for all $(\eta, y) \in \mathcal{S}$ and all $\varrho \in \widetilde{H}^{1/2}(\Gamma_0)$.

To discuss the existence and uniqueness of solution to the coupled system (SM34), it will be useful to rewrite it as an operator equation. We define the linear operators

$$A_0 : \mathcal{S} \to \mathcal{S}', \qquad A_1 : (L^2(\Omega_0))^5 \to (L^2(\Omega_0))^5 \tag{SM35a}$$

$$B : L^2(\Gamma_0) \to (L^2(\Omega_0))^5, \qquad C : (L^2(\Gamma_0))^5 \to L^2(\Gamma_0) \tag{SM35b}$$

such that

$$\langle A_0(\theta, z), (\eta, y)\rangle_{\mathcal{S}',\mathcal{S}} = K(\theta, z; \overline{\eta}, \overline{y}) + k_0 \int_{\Omega_0} \theta(x) \cdot \overline{\eta}(x) + z(x) \cdot \overline{y}(x)\, dx \tag{SM35c}$$

for all $(\theta, z), (\eta, y) \in \mathcal{S}$, and

(SM35d)
$$\langle A_1(\theta, z), (\eta, y)\rangle_{(L^2(\Omega_0))^5} = -\omega^2 M(\theta, z; \overline{\eta}, \overline{y}) - k_0 \int_{\Omega_0} \theta(x) \cdot \overline{\eta}(x) + z(x) \cdot \overline{y}(x)\, dx,$$

(SM35e)
$$\langle B[\varphi], (\eta, y)\rangle_{(L^2(\Omega_0))^5} = \int_{\Omega_0} i\omega\rho_0[\varphi](\phi(x))\, \overline{y}(x) \cdot n(\phi(x))\sqrt{a(x)}\, dx,$$

$$\langle C(\theta, z), \varrho\rangle_{L^2(\Gamma_0)} = \int_{\Gamma_0} i\omega z(x) \cdot n(\widetilde{x})\, \overline{\varrho}(\widetilde{x})\, d\widetilde{x} \tag{SM35f}$$

for all $(\theta, z), (\eta, y) \in (L^2(\Omega_0))^5$ and all $\varphi, \varrho \in L^2(\Gamma_0)$. As before D_{Γ_0} is the hypersingular operator defined in (SM28). Furthermore, given $f \in L^2(\Omega_0)^3$ we define the linear functionals

$$F((\eta, y)) = \int_{\Omega_0} f(x) \cdot \overline{y}(x)\sqrt{a(x)}\, dx \quad \text{and } G = 0$$

for all $(\eta, y) \in (L^2(\Omega_0))^5$.

It is easy to show that under assumptions (A1) and (A2) on the midsurface parametrization ϕ, the sesquilinear forms on the right hand sides in (SM35d-f) are bounded and that therefore the operators A_1, B and C are bounded. Furthermore, F and G are bounded linear functionals. Under assumptions (A1)–(A4) from §SM2.1, Lemma SM2.3 and Theorem SM2.7 imply that A_0 and D_{Γ_0} are continuously invertible. Therefore, the system (SM34) can be written as an operator equation

$$\Big(I + T\Big) \begin{pmatrix} S \\ [\varphi] \end{pmatrix} = \begin{pmatrix} A_0^{-1} F \\ D_{\Gamma_0}^{-1} G \end{pmatrix} \tag{SM36}$$

in $(L^2(\Omega_0))^5 \times L^2(\Gamma_0)$, where

$$T \stackrel{\text{def}}{=} \begin{pmatrix} A_0^{-1} A_1 & A_0^{-1} B \\ -D_{\Gamma_0}^{-1} C & 0 \end{pmatrix} \in \mathcal{L}\Big((L^2(\Omega_0))^5 \times L^2(\Gamma_0)\Big). \tag{SM37}$$

Theorem SM2.11. *Let assumptions (A1)–(A4) from §SM2.1 hold and let* $f \in L^2(\Omega_0)^3$. *If there exist no Jones modes, then there exists a unique solution* $(\theta, z) \in \mathcal{S}$, $[\varphi] \in \widetilde{H}^{1/2}(\phi(\Omega_0))$ *to (SM34). Furthermore, the solution depends continuously on* f, *i.e., there exists a constant* $C(\phi)$ *such that*

$$\|(\theta, z)\|^2_{\mathcal{S}} + \|[\varphi]\|^2_{\widetilde{H}^{1/2}(\Gamma_0))} \le C(\phi)\|f\|^2_{L^2(\Omega_0)^2}$$

for all $f \in L^2(\Omega_0)^3$.

