Preference Based Scheduling in a Healthcare Provider Network

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Healthcare services are often provided by a large network of physicians and clinic facilities to patients with various levels of health conditions and preferences. Appointment scheduling is used to manage access to these services by matching patient demand with physician availability. This raises tremendous challenges for providers due to the heterogeneity in patient preference and physician availability. We propose a preference-based "nested" network model that consists of most practical operational constraints. Our model considers patients with varied priorities who can visit any clinic location and provider of their preference, and request the day and time of the appointment of their choice. The common challenges of patient no-show, cancellation, and uncertainty of physician availability are taken into account. We formulate this model as a Markov decision process and propose an approximate dynamic programming approach to provide robust scheduling policies. We also analyze the joint appointment scheduling and physician capacity planning problem as a mixed-integer nonlinear program. The proposed scheduling policies maximize revenue and minimize physician overtime and idle time while satisfying patient preferences. These policies are shown to perform within a close bound to the best achievable policy. They are also robust under patient demand uncertainties. We highlight the importance of considering patient heterogeneity and preference as well as systematic uncertainties to provide an optimal set of appointments.

Key words: OR in Health Services, Healthcare Provider Network, Patient Choice, Mixed Multinomial Logit Model, Joint Appointment Scheduling and Capacity Planning

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1. Introduction

Outpatient services contributed to 28.2% of the total spending per capita on subservice categories in 2018, which is the second highest per capita (Frost et al. 2018). This population includes Americans younger than age 65 and covered by employer-sponsored insurance. Every hospital outpatient service system is complex and consists of an integrated network of multiple physicians, clinic facilities, patients and other healthcare professionals. This complex network is known as a healthcare provider network (HPN). HPNs are an integral part of the healthcare delivery system. In many healthcare settings around the world, outpatient clinics are part of a provider network. In this HPN, a patient can be sent to multiple clinic facilities for any number of tests and treatments. A physician can see patients at different clinics on different days of the week and there can be more than one physician at the same location. Similarly, patients can schedule appointments at a clinic of their choice with a physician of their preference or urgent patients (walk-in patients) can visit without making an appointment. Despite the interdependence among the facilities and the care providers, they are rarely studied as a complex network system. There is limited research with a focus on this multi-clinic, multi-provider problem. Ignoring this complex interrelationship can lead to unfilled appointment slots and a long waiting list. At a major healthcare system in Pennsylvania during 15 months period, 30 outpatient clinics of a single specialty service suffered from more than 20% uncompleted appointments slots and 650 overdue (waitlisted) patients. In a setting like this, which includes a network of more than 50 outpatients clinics and 6 hospitals with more than 3.8 million outpatient clinic visits annually, it becomes even more critical to consider the network effect in the scheduling model.

Appointment schedulers use a two-step design process to book appointments (Wang and Gupta 2011): *clinic profile setup* and *appointment booking step*. In the first step, the available physicians' time on each workday is divided into appointment slots based on physicians' preference and their administrative duties. Here, we first focus on the second step of *Appointment Booking*. Once we have available time slots for the physicians, this capacity needs to be allocated when a patient requests an appointment. We then propose a framework to solve both steps simultaneously. Gupta and Denton (2008) proposed a road map of the state of the art in the design of appointment management systems and identified major challenges in the development of a practical optimization system for appointment scheduling.

In traditional appointment scheduling research, patients are given a fixed day and time for the appointment to maximize their health system's operational objectives. But this lack of flexibility for patients leads to patient dissatisfaction and lower revenues, increased no-show rates, and discontinuity in care. Studies have shown that scheduling patients while matching their preferences can benefit clinics. For example, matching patients with their preferred Primary Care Physicians (PCP) (O'Hare and Corlett 2004) and offering them a convenient time (Lacy et al. 2004) can lead to higher revenue and increased efficiency due to a lower no-show rate. While there is a vast literature in the service industry (airlines, hospitality, and manufacturing (Zhang and Adelman 2009, Verma 2010) the healthcare operations research literature has mainly ignored patient preference in appointment booking (Gupta and Denton 2008). To our knowledge, Rohleder (2000), Wang and Gupta (2011), Feldman et al. (2014), and Liu et al. (2019) are the only studies that consider patient preference while generating optimal scheduling policies. We highlight the similarities and differences to these studies in the literature review section.

Patient appointment preference consists of the patients' choice of physician, day of the appointment, time of the day, and clinic location (Douglas et al. 2005, Hunter et al. 2009). In addition, patient preferences are non-homogeneous and vary by patients. Research has shown that many patients prefer the continuity of care (physicians they know), while others prefer faster access to care (Baker et al. 2005, Gerard et al. 2008). When a request for an appointment is made it can be categorized into different classes based on patients' history and request. In this paper, based on the request type, we divide the appointment requests into three classes, *Dedicated, Flexible, and Urgent*. Patients with dedicated requests are more likely to wait for a specific provider. These patients might be the ones who have assigned particular physicians as their PCP and therefore, will wait for any available appointment they can get. Patients with flexible requests do not have any preference for a specific provider but will have a preference for the day and time of the appointment. On the other hand, *Urgent* or same-day requests need to be fulfilled on the same day.

In our model, we consider an HPN with multiple clinics and multiple physicians at each clinic with patient preference being governed by a Mixed Multinomial Logit (MMNL) model (McFadden and Train 2000). Unlike traditional appointment scheduling models, this model offers a set of appointments for the customers to choose from based on patients' preferences. Traditionally, the appointment scheduling is driven by the objective of minimizing indirect (lead time) and direct (time spent at the clinic) waiting time and overbooking. However, in reality, patients have a preference for their appointment and some patients can wait longer to have appointments of their choice. Therefore, minimizing waiting time alone may not be sufficient. Similarly, in a network setting, while overbooking is important to increase revenue, equal attention should be paid to physician idle time. Since physicians have other duties besides clinic appointments, they can perform those duties instead of sitting idly. Physician capacity uncertainty is another important yet not generally considered factor in the appointment scheduling literature. According to Gupta and Denton (2008), in the case of advance scheduling, primary clinics also need to decide how to respond to any unplanned shortfall in capacity. This uncertainty might arise due to physicians' illness or emergency which can impact the *indirect waiting times* of patients, sometimes resulting in revenue losses due to patients leaving the system.

Here, we address the problem of finding the optimal sets of appointments to offer which maximize network profit while satisfying patient preferences. We identify the network architecture of the problem and formulate it as a Markov decision process (MDP). We present an approximate dynamic programming (ADP) approach to solve this MDP. This ADP makes use of the properties of a single period scheduling problem which allocates patients without considering the current state of the system. We show that this single period problem is convex and can be solved to global optimality. We develop theoretical bounds on the performance of the single period policy and show that in certain scenarios the performance bound is $\approx 1 - 1/\sqrt{2\pi}$. Our proposed policies outperform the simple greedy approaches used in practice in almost all scenarios and this performance improvement can be up to 55%. We also show that the proposed dynamic scheduling model is robust under more general demand distributions. For this, we assume that the patient demand follows a doubly stochastic Poisson arrival process. Finally, we propose a mixed integer non linear program (MINLP) for the joint capacity planning and appointment scheduling problem. We also show that there exists a globally optimal solution to the joint capacity planning and scheduling problem. This model can be used for the two-step design process on a daily basis.

We make the following distinctive contributions to the appointment scheduling literature:

- We consider patient priority and multi-dimensional patient preferences for appointment scheduling as a network. This network consists of multiple physicians and multiple clinic locations. Our numerical experiments show that with proposed heuristic policies healthcare systems can achieve improvements in profit of up to 55%.
- Our proposed policies can be used with more general stochastic demand. Therefore, they can be implemented in practice without significant loss of performance.
- For the simultaneous two-step design process, we propose a novel joint capacity and appointment scheduling model. Our numerical results show significant improvements in the objective function can be achieved.

The remainder of the paper is organized as follows. We present an in-depth comparison of our approach and the models developed in the literature in section 2. The literature review is divided based on the different features of the proposed model. Section 3 describes the network model in detail. In section 4, we formulate the model as an MDP and present different components of the MDP. We study the properties of the single period problem in section 5 and present the approximate dynamic programming heuristic in section 6. In section 7, we present the advantage of the heuristic approach using numerical experimentation. Section 8 presents the robustness of the model under general demand distribution and discusses the joint capacity planning and appointment scheduling model. Finally, we conclude by discussing future possibilities to improve the model. Appendix A can be referred for a list of notation used in the paper. Appendix B and C present the proofs for lemmas and propositions. The appendices are provided as an online supplement.

2. Literature Review

Appointment scheduling literature can be divided into inter-day and intra-day scheduling. In case of inter-day scheduling appointments are booked on a multi-day planning horizon. The scheduler decides how to allocate demand arriving on the current day into future days (Patrick et al. 2008, Liu et al. 2010, Feldman et al. 2014). Under intra-day scheduling patient demand is known, service time is stochastic and service order is determined by maximizing physician utilization and minimizing patient wait time. The problem of intra-day scheduling is similar to that of job-shop scheduling (machine scheduling) and has been extensively studied (Leung 2004, Chakraborty et al. 2010). Inter-day scheduling, on the other hand, has been less explored but has been a focus of research in recent years. In this research, we focus on the problem of inter-day and intra-day scheduling simultaneously. For a more detailed background on appointment scheduling and the challenges we refer to Cayirli and Veral (2003) and Gupta and Denton (2008).