Proof. The proof follows the method of [SM3], as used for 3d elasticity. Because of Lemma SM2.3 and Theorem SM2.7 the application of T to $S \in L^2(\Omega_0))^5$ and $[\varphi] \in L^2(\Gamma_0)$ gives $T\big(S, [\varphi]\big) \in \mathcal{S} \times \widetilde{H}^{1/2}(\Gamma_0)$. The compact embeddings of $H^1(\Omega_0)$ and $\widetilde{H}^{1/2}(\Gamma_0)$ into $L^2(\Omega_0)$ and $L^2(\Gamma_0)$, respectively [SM18], [SM24], imply that the operator $T \in \mathcal{L}((L^2(\Omega_0))^5 \times L^2(\Gamma_0))$ is compact.

The operator on the left-hand side of (SM36) is a compact perturbation of the identity and, thus, the Fredholm alternative applies. The system (SM36) is equivalent to (SM32) and (SM34). In the absence of Jones modes, Lemma SM2.10 implies uniqueness of the solution. Hence -1 is not an eigenvalue of T and, by the Fredholm alternative $I + T \in \mathcal{L}\Big((L^2(\Omega_0))^5 \times L^2(\Gamma_0)\Big)$ is continuously invertible. In particular,

$$\|(\theta, z)\|^2_{(L^2(\Omega_0))^5} + \|[\varphi]\|^2_{L^2(\Gamma_0)} \le C(\phi)\|f\|^2_{L^2(\Omega_0)^2}$$

for all $f \in L^2(\Omega_0)^3$. Applying Lemma SM2.3 to (SM34a) implies

$$\|(\theta, z)\|_{\mathcal{S}} \le C(\phi)\big(\|f\|_{L^2(\Omega_0)^2} + \|[\varphi]\|_{L^2(\Gamma_0)}\big) \le C(\phi)\|f\|_{L^2(\Omega_0)^2}$$

and applying Theorem SM2.7 to (SM34b) implies

$$\|[\varphi]\|_{\widetilde{H}^{1/2}(\Gamma_0))} \le C(\phi)\|z\|_{(L^2(\Omega_0))^3} \le C(\phi)\|f\|_{L^2(\Omega_0)^2}$$

for all $f \in L^2(\Omega_0)^3$. ∎

For 3d elasticity coupled to the Helmholtz equation (or equivalently, to boundary integral equations), there exist geometries for which Jones modes preclude uniqueness. For shells, this can also happen. As a very simple example, if the shell is flat, then the in-plane motions decouple from the out-of-plane motions. The in-plane problem is elliptic, so there will be an infinite sequence of positive increasing eigenvalues, corresponding to purely in-plane motions, which do not drive the acoustics through (SM34b). If the forcing excites one of these motions, then uniqueness will fail for the coupled problem (SM34). This situation seems exceedingly unlikely for general curved shells, or for joined shells, where the in-plane motions of one would drive out-of-plane motions of the other.

Theorem SM2.11 states the well-posedness (existence, uniqueness, and continuous dependence on the data f) of the coupled system (SM34) in the Sobolev space $\mathcal{S} \times \widetilde{H}^{1/2}(\phi(\Omega_0))$. Notice that $\widetilde{H}^{1/2}(\phi(\Omega_0))$ depends on ϕ, and thus it is inconvenient to compute the shape derivatives with respect to the chart function. To avoid this, following the Naghdi geometry, we shall rewrite the entire coupled system (SM34) in the reference coordinates representing Ω_0 (we use (SM31)): Find $(\theta, z) \in \mathcal{S}$ and $\psi \in \widetilde{H}^{1/2}(\Omega_0)$ such that

$$K(\theta, z; \overline{\eta}, \overline{y}) - \omega^2 M(\theta, z; \overline{\eta}, \overline{y}) = \int_{\Omega_0} (f \cdot \overline{y} - i\omega\rho_0\psi\overline{y} \cdot n)\, \sqrt{a}\, dx \tag{SM38a}$$

$$\langle D(\phi)\psi, \overline{\varrho}\rangle_{H^{-1/2}(\Omega_0)\times\widetilde{H}^{1/2}(\Omega_0)} = \int_{\Omega_0} i\omega z \cdot a_3\, \overline{\varrho}\sqrt{a}\, dx \tag{SM38b}$$

for all $(\eta, y) \in \mathcal{S}$ and all $\varrho \in \widetilde{H}^{1/2}(\Omega_0)$. See (SM30) for the definition of the differential operator $D(\phi)$ in (SM38b).

COROLLARY SM2.12. *Under the assumptions of Theorem SM2.11 there exist a unique solution $(\theta, z) \in \mathcal{S}$ and $\psi \in \widetilde{H}^{1/2}(\Omega_0)$ solving (SM38). In addition, the solution depends continuously on data, i.e., there exists a constant $c(\phi) > 0$ such that*

$$\|(\theta, z)\|^2_{\mathcal{S}} + \|\phi\|^2_{\widetilde{H}^{1/2}(\Omega_0))} \le c(\phi)\|f\|^2_{L^2(\Omega_0)^2}.$$

Proof. The well-posedness of (SM38) is due to Theorem SM2.11 in conjunction with Proposition SM2.1 and Corollary SM2.8. We omit the details for brevity. ∎

Equation (SM38) will be the state equation in the shape optimization problem that we study in §4.

SM3. Shell and Integral Operator Differentiability. Here we only state the statement and proof of Lemma 3.1 from the paper.

LEMMA SM3.1. *Let $\phi \in \mathcal{C}$ be one-to-one on $\overline{\Omega}_0$. If there exists $\sigma > 0$ such that*

$$\left|\left(\nabla\phi(x)^T\nabla\phi(x)\right)^{-1}\right| \leq \sigma \qquad \forall x \in \Omega_0, \quad \textit{then:}$$

1. *The function ϕ has a Lipschitz-continuous inverse, i.e., there exists a $c_2 > 0$ such that $|x - x'| \leq c_2|\phi(x) - \phi(x')|$ for all $x, x' \in \Omega_0$.*
2. *There exists a constant $c_1 = c_1(\sigma) > 0$ such that for any $h \in W^2_\infty(\Omega_0)^3$ with $\|h\|_{W^2_\infty(\Omega_0)^3} < c_1$, the sum $\phi + h \in \mathcal{C}$.*

Proof. 1. Suppose that there is no $c_2 > 0$ such that $|x - x'| \leq c_2|\phi(x) - \phi(x')|$ for all $x, x' \in \Omega_0$. Then there exists sequences $\{x_k\}, \{x'_k\}$ in Ω_0 such that

$$\frac{|x_k - x'_k|}{|\phi(x_k) - \phi(x'_k)|} \to \infty. \tag{SM39}$$

Since Ω_0 is bounded, there exist converging subsequences of $\{x_k\}, \{x'_k\}$. To simplify the presentation, let $\lim_{k\to\infty} x_k = x_* \in \overline{\Omega}_0$ and $\lim_{k\to\infty} x'_k = x'_* \in \overline{\Omega}_0$.

If $x_* \neq x'_*$, then (SM39) implies $\phi(x_*) = \phi(x'_*)$. Under the assumption that ϕ is one-to-one on $\overline{\Omega}_0$, we get a contradiction.

Now consider the case $x_* = x'_*$. Taylor expansion $\phi(x_k) - \phi(x'_k) = \nabla\phi(x'_k)(x_k - x'_k) + o(|x_k - x'_k|)$ implies

$$\begin{aligned}|\phi(x_k) - \phi(x'_k)|^2 &= (x_k - x'_k)^T\nabla\phi(x'_k)^T\nabla\phi(x'_k)(x_k - x'_k) - o(|x_k - x'_k|^2)\\ &\geq \frac{1}{\sigma}|x_k - x'_k|^2 - o(|x_k - x'_k|^2) \geq \frac{1}{2\sigma}|x_k - x'_k|^2\end{aligned}$$

for k sufficiently large. This inequality contradicts (SM39).