In the following paragraphs we discuss the relevant literature on network effect in appointment scheduling, customer preference, patient class, and no-show rate and highlight our contributions in comparison to the literature.

According to our literature research, *network effect in appointment scheduling* has rarely been studied before. The only relevant paper is Liu et al. (2019). They focused on non-sequential and sequential offerings of appointments to the patients of different types. Where in the non-sequential setting the patient is offered a set of appointments at the same time and the patient is supposed to choose an appointment from them or leave without making an appointment. In the later setting, each patient is offered appointments one at a time. If the patient does not like an appointment, another option is offered or else the patient can leave without making an appointment. However, in their model, the network structure is generated due to the flexibility of scheduling appointments at different time slots on a given day. We consider a different network of clinics and physicians.

Customer preference has been studied extensively in revenue management, hospitality industry, and assortment planning. In many ways, the assortment planning problem is similar to appointment scheduling. Assortment planning problems study the optimal offering of a set of products to customers based on their choices. Bront et al. (2009) and Rusmevichientong et al. (2010) showed that the assortment planning problem with multiple customer types where customers chose an option from the offer set using a multinomial logit (MNL) model is NP-Hard. Gallego et al. (2011) proposed a choice-based linear program which can be used to approximate the assortment planning problem. We use a similar approach to solve the appointment scheduling problem. We refer to Talluri and Van Ryzin (2004) for a more detailed review of assortment planning literature. Gallego and Topaloglu (2014) and Rayfield et al. (2015) can be referred for a detailed overview of the literature on the use of logit models in revenue management and product pricing problem. While a substantial body of literature exists in healthcare appointment scheduling which focuses on determining choice model parameters, very few focus on the use of these models to maximize patient access. To our knowledge only Rohleder (2000), Wang and Gupta (2011), Feldman et al. (2014), Liu et al. (2019) and Liu et al. (2020) have considered patient preference in their appointment scheduling models. Wang and Gupta (2011) also considered intra-day scheduling but overall patient preference for physician and time slot is independently captured using fixed probabilities. Feldman et al. (2014), on the other hand, considered inter-day patient preference with an MNL choice model

(McFadden 1973). Liu et al. (2019) assumed that patients choose from available appointments uniformly. Liu et al. (2020) focus on sequential offering of appointments where patient choice follows MNL distribution. Whereas, our study focuses on multi-dimensional patient preference for physician, day and time of the appointment as well as the location of the clinics, for which we use an MMNL model (McFadden and Train 2000) to represent heterogeneous patient preferences. The MNL choice model estimates the parameter for every individual choice maker. However, a large number of choice situations per individual is needed to estimate these parameters consistently. On the other hand, MMNL can utilize any distribution for the random coefficients and allow for "random taste variations and correlation in unobserved factors over time" (Train 2009). Rohleder (2000) studied intra-day scheduling with three levels of patient preference, i.e., with three different probabilities patients request special slots and these special slots can be determined from two distributions (uniform and end-of-period). In the end-of-period policy patients request any of the last five slots with equal probability.

Most of the literature in appointment scheduling is limited to only one or two *patient classes*. However, in practice, patients can have multiple priorities and ignoring that can have significant operational impacts. Patrick et al. (2008), Sauré et al. (2012) studied advanced appointment scheduling for diagnostic resource and radiation therapy respectively, under the multi-priority patient setting. Astaraky and Patrick (2015) considered multi-class surgical scheduling with multi-resources. However, in all of these studies, the authors did not consider patient preferences, which plays an important role in outpatient clinic scheduling. Wang et al. (2020) divided patients into scheduled and walk-in classes. Lee et al. (2018) considered two patient types namely, returning and follow-up. However, they did not consider walk-in patients. Stein et al. (2020) proposed a general framework which considers n patient classes where each patient may consume variable capacity from available resources. However, they do not consider no-show and overbooking in their model. Feldman et al. (2014) proposed appointment scheduling policies for single patient class, making their analysis simpler than ours. They explained how their approach of appointment scheduling for single patient class can be extended to multiple patient class having heterogeneous preferences. However, in this paper, we build upon their framework and propose appointment scheduling policies for the case of multiple appointment request classes as well as multiple physicians and multiple clinic locations, thus a "nested" network. We consider three specific appointment request types, Urgent (Walk-in), Flexible and Dedicated. In addition, with analytical and numerical results we show that the proposed policies perform well in various conditions including more general demand distributions. We also propose a framework for two-step clinic design process. In this framework, we solve the joint appointment scheduling and physician capacity planning problem.

The topic of *no-show* has been a focus of research for more than a decade now, because of its impact on the overall healthcare system. Zeng et al. (2010) studied a problem with overbooking for patients with heterogeneous no-show probabilities. They did not consider urgent patient demand and only consider independent and identically distributed (iid) service times across patients. Liu et al. (2010) proposed heuristic dynamic policies to appointment scheduling while considering patient no-show and cancellation. But in their model, they did not consider patient preference and assume that patients accept any available appointment given to them. Feldman et al. (2014) proposed a general formulation for advanced appointment scheduling with no-show and cancellation. They assumed a probability with which the clinics retain their patients until the day of the appointment. This probability depends on the number of days between the scheduling day and the appointment day (indirect waiting time). As patients schedule appointments further into the future, they are less likely to be retained. Since no-show and cancellation have become standard modeling components in patient scheduling, we include them in our model formulation.

3. The Preference-based "Nested" Network Model

We consider $\mathbb{K} = \{1, ..., K\}$ care providers who can treat patient at $\mathbb{L} = \{1, ..., L\}$ clinic locations. Each clinic is covered by at least one care provider. We assume that the following sequence of events occurs on each epoch. First, the appointment system checks the current schedule (state) and the set of available appointments. This set is comprised of available physicians, days, time slots (same day or future), and clinic locations. Second, arriving appointment requests are categorized into one of the three classes (*Dedicated, Flexible, Urgent*). Third, a subset of appointments is offered to the patients and they either choose to schedule an appointment or leave without scheduling. Fourth, some patients with scheduled appointments may cancel their appointments or have a no-show. Fifth, some physicians can cancel an already scheduled appointment, which will need to be rescheduled in the future. We build upon the framework proposed by Liu et al. (2010) and Feldman et al. (2014). Our objective is to offer the optimal set of appointments to patients based on their preference for a physician, location, day and time, giving more flexibility to the patients. Note that we do not consider access time and direct waiting time in the model.

The patient demand arises from three sources, denoted as $c \in \mathbb{C} = \{\mathfrak{D}, \mathfrak{F}, \mathfrak{U}\}$ for *Dedicated, Flexible* and *Urgent* requests, respectively. These requests arrive independently following a Poisson process. Let $\lambda_l^{\mathfrak{D}}$ be the total arrival rate for *Dedicated* requests at location l. This dedicated request has different arrival rates for different physicians. Therefore, by the Poisson thinning (sampling) property, it can be considered as a separate subclass $\mathfrak{D}_1, \mathfrak{D}_2, ..., \mathfrak{D}_K$, represented as $\lambda_{lk}^{\mathfrak{D}}$. Let $\lambda_l^{\mathfrak{U}}$ be the arrival rate of *Urgent* requests at location $l \in \mathbb{L}$. In practice, *Urgent* requests do not need to schedule appointments, but this demand needs to be accounted for in the scheduling model. Let

$\lambda^{\mathfrak{F}}$ be the total arrival rate of *Flexible* requests to the network. Patients with flexible requests can be treated at any location and by any physician. This flexibility can be exploited by the network provider so that more patients with *Dedicated* and *Urgent* requests can be treated at each location. (Full flexibility is assumed. If *Flexible* requests can be fulfilled at only a subset of the locations, then the network model can be decomposed into two disconnected networks.) Given any scheduling policy, we let $\lambda_l^{\mathfrak{F}}$ be the rate of *Flexible* requests that are "allocated" to location $l \in \mathbb{L}$. Evidently, we must have $\lambda^{\mathfrak{F}} = \sum_{l\in\mathbb{L}} \lambda_l^{\mathfrak{F}}$.

The proposed appointment scheduling framework can be depicted by a 'nested' network. The overall appointment scheduling model has a network structure due to multiple locations (Fig 1). On the other hand, the location specific appointment scheduling model has a slightly different network structure due to the flexibility of physicians (Fig 2).

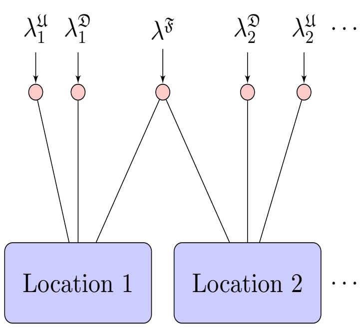

Figure 1 **Overall scheduling network.** $\lambda_l^{\mathfrak{D}}$ **= average arrival rate of Dedicated requests at location** l**.** $\lambda^{\mathfrak{F}}$ **= average arrival rate of Flexible requests.** $\lambda_l^{\mathfrak{U}}$ **= average arrival rate of Urgent requests for location** l

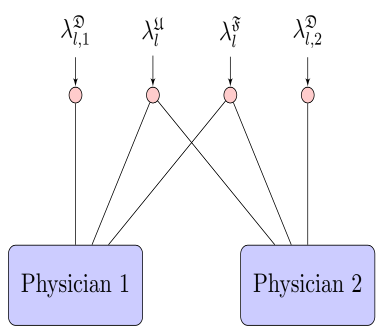

Figure 2 **Location specific scheduling network.** $\lambda_{l,k}^{\mathfrak{D}}$ **= average arrival rate of Dedicated requests of at location** l **for provider** k**.** $\lambda_l^{\mathfrak{F}}$ **= average arrival rate of Flexible requests.** $\lambda_l^{\mathfrak{U}}$ **= average arrival rate of Urgent requests for location** l

In this network, patients with *Urgent* requests can go to any clinic location and since they have to be treated at that particular location we can consider them as l separate streams of patients corresponding to the l clinic locations. Similarly, *Dedicated* requests can only be fulfilled at the locations where their respective physicians are available. Finally, *Flexible* requests can be fulfilled at any of the l clinic locations. At location level, *Urgent* and *Flexible* requests can be completed by any physician, while patients with *Dedicated* requests have to be seen by their respective physicians. This pooling of physician capacity at multiple clinics to serve *Flexible* requests provide significant benefit to healthcare providers. This provides more flexibility to patients, as well as helps with better utilization of resources. In this paper, we assume complete flexibility of *Flexible* requests (we discuss its extensions to limited flexibility in Section 10). We illustrate the advantages of network effect through a numerical example in Section 7.3.3.