2. The matrix $\nabla\phi(x)+\nabla h(x)$ has full rank if and only if $(\nabla\phi(x)+\nabla h(x))^T(\nabla\phi(x)+\nabla h(x)) = \nabla\phi(x)^T\nabla\phi(x) + \nabla h(x)^T\nabla\phi(x) + \nabla\phi(x)^T\nabla h(x) + \nabla h(x)^T\nabla h(x)$ is invertible. Since

$$\begin{aligned}&\left|\left(\nabla\phi(x)^T\nabla\phi(x)\right)^{-1}\left(\nabla h(x)^T\nabla\phi(x) + \nabla\phi(x)^T\nabla h(x) + \nabla h(x)^T\nabla h(x)\right)\right|\\ &\leq \left|\left(\nabla\phi(x)^T\nabla\phi(x)\right)^{-1}\right|\left(2|\nabla h(x)||\nabla\phi(x)| + |\nabla h(x)|^2\right)\\ &\leq \sigma\left(2\|h\|_{W^2_\infty(\Omega_0)^3}\|\phi\|_{W^2_\infty(\Omega_0)^3} + \|h\|^2_{W^2_\infty(\Omega_0)^3}\right)\end{aligned}$$

for all $x \in \Omega_0$, the matrix $(\nabla\phi(x) + \nabla h(x))^T(\nabla\phi(x) + \nabla h(x))$ is invertible for all $x \in \Omega_0$ if for some $\delta \in (0, 1)$, $\sigma\left(2\|h\|_{W^2_\infty(\Omega_0)^3}\|\phi\|_{W^2_\infty(\Omega_0)^3} + \|h\|^2_{W^2_\infty(\Omega_0)^3}\right) \leq \delta$. The latter inequality is satisfied for all $h \in W^2_\infty(\Omega_0)^3$ with

$$\|h\|_{W^2_\infty(\Omega_0)^3} < -\|\phi\|_{W^2_\infty(\Omega_0)^3} + \sqrt{\|\phi\|^2_{W^2_\infty(\Omega_0)^3} + \delta/\sigma} =: c_1.$$

Moreover, for such h,

$$\left|\left((\nabla\phi(x) + \nabla h(x))^T(\nabla\phi(x) + \nabla h(x))\right)^{-1}\right| \leq \frac{\sigma}{1-\delta} \qquad \forall x \in \Omega_0.$$

By part 1, $\phi + h$ has a Lipschitz continuous inverse and, consequently, is one-to-one. ∎

SM4. Transfer Function Optimization. This section includes the example omitted from Section 4.1. This example illustrates that if there exists a shape g_* that produces exactly the response data the objective function seeks to match, and the shape g_* has a Jones mode that is driven by the forcing, $g_k \to g_*$, and $c(g_k, U_k; \omega) = 0$, then $\|U_k\|_{\mathcal{U}}$ will be unbounded.

SM4.1. Existence of Optimal Solutions.

EXAMPLE SM4.1. *Consider the boundary value problem* $-u''(y) - \mu u(y) = f(y)$, $u(0) = u(\rho) = 0$ *on the interval* $[0, \rho]$. *The parameter* $\mu > 0$ *plays the role of the driving frequency* ω^2. *Using the transformation to reference coordinates* $x = y/\rho$, *this can be recast as*

$$-1/\rho^2 u''(x) - \mu u(x) = f(\rho x), \tag{SM40a}$$
$$u(0) = u(1) = 0. \tag{SM40b}$$

We define $c : \mathbb{R} \times H_0^1(0,1) \to H^{-1}(0,1)$ *as*

$$\langle c(\rho, u), v\rangle = \int_0^1 \frac{1}{\rho^2} u'(x)v'(x) - \mu u(x)v(x) - f(\rho x)v(x)\, dx.$$

The weak form of (SM40) *is: Find* $u \in H_0^1(0,1)$ *such that* $\langle c(\rho, u), v\rangle = 0$ *for all* $v \in H_0^1(0,1)$.