Let *scheduling horizon* be $\mathbb{D} = \{0, 1, ..., D\}$, where day 0 corresponds to the current day, and each day is divided into $\mathbb{T} = \{1, ..., T\}$ time slots. We have, $\mathbb{S} = \{(k, d, t, l) | k \in \mathbb{K}, d \in \mathbb{D}, t \in \mathbb{T}, l \in \mathscr{L} \subset \mathbb{L}\}$ is a set of all possible appointments. We offer a choice set $\mathscr{S} = \{(k, d, t, l) | k \in \mathscr{K} \subset \mathbb{K}, d \in \mathscr{D} \subset \mathbb{D}, t \in \mathscr{T} \subset \mathbb{T}, l \in \mathscr{L} \subset \mathbb{L}\}$, comprising of available physicians, days, time slots, and clinic locations. Choice set $\mathscr{S}$ offered to patients is a subset of the all appointments $\mathbb{S}$. This choice set differs by patient class. Therefore, the offer set is represented as $\mathscr{S}^c$, for $c \in \mathbb{C}$. The probability that a patient schedules an appointment $s = (k, d, t, l) \in \mathbb{S}$ at location l with physician k at time slot t, d days into the future is given as a joint probability function $P_s(\mathscr{S}^c)$. Note that, all patients except ones having *Urgent* requests have an option of not scheduling an appointment.

The patient choices in outpatient networks are modeled by an MMNL model. In MMNL models, entities can be divided into distinct classes and each of these classes will have their own attribute parameter values. For example, in this model patients requesting appointments for the same class will have the same parameter values. However, these parameters vary across appointment class. One of the advantages of MMNL over MNL is that it allows for different patients to have different choice parameters. Given that we have divided appointment requests into three classes $\mathbb{C} = \{\mathfrak{D}, \mathfrak{F}, \mathfrak{U}\}$ and availability set $\mathscr{S}^c$, we can write the choice probability as

$$P_s(\mathscr{S}^c) = \begin{cases} \dfrac{v_s^c}{1 + \sum_{s' \in \mathscr{S}^c} v_{s'}^c}, & c \in \{\mathfrak{D}, \mathfrak{F}\}, \quad (1a) \\ \dfrac{v_s^c}{\sum_{s' \in \mathscr{S}^c} v_{s'}^c}, & c \in \{\mathfrak{U}\}. \quad (1b) \end{cases}$$

Here, each patient associates a preference weight of v_s^c with the option of scheduling an appointment s. The option of not scheduling an appointment has a nominal weight of 1 for *Dedicated* and *Flexible* patient classes. These weights v_s^c can be calculated using discrete choice experiments similar to Hole (2008). The availability sets are described as follows. For *Dedicated* requests of physician $k \in \mathbb{K}$ and location $l \in \mathbb{L}$, $\mathscr{S}_{k,l}^{\mathfrak{D}} = \{(d, t) | d \in \mathscr{D} \subset \mathbb{D}, t \in \mathscr{T} \subset \mathbb{T}\}$; for Urgent requests at location $l \in \mathbb{L}$, $\mathscr{S}_l^{\mathfrak{U}} = \{(k, d, t) | k \in \mathscr{K} \subset \mathbb{K}, d = 0, t \in \mathscr{T} \subset \mathbb{T}, (k, d, t, l) \notin \{\phi\}\}$; where $(k, d, t, l) \notin \{\phi\}$ means that *Urgent* requests have to be fulfilled. Finally for the *Flexible* requests we have $\mathscr{S}^{\mathfrak{F}} = \{(k, d, t, l) | k \in \mathscr{K} \subset \mathbb{K}, d \in \mathscr{D} \subset \mathbb{D}, t \in \mathscr{T} \subset \mathbb{T}, l \in \mathscr{L} \subset \mathbb{L}\}$.

Henceforth unless specified we will use the generic offer set $\mathscr{S}^c$. Using the Poisson thinning property we can divide the patient arrival at different nodes as $\lambda^c P_s(\mathscr{S}^c)$. The probability that a patient leaves the system without making an appointment, can be given as

$$N(\mathscr{S}^c) = 1 - \sum_s P_s(\mathscr{S}^c) = \begin{cases} \dfrac{1}{1 + \sum_{s' \in \mathscr{S}^c} v^c_{s'}}, & c \in \{\mathfrak{D}, \mathfrak{F}\}, \quad (2a) \\ 0, & c \in \{\mathfrak{U}\}. \quad (2b) \end{cases}$$

Once the appointments have been scheduled, a patient can cancel or may not show up. We assume that the cancellation probability depends on the indirect waiting time and the patient class. Similar to Feldman et al. 2014, we represent the retention probability (1 - cancellation probability) of a patient of class c who called i days ago to make an appointment d days into the future from today as δ^c_{id}. Similarly, we represent the show-up probability (1 - no-show probability) for a patient of class c who called i days prior to make an appointment for today and has not canceled it until the current day as r^c_i. These probabilities can be a general function of i, d and c.

4. The MDP Formulation

We formulate the problem of determining optimal probabilities with which to offer the appointment scheduling problem as an MDP. Given the appointment requests on each day, patients are offered a set of appointments. The Markov Decision process and its properties are presented below.

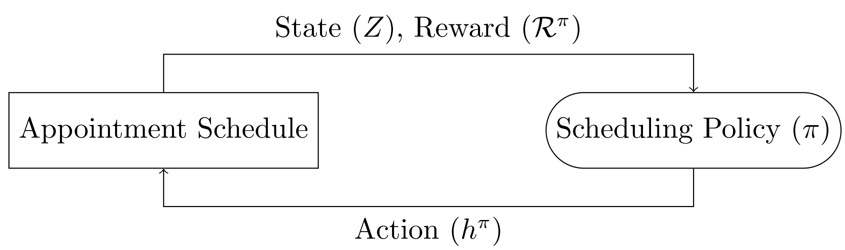

Figure 3 **Markov Decision Process: Schematic of policy-environment interaction.**

State Space – Let $Z(\mathfrak{d}) = \{Z_{i,s,c}(\mathfrak{d}) | k \in \mathbb{K}, 0 \leq i \leq d \leq D, t \in \mathbb{T}, l \in \mathbb{L}, c \in \mathbb{C}\}$ be the appointment schedule at the beginning of the day $\mathfrak{d}$, where $Z_{i,s,c}(\mathfrak{d})$ is the number of requests of class c who called in i days ago and booked an appointment $s = (k, d, t, l)$, for d days into the future or $(\mathfrak{d} - i + d)$ from day $\mathfrak{d}$. We assume that all appointments are of equal length.

Action Set – The task of the scheduler (an online system or a person) is to offer a set of appointments to each patient at each decision epoch. Let $h^\pi(Z^\pi(\mathfrak{d}), \mathscr{S}^c)$ denote the probability with which the subset $\mathscr{S}^c$ of appointments are offered to the patients having requesting a class c appointment on day $\mathfrak{d}$, when the system state is $Z(\mathfrak{d})$ under policy π. To be valid, any action must satisfy the following constraints for all states $Z^{\pi}(\mathfrak{d})$:

$$\sum_{\mathscr{S}^c} h^{\pi}(Z^{\pi}(\mathfrak{d}), \mathscr{S}^c) = 1, \quad h^{\pi}(Z^{\pi}(\mathfrak{d}), \mathscr{S}^c) \geq 0, \quad c \in \mathbb{C}. \tag{3}$$

Here, the first constraint in (3) states that requests of all classes have to be assigned a slot from the set of available assignment slots corresponding to their class including the null set (no appointment). For example, patients with *Dedicated* requests will either be offered a set of appointments to see their specific provider or none at all. On the other hand, patients with *Urgent* requests will only be offered appointments on the same day, and they cannot leave without making an appointment. Finally, the offering probability should be non-negative, i.e., the second constraint in (3). For notational brevity, we can represent the set of constraints as $\mathcal{H}$.