The length of the domain, $\rho > 0$, *now acts as a shape parameter. The operator* $-1/\rho^2 d^2/dx^2$ *has eigenfunctions and eigenvalues*

$$\varphi_n(x) = \sin(n\pi x), \quad \lambda_n(\rho) = (n\pi/\rho)^2, \quad n = 1, 2, \cdots.$$

The Fredholm alternative applies to this problem: Unless

$$\mu = \lambda_n(\rho) \quad \text{and} \quad (f(\rho\,\cdot), \varphi_n)_{L^2} \neq 0$$

for some n, *the solution of* (SM40) *is* $u(x) = \sum_{n=1}^{\infty} \alpha_n(\rho)\varphi_n(x)$, *where*

$$\alpha_n(\rho) = \begin{cases} 2\frac{(f(\rho x), \varphi_n)_{L^2}}{\lambda_n(\rho) - \mu} & \text{if } \mu \neq \lambda_n(\rho) \\ \text{arbitrary} & \text{otherwise.} \end{cases}$$

Suppose that $\rho \in R = [3/4, 5/4]$, *and* $\mu = \pi^2$. *Then,* $\lambda_n = \mu$ *if and only if* $\rho = 1$ *and* $n = 1$. *Thus, uniqueness (and, depending on* f, *possibly also existence) of the solution can only fail at* $\rho = 1$.

Let $f(x) = \sin(\pi x) + \sin(2\pi x)$. *For* $\rho \in R$ *with* $\rho \neq 1$ *the unique solution of* $c(\rho, u) = 0$ *is* $u(\rho, x) = \sum_{n=1}^{\infty} \alpha_n(\rho)\varphi_n(x)$ *with*[2]

$$\alpha_n(\rho) = \frac{(-1)^{n+1} 2n\rho^2}{(n^2 - \rho^2)^2\pi^3} \sin(\rho\pi) + \frac{(-1)^{n+1} 2n\rho^2}{(n^2 - 4\rho^2)(n^2 - \rho^2)\pi^3} \sin(2\rho\pi).$$

Note that $|\alpha_1(\rho)| \to \infty$ *as* $\rho \to 1$ *and, consequently,* $\|u(\rho)\|_{L^2} \to \infty$ *as* $\rho \to 1$.

[2] We use the identity

$$\int_0^1 h(x)\sin(n\pi x)dx = \frac{(-1)^{n+1}h(1) + h(0)}{n\pi} - \frac{1}{n^2\pi^2}\int_0^1 h''(x)\sin(n\pi x)dx.$$

With $h(x) = \sin(k\rho\pi x)$, $k \in \mathbb{N}$, $\rho \neq 1$, this implies $\int_0^1 \sin(k\rho\pi x)\sin(n\pi x)dx = \frac{n}{(n^2 - k^2\rho^2)\pi}\sin(k\rho\pi)$.

Now we consider the optimization problem

$$\text{(SM41a)} \qquad \min \quad j(\rho, u)$$

$$\text{(SM41b)} \qquad s.t. \quad c(\rho, u) = 0, \ \rho \in R, \ u \in H_0^1(0,1),$$

with objective function

$$j(\rho, u) = \left(2(u, \varphi_2)_{L^2} - 2\frac{(f, \varphi_2)}{3\pi^2} \right)^2 = \left(\alpha_2(\rho) - \frac{1}{3\pi^2} \right)^2.$$

Since

$$\lim_{\rho \to 1} \alpha_2(\rho) = \frac{1}{3\pi^2},$$

any sequence $\rho_k \to 1$, $\rho_k \neq 1$, *gives*

$$\lim_{k \to \infty} j(\rho_k, u(\rho_k)) = 0 = \inf \left\{ j(\rho, u(\rho)) : c(\rho, u) = 0, \ \rho \in R, \ u \in H_0^1(0,1) \right\},$$

but the optimization problem (SM41) *does not have a solution. The fundamental issue is that* f *drives both of the first two eigenfunctions, but the objective function is sensitive only to the second.*

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