Transition Probabilities – Under this policy the state, $Z^{\pi}_{i,s,c}(\mathfrak{d})$, evolves when appointments are made and canceled. On any given day, if patients call to make appointments then we can update the state by the expected number of appointments which will be scheduled. This expected number can be calculated as a product of the arrival rate of the requests, the offer probability of set $\mathscr{S}^c$ and the selection probability of the patient. We also consider the retention probability (1 - cancellation probability) δ^c_{0d} for a patient of class c who called on the current day and scheduled an appointment for day d. Specifically, we can write

$$Z^{\pi}_{i,s,c}(\mathfrak{d}+1) = \begin{cases} \text{Pois}\left(\sum_{\mathscr{S}^c} \lambda^c \delta^c_{0d} P_s(\mathscr{S}^c) h^{\pi}(Z^{\pi}(\mathfrak{d}), \mathscr{S}^c)\right) & \text{if } 1 = i \leq d \leq D \\ \text{Bin}\left(Z^{\pi}_{i-1,s,c}(\mathfrak{d}), \delta^c_{i-1,d}\right) & \text{if } 2 \leq i \leq d \leq D \end{cases} \tag{4}$$

Similarly, we assume that every day some random number of appointments are canceled. We assume that this number follows a binomial distribution which can be computed as shown in (4). $\text{Bin}(n,p)$ represents a binomial random variable with parameters $n = Z^{\pi}_{i-1,s,c}(\mathfrak{d})$ and $p = \delta^c_{i-1,d}$. Here, n represents the number of appointments booked $i-1$ days ago to book an appointment s for class c and p represents the retention probability of patients who called $i-1$ days ago to book an appointment s. Similarly, there are some patients who call in on day $\mathfrak{d}$ to make the same day appointment. We can express this number as

$$Z^{\pi}_{i,s,c}(\mathfrak{d}) = \text{Pois}\left(\sum_{\mathscr{S}^c} \lambda^c P_s(\mathscr{S}^c) h^{\pi}(Z^{\pi}(\mathfrak{d}), \mathscr{S}^c)\right) \qquad \text{if } i = d = 0 \tag{5}$$

Net Reward – The total expected profit associated with a given state-action pair consists of three components, revenue generated from completed appointments (R), overtime (C_o) and idle time cost ($C_\imath$) from appointments scheduled above and below the regular physician capacity respectively. The regular capacity ($\mathscr{C}_{kl}$) represents the number of appointments a provider k can complete in regular time at a clinic location l on a given day. To model the resource uncertainty, we consider a probability γ_{kl} with which a physician retains any scheduled appointment, i.e., $1-$ *physician cancellation probability*. We assume that $\sum_k \gamma_{kl} > 0$, i.e. not all physicians can cancel the appointments at the same time at the same locations. There will be at least one physician to cover a clinic location. This step can be ensured at the time of capacity allocation. Since patients with *Flexible* and *Urgent* requests do not have a strict preference for physicians the capacity uncertainty does not affect the revenue generated from appointment scheduled by them as long as there is another physician who can complete the appointments. Additionally, patients with *Dedicated* requests may want to reschedule the appointment. We further assume that with probability β_c patients with *Dedicated* requests may not reschedule the appointment. This results in a loss of net revenue by $\beta_c(1-\gamma_{kl})$. We assume $\beta_c = 0$ for patients with requests of classes $c \in \{\mathfrak{F}, \mathfrak{U}\}$. The rescheduled requests can be considered as new *Dedicated* request. This capacity uncertainty also poses an extra burden on other physicians. Therefore, there will be an additional cost or penalty. The new expected cost can be calculated by reducing the regular capacity of a particular physician by a factor of γ_{kl}.

We formulate the three components of the objective as below. For each patient served, a nominal revenue of 1 unit is generated. Therefore, the first component R on a given decision epoch $\mathfrak{d}$, can be represented as $R := \sum_{ktcli} Y^{\pi}_{ktlci}(\mathfrak{d})$, the number of appointments booked with provider k at location l for day $\mathfrak{d}$ and slot t on day $\mathfrak{d}-i$ and the corresponding patients showed up for their appointment. For $i=0$ this is equal to the total number of same day appointment requests. For $i>0$ we can compute the number of requests resulting in patient showing up for their appointments using a binomial distribution $Bin(n,p)$. Here, $n = Z^{\pi}_{i,kitl,c}(\mathfrak{d})$ is the number of booked appointments at the beginning of day $\mathfrak{d}$ and $p = r^c_i$ is the probability that a patient will show up. Note that the no-show rate depends on the patient type. We can represent $Y^{\pi}_{ktlci}(\mathfrak{d})$ as

$$Y^{\pi}_{ktlci}(\mathfrak{d}) = \begin{cases} \text{Pois}\left(\sum_{\mathscr{S}^c} \lambda^c r^c_0 (1-\beta_c(1-\gamma_{kl})) P_{k0tl}(\mathscr{S}^c) h^{\pi}(Z^{\pi}(\mathfrak{d}), \mathscr{S}^c)\right) & \text{if } i=0, \\ \text{Bin}\left(Z^{\pi}_{i,kitl,c}(\mathfrak{d}), r^c_i\right) & \text{if } 1 \le i \le D. \end{cases} \tag{6}$$

We assume that the system incurs a unit cost of $\theta_\imath$ for the idle time and θ_o for the overtime a provider spends at a clinic. The second and the third components are due to overtime (C_o) and idle time cost ($C_\imath$) and are represented as

$$C_o = \theta_o \sum_{k,l} \left[\left(\sum_t \left(\sum_{i=1}^{D} Z^{\pi}_{i,kitl,c}(\mathfrak{d}) \right) + Z^{\pi}_{0,k0tl,c}(\mathfrak{d}) \right) - \gamma_{kl} \mathscr{C}_{kl} \right]^+,$$

$$C_\imath = \theta_\imath \sum_{k,l} \left[\left(\sum_t \left(\sum_{i=1}^{D} Z^{\pi}_{i,kitl,c}(\mathfrak{d}) \right) + Z^{\pi}_{0,k0tl,c}(\mathfrak{d}) \right) - \gamma_{kl} \mathscr{C}_{kl} \right]^-,$$

respectively, where $[.]^+ = \max\{.,0\}$ and $[.]^- = -\min\{.,0\}$. The cost is positive if the total number of appointments scheduled for day $\mathfrak{d}$ are higher or less than the expected physician capacity.

Let $\mathcal{R}^\pi(h^\pi(Z^\pi(\mathfrak{d}), \mathscr{S}^c))$ be the immediate reward received after the transition from day $\mathfrak{d}$ to $\mathfrak{d}+1$. Then we have $\mathcal{R}^\pi(h^\pi(Z^\pi(\mathfrak{d}), \mathscr{S}^c)) = R - C_o - C_\imath$. The objective is to find the best policy π that maximizes the steady-state expected profit per day, i.e., solve the following maximization problem

$$\sup_{\pi} \lim_{\bar{\mathfrak{d}}\to\infty} \frac{1}{\bar{\mathfrak{d}}+1} \mathbb{E}\left\{ \sum_{\mathfrak{d}=0}^{\bar{\mathfrak{d}}} \mathcal{R}^\pi(h^\pi(Z^\pi(\mathfrak{d}), \mathscr{S}^c)) \middle| Z^\pi(0) = z \right\}, \qquad \forall z, h^\pi \in \mathcal{H} \tag{7}$$

where z is the initial state of the system.

We remark that the MDP formulation gives strict priority to *Urgent* requests so that they receive treatment on the same day, and also gives priority to *Dedicated* requests in terms of their preference of location and physician. However, *Flexible* requests are accommodated to the largest extent at various locations of the hospital network while minimizing overtime and idle time cost.

The MDP problem formulation presented here suffers from the "curse of dimensionality" and is intractable. First, the state space $Z^\pi(\mathfrak{d})$ is multidimensional and can alone grow large as the decision epoch increases. Second, the decision variable $h^\pi(Z^\pi(\mathfrak{d}), \mathscr{S}^c)$ is multidimensional in state space, available appointments and patient class. Therefore, the number of decision variables will increase exponentially. Using conventional dynamic programming algorithms such as value or policy iterations becomes infeasible. In the following sections, we will study the properties of a single period model and show how we can use this model to develop efficient ADP approach to solve the appointment scheduling problem.

5. Properties of the Single Period Model

We now consider a single period model (static model) that makes each subset of choices in the scheduling horizon available for appointments with a fixed probability without considering the current state of the system. Therefore, we can define $h(\mathscr{S}^c)$ as the fixed probability with which we make the subset $\mathscr{S}^c$ available to appointment requests of class c.

It is shown that retention probability is decreasing in d (day of appointment requested), so that patients who schedule appointments further into the scheduling horizon are more likely to cancel the appointments (Gallucci et al. 2005). Therefore, the overall probability that the appointment is retained until the day of the appointment can be given as $\bar{\delta}^c_d = \delta^c_{0d}\delta^c_{1d}\cdots\delta^c_{dd}$, where $\delta^c_{d'd}$ is the retention rate for a patient of class c who called d' day before and scheduled an appointment for day d. This represents decreasing retention probability as d increases. Given that the patient has not canceled her/his appointment until the day of the appointment, we assume a show-up rate (= 1 - *No-show rate*) of r^c_d for the patient. The offer probability $h(\mathscr{S}^c)$ is determined by solving the optimization problem that maximizes the net reward per decision epoch:

$$\max_{h\in\mathcal{H}} R - C_o - C_\imath \tag{8}$$

where
$$R = \sum_{c,s,\mathscr{S}^c} \lambda^c \alpha^c_{kdl} \bar{\delta}^c_d P_s(\mathscr{S}^c) h(\mathscr{S}^c), \quad C_o = \theta_o \sum_{k,l} \mathbb{E}[\chi^+_{k,l}], \quad C_\imath = \theta_\imath \sum_{k,l} \mathbb{E}[\chi^-_{k,l}],$$

with $\alpha^c_{kdl} = r^c_d(1-\beta_c(1-\gamma_{kl}))$, and $\chi_{k,l} = \text{Pois}\left(\sum_{c,d,t,\mathscr{S}^c} \lambda^c \bar{\delta}^c_d P_s(\mathscr{S}^c) h(\mathscr{S}^c)\right) - \gamma_{kl}\mathscr{C}_{kl}$.

Note that the constraints of this single period policy (8) are the same as in the multi-stage MDP (7). The optimization program in (8) has $2^{|\mathbb{K}||\mathbb{D}||\mathbb{T}||\mathbb{C}|}$ decision variables, which can grow exponentially in practical situations. For example, for a clinic with 5 providers, a scheduling horizon of a week and 4 time blocks per day, the total number of choices becomes $2^{5*7*4*3} = 2^{420}$. We present an equivalent reformulation (9)-(15) below. The number of decision variables in the reformulation is $|\mathbb{S}||\mathbb{C}|+1$, which makes the problem tractable. This approach was first used by Topaloglu (2013) to study joint stocking and product offer decisions under the multinomial logit model. Feldman et al. (2014) then use this approach for appointment scheduling for a single clinic single physician with patients following the MNL choice model. Here, we adapt that approach to our network model under the MMNL and multidimensional patient preferences.

Consider the following optimization problem where $x_s(c)$ can be interpreted as the probability that a patient of class c schedules an appointment s and $u_l(c)$ represents the probability that a patient of class c leaves without making an appointment at a clinic location l.

$$\max_{x,u} \tilde{R} - \tilde{C}_o - \tilde{C}_\imath \tag{9}$$

where

$$\tilde{R} = \sum_{c,s} \lambda^c \alpha^c_{kdl} \bar{\delta}^c_d x_s(c), \quad \tilde{C}_o = \theta_o \sum_{k,l} \mathbb{E}[\tilde{\chi}^+_{k,l}], \quad \tilde{C}_\imath = \theta_\imath \sum_{k,l} \mathbb{E}[\tilde{\chi}^-_{k,l}], \quad \tilde{\chi}_{k,l} = \text{Pois}\left(\sum_{c,d,t} \lambda^c \bar{\delta}^c_d x_s(c)\right) - \gamma_{kl}\mathscr{C}_{kl}$$

subject to

$$\sum_{d,t} x_s(c) + u_l(c) = 1 \qquad c = \mathfrak{D}_k, \forall k \in \mathbb{K}, \forall l \in \mathbb{L} \tag{10}$$
$$\sum_s x_s(c) + \sum_l u_l(c) = 1 \qquad c = \mathfrak{F} \tag{11}$$
$$\sum_{k,d,t} x_s(c) = 1 \qquad d = 0, \forall l \in \mathbb{L}, c = \mathfrak{U} \tag{12}$$
$$u_l(c) = 0 \qquad \forall l \in \mathbb{L}, c = \mathfrak{U} \tag{13}$$
$$x_s(c) - v^c_s u_l(c) \leq 0 \qquad s \in \mathbb{S}, c \in \{\mathfrak{D}, \mathfrak{F}\} \tag{14}$$
$$x_s(c), u_l(c) \geq 0 \qquad \forall c \in \mathbb{C}, \forall s \in \mathbb{S}. \tag{15}$$

Scheduling probabilities in the case of *Dedicated* patient class should sum up to 1 over all days and time slots for all physician and clinic locations, as shown in (10). Constraint (11) represents that the sum of probabilities for *Flexible* requests should be equal to 1 over all physicians, days, time slots and clinic locations. Similarly, (12) represents that the sum of probabilities for *Urgent* class should be equal to 1 over all physicians and time slots for all locations on the same day (d=0). *Urgent* class requests cannot remain unfulfilled (13). Constraint (14) defines the relationship between the probabilities of scheduling and not scheduling an appointment for *Dedicated* and *Flexible* class requests. For notational brevity we represent these feasibility constraints (10)–(15) as $\mathscr{X}$. We derive following theorem to show the equivalence of (8) and (9)-(15).

THEOREM 1. $h^* = \{h^*(\mathscr{S}^c) : \mathscr{S}^c \subset \mathbb{S}, c \in \mathbb{C}\}$ *is an optimal solution to the problem* (8)*, if and only if* $(x_s^*(c), u_l^*(c))$ *is a feasible solution to the problem* (9)–(15)*, where*

$$x_s^*(c) = \sum_{\mathscr{S}^c} P_s(\mathscr{S}^c) h^*(\mathscr{S}^c), \quad \textit{and} \quad u_l^*(c) = \sum_{\mathscr{S}^c} N(\mathscr{S}^c) h^*(\mathscr{S}^c), \quad \forall c \in \mathbb{C}.$$

The proof (Appendix C) follows by showing that if h^* is an optimal solution to the original problem, we can construct a solution (x^*, u^*) using the definition given above, which is feasible and has an optimal objective equal to the original single period problem. Similarly, given the optimal solution (x^*, u^*) we can construct an optimal solution h^* as follows. We reorder and re-index the appointment sets such that we have $x^*_{s_1}(c)/v^c_{s_1} \geq x^*_{s_2}(c)/v^c_{s_2} \geq \cdots \geq x^*_{s_n}(c)/v^c_{s_n}$, where n is the number of all available slots. We define the subsets $\mathscr{S}^c_{s_1}, \mathscr{S}^c_{s_2}, ..., \mathscr{S}^c_{s_n}$ as $\mathscr{S}^c_j = \{s_1, s_2, ..., s_j\}$. We set $x_{s_{n+1}} = 0$ for the scheduling probability for any appointment outside the scheduling horizon. We can then let

$$h^*(\phi) = \begin{cases} u_l^*(c) - \frac{x^*_{s_1}(c)}{v^c_{s_1}}, & c = \mathfrak{D}_k, \forall k \in \mathbb{K}, \forall l \in \mathbb{L} \\ \sum_l u_l^*(c) - \frac{x^*_{s_1}(c)}{v^c_{s_1}}, & c = \mathfrak{F} \\ 0, & \text{for } c = \mathfrak{U}, \end{cases}$$

$$h^*(\mathscr{S}^c_{s_j}) = \begin{cases} \left[1 + \sum_{s \in \mathscr{S}^c_j} v^c_{s_j}\right] \left[\frac{x^*_{s_j}(c)}{v^c_{s_j}} - \frac{x^*_{s_{j+1}}(c)}{v^c_{s_{j+1}}}\right], & \forall c \in \{\mathfrak{D}, \mathfrak{F}\} \\ \left[\sum_{s \in \mathscr{S}^c_j} v^c_{s_j}\right] \left[\frac{x^*_{s_j}(c)}{v^c_{s_j}} - \frac{x^*_{s_{j+1}}(c)}{v^c_{s_{j+1}}}\right], & \forall c = \mathfrak{U} \\ 0, & \text{for } \mathscr{S}^c \not\subset \mathbb{S}. \end{cases} \tag{16}$$

The resulting problem is nonlinear with convex and differentiable objective (Lemma 1) and linear constraints. Additionally, the first and second derivatives of the objective function can be analytically computed (see proof of Lemma 1 in Appendix C). Therefore, we can find a global optimum for this problem using gradient descent approaches such as sequential quadratic programming or interior-point method.

LEMMA 1. $\theta_o \mathbb{E}[\tilde{\chi}^+_{k,l}] + \theta_\iota \mathbb{E}[\tilde{\chi}^-_{k,l}]$ *is convex and differentiable with respect to* $x_s(c)$*.*

The static model above (9)–(15) proposes a simple strategy to assign appointment requests to slots. We now develop bounds on the performance of this static model. To this end, we study a deterministic approximation of the static problem (9)–(15), where the objective function is the deterministic analog of the net expected revenue generated from appointment completion (9). In other words, we assume that the demand is deterministic and equal to the average arrival rate.

Consider the optimization problem:

$$\max_{(x,u)\in\mathscr{X}} \tilde{R} - \breve{C}_o - \breve{C}_\imath \tag{17}$$

where

$$\breve{C}_o = \theta_o \sum_{k,l} \breve{\chi}^+_{k,l}, \quad \breve{C}_\imath = \theta_\imath \sum_{k,l} \breve{\chi}^-_{k,l}, \quad \breve{\chi}_{k,l} = \sum_{c,d,t} \lambda^c \bar{\delta}^c_d x_s(c) - \gamma_{kl}\mathscr{C}_{kl}.$$

This deterministic formulation (17) provides appointments to patients with a fixed probability under the assumption that all random variables take their expected values. Our approach is to first construct an upper bound on the expected profit per day generated by the optimal policy (V^*), which depends on the state of the system, given in (7). We use Lemma (2) and Proposition (1) to give a performance bound for the scheduling probabilities from solving problem (9)–(15).

Lemma 2. *Let Z^* be the optimal objective of the problem* (17)*. Then $Z^* \geq V^*$ holds.*

Let $\Pi(x,u)$ be the objective function of this static problem. If (x^*,u^*) is an optimal solution to this problem, then by the definition of h^* (Theorem 1) the static policy generates an expected profit of Π^* per day. Because V^* is the optimal expected profit, we have $\Pi^*/V^* \leq 1$. The following proposition provides a lower bound on the ratio Π^*/V^*.

Proposition 1. *If Π^* is optimal objective to the problem* (8)*,*

$$\frac{\Pi^*}{V^*} \geq 1 - \frac{\theta_o+\theta_\imath}{\sqrt{2\pi}} \max\left(\frac{\sum_{kl}(\sum_{c,d,t}\lambda^c\bar{\delta}^c_d v^c_s/\nu^c)/\sqrt{\gamma_{kl}\mathscr{C}_{kl}-1}}{\sum_{c,s}\lambda^c\bar{\delta}^c_d(\alpha^c_{kdl}+\theta_\imath)v^c_s/\nu^c - \theta_\imath\sum_{kl}\gamma_{kl}\mathscr{C}_{kl}}, \frac{1}{\alpha^*_{kdl}\sqrt{(\gamma_{kl}\mathscr{C}_{kl})^*-1}} \right) \tag{18}$$

where $\nu^c = 1+\sum_{s\in\mathscr{S}^c} v^c_s$ and $\sum_{s\in\mathscr{S}^c} v^c_s$ for $c\in\{\mathfrak{D},\mathfrak{F}\}$ and $c=\mathfrak{U}$, respectively.

This theoretical bound has interesting properties. If a physician's capacity is larger than the total expected demand for that physician ($\sum_{c,d,t}\lambda^c\bar{\delta}^c_d v^c_s/\nu^c$), then the denominator of the first term in $\max(\cdot,\cdot)$ can be simplified to $\sum_{c,s}\lambda^c(\alpha^c_{kdl}+\theta_\imath)v^c_s/\nu^c - \theta_\imath\sum_{kl}\gamma_{kl}\mathscr{C}_{kl}$. This represents the revenue due to scheduled appointments. If the demand decreases or the capacity increases, the idle time will increase or the number of appointments scheduled will decrease, resulting in the lower bound. In contrast, if the demand increases or the capacity decreases, the single period objective value approaches the optimal dynamic policy. Similarly, when the physician capacity is smaller than the expected demand, the denominator can be simplified to $\alpha^*_{kdl}\sqrt{(\gamma_{kl}\mathscr{C}_{kl})^*-1}$. In this case, as the physician capacity increases, the objective value of single period scheduling policy approaches the objective value of the dynamic optimal policy. We can further simplify the bound when the minimum no-show rate is 0 or $\alpha^*_{kdl}=1$. In cases when $(\theta_o+\theta_\imath)/(\alpha^*_{kdl}\sqrt{(\gamma_{kl}\mathscr{C}_{kl})^*-1}) < 1$, we can simplify the bound as $1 - 1/\sqrt{2\pi}$. In a real clinic setting with a single specialty having 3 and 2 physicians at two clinics, an overtime and idle time cost of 1.5 times the regular pay, a uniformly decreasing choice parameter $\max(1.5, 5 - 0.5d)$ and a scheduling horizon of 7 days, the single period policy objective is theoretically at least 40.6% (62.8% practical bound) of the optimal dynamic policy objective. Therefore, the single period scheduling policy can be used to get good performance.

6. Development of Approximate Scheduling Policy

In the previous section, we discussed some structural properties of the single period scheduling model. While this can be used in practice with a "good enough" solution quality, it does not consider the current state of the system. In this section, we develop an approximate dynamic policy which addresses this problem by proposing a dynamic policy. The main idea behind this policy is that on each decision epoch patients will be offered a set of appointments optimally while assuming that future appointments are offered using a fixed probability, $h(\mathscr{S}^c)$ for $\mathscr{S}^c \in \mathbb{S}$ and $c \in \mathbb{C}$. We present a general derivation which is independent of the fixed probabilities used.

At any decision epoch we start with an appointment schedule $z = \{z_{i,s,c} | 1 \le i \le d \le D\}$. This represents the number of appointment requests of class c who called in i days ago and will book an appointment s, d days into the future from today. Our objective is to find an optimal policy π that allocates the requests arriving on the current decision epoch with probability q and assumes a fixed probability h for requests which arrive on the future decision epochs.

We can rewrite the dynamic problem (7) as maximizing the difference of the long run total expected rewards between the dynamic policy π and the static policy which uses fixed offer policy h. Because both policies offer appointments using identical probabilities after the current decision epoch. Since the appointments on current decision epoch can only be scheduled till day D (scheduling horizon), the appointment schedules are stochastically identical beyond the scheduling horizon. Therefore, the objective of the MDP (7) can be approximated by the difference between the total expected rewards generated over next $D+1$ days under the dynamic policy (L_π) and the static policy (L). This difference can be written as $\max_{q \in \mathcal{H}} L_\pi(z, q, h) - L(z, h)$. Since the second term in the objective is independent of q, we can further simplify it as $\max_{q \in \mathcal{H}} L_\pi(z, q, h)$. Give the current state of the system we can calculate the objective as

$$\max_{q \in \mathcal{H}} \sum_{c,s} G_{c,s}(z, q, h)$$

where $\mathcal{H}$ is defined in (3) and G is defined by

$$G_{c,s}(z,q,h) = \mathbb{E}\left\{ \sum_{i=1}^{D-d} \bar{A}_{i,i+d} + \bar{B}_d(q) + \sum_{i=1}^{d} \bar{C}_{id}(h) - \theta_o \left[\sum_{i=1}^{D-d} A_{i,i+d}(z) + B_d(q) + \sum_{i=1}^{d} C_{id}(h) - \gamma_{kl} \mathscr{C}_{kl} \right]^+ \right.$$

$$- \theta_\imath \left[\sum_{i=1}^{D-d} A_{i,i+d}(z) + B_d(q) + \sum_{i=1}^{d} C_{id}(h) - \gamma_{kl}\mathscr{C}_{kl} \right]^- \Bigg\}. \tag{19}$$

Here, $A_{i,i+d}$ is the number of requests arriving i days ago for an appointment d days from today and will not be canceled by the morning of their appointments. $\bar{A}_{i,i+d}$ is the number of requests resulting in patient show up. Similarly, B_d is the number of requests arriving today to make an appointment for d days from today and will not result in cancellation till the day of the appointment. $\bar{B}_d$ is the number of requests resulting in patient show up. C_{id} is the number of appointments arriving on i^{th} day in the scheduling horizon for an appointment on d^{th} day and will not be canceled till the day of the appointment. Finally, $\bar{C}_{id}$ is the number of requests resulting in patient show up. We can define these variables as below

$$\begin{aligned}
&A_{i,i+d} = \text{Bin}(z_{i,i+d}, \delta^c_{i,i+d}), && \bar{A}_{i,i+d} = \text{Bin}(z_{i,i+d}, \alpha^c_{kdl}\delta^c_{i,i+d}),\\
&B_d = \text{Pois}\Big(\sum_{\mathscr{S}^c} \lambda^c \bar{\delta}^c_d P_s(\mathscr{S}^c) q(\mathscr{S}^c)\Big), && \bar{B}_d = \text{Pois}\Big(\sum_{\mathscr{S}^c} \lambda^c \alpha^c_{kdl} \bar{\delta}^c_d P_s(\mathscr{S}^c) q(\mathscr{S}^c)\Big),\\
&C_{id} = \text{Pois}\Big(\sum_{\mathscr{S}^c} \lambda^c \bar{\delta}_{d-i} P_{k,d-i,t}(\mathscr{S}^c) h(\mathscr{S}^c)\Big), && \bar{C}_{id} = \text{Pois}\Big(\sum_{\mathscr{S}^c} \lambda^c \alpha^c_{kdl} \bar{\delta}^c_{d-i} P_{k,d-i,t}(\mathscr{S}^c) h(\mathscr{S}^c)\Big),
\end{aligned} \tag{20}$$

where $z_{i,i+d} = z_{i,k(i+d)tl,c}$ is the number of requests arriving i days ago for an appointment d days from today $\alpha^c_{kdl} = r^c_d(1 - \beta_c(1 - \gamma_{kl}))$, and $\delta^c_{i,i+d}$ is the probability that a patient who called on day i requests to make an appointment on day $i + d$. $\bar{\delta}_d$ is the probability that the appointment is retained until the day of the appointment. $\bar{\delta}_d$ is defined in section 5. The optimal policy q^* depends on the current state of the system z and can give the optimum appointment set offer probability. Note that the number of decision variables in the current system still grows exponentially. Since the problem structure is similar to the original single period problem (8), it can be reformulated to be tractable by approximating the binomial random variables A and $\bar{A}$ as Poisson random variables with their corresponding means.

7. Computational Experiments

7.1. Description of other heuristic policies

Here, we define a total of five scheduling policies which can be used for appointment scheduling while considering heterogeneous patients preference. Some of these are discussed above and some are aimed to capture real world appointment scheduling practices.

Current Policy (CP) is aimed to replicate the real world scheduling practices. In practice, when a patient calls for an appointment, the scheduler asks for her/his preference and if available, it is allocated to her/him. Otherwise, the patient will be put on a waiting list or she/he can leave without making an appointment. In this approximation, we ignore any waiting list. Therefore, in this policy, we assume that all of the available appointments can be offered to patients according to the choice probability of their request class.

Single Period Policy (SP) is obtained by solving the single period optimization problem (8) or its tractable reformulation (9)–(15). The optimum offer set probability h^* can be calculated from the optimum scheduling probability (x^*, u^*) by using Theorem 1.

Dynamic Policy (DP) is obtained by solving the approximate dynamic policy problem $\max_{q\in\mathcal{H}} \sum_{cs} G_{cs}(z, q, h)$. Given the state of the system z and single period scheduling policy h, it will give the scheduling probability with which an appointment is scheduled. This probability is calculated at every decision epoch. Once the scheduling probability is obtained, we can calculate the offer set probability $h^\pi(Z(\mathfrak{d}), \mathscr{S}^c)$. In this policy, the single period offer probability is calculated by solving the problem (8) and it stays constant throughout.

Dynamic - Current Policy (DCP) is similar to DP. The only difference is that we use the choice probability (same as CP) to calculate the fixed scheduling policy h. Since we are comparing the CP and SP it makes sense to create a dynamic policy which uses CP to schedule patients.

Open Access Policy (OAP) is widely used in many clinic setups. In this policy appointments are only scheduled for short term, i.e., just on the same-day or in some cases same day and next-day. We can develop a dynamic version of the OA policy with patient choice by changing the scheduling horizon to 1 day, i.e., day 0 and 1. The rest of the procedures follow same as DP.

7.2. Data Description

All computational experiments were conducted using one year scheduling data from a single specialty clinics at a major healthcare system in the east coast of USA. In this specialty, 19 physicians serve 7 clinics.

Arrival data and classification. In the historical scheduling data the appointments were categorized into 3 classes, namely, new, returning and walk-in (same day). We assume that these appointments belong to 3 classes of appointment requests, *Urgent*, *Flexible*, and *Dedicated* corresponding to the appointments in the historical data. The average arrival rates for dedicated, flexible and urgent requests were derived from the historically booked appointment and the patients in the waiting list for each class. Due to the unavailability of categorized appointment requests in the historical data we used the following classification as an estimate. New requests were assigned as *Flexible*, the follow up requests were assigned as *Dedicated*, and requests without a prior appointment were classified as *Urgent*. This classification of appointment requests can be improved by collecting data on patient history and the type of healthcare service requested as future appointment requests arrive. For the numerical experiments in this paper, the arrival rates for urgent requests vary between 2.5 and 6.5 across different clinics, the arrival rate for all flexible appointments is 80, and the arrival rates for dedicated appointments vary between 5.5 and 7 across all physicians.

Capacity. The capacity C_{kl} was determined as the average number of appointment slots booked by the physicians at the clinic location in the past one year. The average capacity for across all providers was 7.5. For the numerical studies in subsections 7.3.2 and 7.3.3, we assume the capacity uncertainty coefficient γ_{kl} to be 1 for all physicians. In subsection 7.3.4, we study the effect of γ (capacity uncertainty) and β (probability that dedicated patients are not going to reschedule an already scheduled appointment once it is canceled by the clinic) on the problem objective.

Cancellation Rate. After discussions with the schedulers, the cancellation rate was assumed to be solely dependent on the difference between the day when appointment was made (i) and the day of the appointment (d). We assume the probability of cancellation of requests $(1-\bar{\delta}_d^c)$ as $\max(0.2, 0.04(d-i))$, which is a slight modification of the one used by Feldman et al. (2014). This cancellation rate has an upper bound of 0.2 to make it more realistic for a longer scheduling horizon. We assume that if an appointment was not canceled by the day of the appointment, the patient will show up for the appointment.

Cost parameters. To see the sensitivity of overtime and idle time cost, we vary θ_o, θ_i as 1.25, 1.5, and 1.75 assuming the regular revenue as 1. The idle time cost is assumed to be the same as overtime cost.

Choice parameter. To see how sensitive the solution is to the choice parameter, we consider two choice parameters, a uniformly decreasing parameter $v^a = \max(1.5, 5-0.5d)$ and a constant parameter $v^b = 1.5$. We take the minimum choice parameter as 1.5 because the choice parameter corresponding to not scheduling any appointment is 1, and we assume patients would prefer scheduling an appointment compared to leaving without making any appointment.

Scheduling horizon. As the scheduling horizon affects the optimum profit and the schedule, we vary the scheduling horizon (D) as 2, 7 and 15 days, which includes the current day. We assumed two slots per day (morning, afternoon) for the computational experiment, since the choice parameters depend on the different population setting.

7.3. Numerical Results

7.3.1. Design of numerical experiments We conducted simulation studies to compare policies described in Section 7.1, with respect to the expected net profit per day. This profit was calculated as the difference of the revenue generated from the total number of appointments completed and the overtime and idle time costs because of the appointments scheduled over and under the daily capacity limit respectively, of each physician. On each day, we determine the offer set for each patient class using the policy definition. Then, we sample the total number of patients requesting appointments for each class. Each patient chooses an appointment from the offer set using the choice probability (1a)–(1b) or leaves without making an appointment. Once scheduled, each patient can cancel the appointment based on the associated cancellation probability or have a no-show on the day of the appointment. Using the updated schedule we calculate the revenue and the number of appointments scheduled and hence the profit for that day. We repeat this process for 100 days out of which the first 40 days are set as the warm up period. We conduct 100 such replications. We keep the same random seed for different policies for a particular iteration to have the same random demand. We used Intel® Xeon® CPU E5-2680 v3 @ 2.50GHz with 10 cores and 32 Gb RAM to conduct these simulations.

Table 1 Benchmark comparison of the scheduling policies

Choice	Horizon	$\theta_o = \theta_\imath$	Expected net reward per day				
			DP	DCP	SP	OAP	CP
v^a	7	1.25	276.29	276.5* (0.07)	267.59 (-3.15)	276.67* (0.14)	238.8 (-13.57)
		1.50	253.01	254.81* (1.06)	242.02 (-4.1)	253.28* (0.33)	212.11 (-16.02)
		1.75	229.21	232.48* (1.43)	216.14 (-5.7)	229.34* (0.06)	185.14 (-19.23)
	15	1.25	276.76	267.07 (-3.5)	266.97 (-3.54)	276.67* (-0.03)	199.36 (-27.97)
		1.50	253.13	243.57 (-3.81)	241.73 (-4.51)	253.27* (0.34)	171.11 (-32.73)
		1.75	229.21	219.47 (-4.25)	215.99 (-5.77)	229.34* (0.06)	142.24 (-37.94)
v^b	7	1.25	263.63	259.81 (-1.45)	247.95 (-5.95)	260.42 (-1.22)	223.2 (-15.34)
		1.50	241.34	237.02 (-1.6)	222.12 (-7.89)	236.99* (-1.42)	195.87 (-18.87)
		1.75	218.28	214.12 (-1.91)	196.12 (-10.15)	213.2 (-2.33)	168.31 (-22.89)
	15	1.25	262.68	246.68 (-6.09)	248.37 (-5.45)	260.42* (-0.86)	160.7 (-38.82)
		1.50	240.68	221.76 (-7.76)	222.68 (-7.42)	236.92* (-1.2)	129.75 (-46.62)
		1.75	218.15	196.36 (-9.99)	196.64 (-9.86)	213.2 (-2.27)	98.17 (-55)

(.) Percentage gap of different policies with respect to DP
* Performance of DP is statistically different except when marked
θ = overtime and idle time cost; v^a, v^b = choice parameters

7.3.2. Benchmark comparison of various policies The benchmark comparison results of the simulation are given in Table 1. The first three columns of the table describe each scenario, which has different scheduling horizons, overtime and idle time cost parameter ($\theta_o = \theta_\imath$) and patient choice parameter. The next five columns give the expected net reward per day by different policies over different scenarios. The result from open access policy (OAP) is constant for different scheduling horizon. We see that performance of DP is significantly better than other policies with 95% confidence level in most scenarios, except in the case of OAP since the performance of DP does not change significantly by the number of days in the scheduling horizon. From the performance gap of different policies with respect to DP, we see that DP performs the best in all but 3 scenarios, where DCP gives better performance than DP. In these scenarios the scheduling horizon is shorter and the choice parameter is decreasing uniformly. DCP likely performs worse because the random scheduling policy CP is a short-sighted policy which does not consider long-term effects and performs much worse for longer scheduling horizon. As the overtime cost and the scheduling horizon increase, it is important to offer the optimal set while considering future uncertainty. The performance gap of SP and CP with respect to DP is up to 9.86% and 55.0%, respectively, making CP the worst performing policy. This might be because these are static policies and cannot adapt to the demand uncertainty and the changes in the schedules. On the other hand, DP and DCP offer the appointment slot based on the appointment schedule on the current day. Since CP is the most accommodating scheduling policy for the customers, it performs the worst. This change in the objective varies by different overtime/idle time costs θ. As θ increases, the performance gap with DP is increasing, meaning that DP performs even better with higher overtime and idle costs. Therefore, in practice, appointment schedulers can see significant gains by using DP or OAP policy.

The choice parameters has significant impact on the objective values. We see that in case of v^b (where the preference weights remains constant as we go further in the booking period), DP performs better compared to other policies. This implies that when patients are willing to wait longer for their appointments, then the performance of DP is significantly better compared to other policies. As the scheduling horizon increases, the performance of DP, SP and OAP remain constant irrespective of the choice parameters. However, the performance of policies derived from CP deteriorate as the scheduling horizon increases. This is more predominant in case of v^a. This might be because with the choice parameter v^a, less and less patients schedule long term appointments. However, in the case of v^b, more patients can schedule appointments later in the horizon, thereby increasing the profit. But eventually, the effect of overtime cost and idle time cost causes the overall objective to reduce faster.

7.3.3. Illustration of the network effect In our scheduling framework, we assume that the *Flexible* appointment requests can be allocated to any clinic location and/or physician. Next we illustrate advantages of using this pooled demand vs when these *Flexible* requests have independent demand for each clinic and they are allocated to them separately without considering this pooling effect. We conduct a numerical study where we compare the pooling effect scenario with a new scenario where *Flexible* requests' demand is equally divided to different clinic and we modify (11) to allow patients to be seen only at corresponding clinic locations.

Table 2 shows that as the overtime cost increases, the improvement in the objective value provided by the pooled effect increases. Static Policy (SP) with the pooling effect can provide a significant improvement of up to 8.6%. In the previous numerical study (Table 1), we saw that the performance of SP remains constant for different scheduling horizon. We see similar pattern here. We also observe that the performance gain provided by the pooling effect is more significant when the choice parameter decrease with the scheduling horizon, compared to the case when the choice parameter remain constant. Therefore, in practice, we should be able to see similar gains in

Table 2 Benchmark comparison of the SP with demand pooling

Choice	Horizon	$\theta_o = \theta_\imath$	Expected net reward per day for SP	
			No Demand Pooling	Demand Pooling
v^a	7	1.25	265.27	267.5 (0.84)
		1.50	232.84	241.82* (3.86)
		1.75	199.01	216.09* (8.58)
	15	1.25	265.34	267.5 (0.82)
		1.50	233.11	241.85* (3.75)
		1.75	199.43	216.09* (8.36)
v^b	7	1.25	247.33	247.33 (-0.0)
		1.50	217.56	221.05* (1.61)
		1.75	186.09	194.91* (4.74)
	15	1.25	247.13	247.53 (0.16)
		1.50	218.17	221.47 (1.51)
		1.75	185.21	195.33* (5.46)

(.) Percentage gap of different policies with respect to SP with no demand pooling
* Performance of SP with no demand pooling is statistically different except when marked
$\theta =$ overtime and idle time cost; v^a, v^b = choice parameters

our objective function. The dynamic policies DP and DCP can provide advantage on top of these gains provided due to the pooling effect. This numerical study shows that network pooling can be a significant advantage, specially when patients are *Flexible* in their choice of clinic location.

7.3.4. Effects of physician uncertainty and rescheduling of Dedicated patients In previous numerical studies, we assumed that physicians cannot cancel their appointment once they are scheduled ($\gamma = 1$). In other words, there is no physician uncertainty. Now, we illustrate how this physician uncertainty (γ) and the rescheduling probability of *Dedicated* requests (β) affects the performance of our scheduling policies. We conduct numerical studies where $\gamma \in \{1, 0.95, 0.85\}$ and $\beta \in \{0.0, 0.5, 1.0\}$. The results of this study are shown in Table 3. Since there are no physician cancellation for $\gamma = 1$, the objective does not vary for different values of β.

We observe from Table 3 that as γ decreases the objective value decreases in general, because the effective physician capacity has reduced. We further see that as β increases the objective value decreases, since less *Dedicated* requests are being rescheduled. We notice that this change is more significant when the choice parameter decreases with scheduling horizon (v^a). We remark that for smaller values of γ and β the objective value in the case of SP does not change significantly. Therefore, in practice, SP will be robust to small changes in physician capacity. In the case of CP, we see a different trend. The objective value first increases and then decreases as γ decreases. This is an indicator of higher idle time cost in the beginning. As γ decreases, the effective physician capacity is reduced, thereby, reducing the idle time cost and increasing the overall objective value. But eventually the physician capacity becomes lower than the average demand. This leads to lower objective value. From this study we can see that even in the case of significant physician uncertainty, the proposed SP outperforms the CP policy in most scenarios.

Table 3 Benchmark comparison of the SP with physician uncertainty

Policy	Choice	Horizon	$\theta_o = \theta_\imath$	Expected net reward per day for SP						
				γ =1.00	γ =0.95			γ =0.85		
				$\beta = 0$	$\beta = 0$	$\beta = 0.5$	$\beta = 1$	$\beta = 0$	$\beta = 0.5$	$\beta = 1$
SP	v^a	7	1.25	266.79	264.74* (−0.77)	261.60* (−1.94)	258.32* (−3.17)	242.49* (−9.11)	233.76* (−12.38)	220.61* (−17.31)
			1.50	241.07	238.76* (−0.96)	235.83* (−2.17)	232.74* (−3.45)	202.41* (−16.04)	188.84* (−21.67)	176.99* (−26.58)
			1.75	215.36	210.21* (−2.39)	204.83* (−4.89)	200.00* (−7.13)	166.09* (−22.88)	155.27* (−27.90)	142.03* (−34.05)
		15	1.25	266.79	264.74* (−0.77)	261.60* (−1.94)	258.32* (−3.17)	242.49* (−9.11)	232.54* (−12.84)	217.65* (−18.42)
			1.50	241.07	238.76* (−0.96)	235.83* (−2.17)	232.74* (−3.45)	201.71* (−16.33)	190.72* (−20.88)	175.19* (−27.33)
			1.75	215.36	209.80* (−2.58)	205.19* (−4.72)	200.25* (−7.02)	165.05* (−23.36)	149.89* (−30.40)	141.62* (−34.24)
	v^b	7	1.25	247.92	249.65* (0.70)	245.59* (−0.94)	242.95* (−2.00)	235.97* (−4.82)	231.51* (−6.62)	219.50* (−11.46)
			1.50	221.92	224.29* (1.07)	220.22* (−0.77)	217.74* (−1.89)	202.55* (−8.73)	188.97* (−14.85)	176.56* (−20.44)
			1.75	195.90	198.62* (1.39)	194.66 (−0.63)	192.30* (−1.84)	164.87* (−15.84)	154.18* (−21.29)	138.86* (−29.12)
		15	1.25	247.79	249.58* (0.72)	245.84* (−0.79)	243.17* (−1.86)	235.43* (−4.99)	231.15* (−6.71)	219.41* (−11.45)
			1.50	221.93	224.20* (1.02)	220.51 (−0.64)	217.78* (−1.87)	202.40* (−8.80)	191.44* (−13.74)	176.87* (−20.31)
			1.75	195.96	198.25* (1.17)	194.76 (−0.61)	192.16* (−1.94)	164.97* (−15.81)	152.18* (−22.34)	139.60* (−28.76)
CP	v^a	7	1.25	239.83	242.30* (1.03)	239.42 (−0.17)	235.76* (−1.69)	231.91* (−3.30)	226.14* (−5.71)	220.63* (−8.00)
			1.50	213.22	216.40* (1.49)	213.67 (0.21)	210.03* (−1.49)	204.42* (−4.13)	199.82* (−6.28)	195.29* (−8.41)
			1.75	186.61	190.49* (2.08)	187.92 (0.70)	184.30* (−1.24)	176.93* (−5.19)	173.50* (−7.03)	169.94* (−8.94)
		15	1.25	199.46	208.92* (4.74)	204.94* (2.75)	201.02* (0.78)	212.41* (6.49)	204.91* (2.73)	197.53* (−0.97)
			1.50	170.86	182.54* (6.83)	178.40* (4.41)	174.27* (2.00)	187.46* (9.72)	180.32* (5.54)	173.17* (1.35)
			1.75	142.26	156.16* (9.77)	151.85* (6.74)	147.52* (3.70)	162.52* (14.24)	155.73* (9.47)	148.82* (4.61)
	v^b	7	1.25	223.38	229.41* (2.70)	225.93* (1.14)	221.79 (−0.71)	224.35 (0.44)	218.28* (−2.28)	211.78* (−5.19)
			1.50	195.93	203.43* (3.83)	199.93* (2.04)	195.64 (−0.15)	198.02* (1.06)	192.76* (−1.62)	186.87* (−4.63)
			1.75	168.49	177.45* (5.32)	173.94* (3.23)	169.49 (0.59)	171.68* (1.89)	167.23 (−0.75)	161.96* (−3.88)
		15	1.25	162.12	175.01* (7.95)	170.93* (5.44)	168.11* (3.69)	188.97* (16.56)	181.47* (11.93)	174.02* (7.34)
			1.50	131.02	146.89* (12.11)	142.56* (8.81)	139.61* (6.55)	164.53* (25.58)	157.05* (19.87)	149.50* (14.11)
			1.75	99.91	118.77* (18.87)	114.18* (14.28)	111.10* (11.19)	140.10* (40.22)	132.63* (32.74)	124.98* (25.09)

(·) Percentage gap of different scenarios with respect to the case of no physician cancellation
* Performance of SP/CP without physician cancellation is statistically different except when marked
θ = overtime and idle time cost; v^a, v^b = choice parameters
γ = probability that a physician retains their appointment
β = probability that a *Dedicated* patient will reschedule an appointment if it has been canceled by the physician

8. Robustness with Stochastic Intensity

We now assume that the demand follows a doubly stochastic Poisson process, where the mean arrival rate is a random variable following a general probability distribution with finite support such that Λ^c has a probability mass function $\zeta_i := \mathbb{P}(\Lambda^c = \lambda_i^c),\ \forall i \in 1,...,n$. In Appendix B, we show that the new static policy objective function (27) is also differentiable and convex. Therefore, we can still achieve a globally optimal solution to (27). We can use similar approach section 6 and derive the results for all of the policies described in section 6. We conducted simulation experiments with arrival intensities following finite support distribution $\lambda^c + 0.1\lambda^c\{\frac{n+1}{2} - i\}$ with probability $\zeta_i = 1/n$, where λ^c is the original arrival intensities used in section 7, $i = 1,...,n$ and $n = 3$. This results in the same expected arrival rate as in Table 1.

Table 4 **Benchmark comparison of the scheduling policies under stochastic arrival intensity**

Choice	Horizon	$\theta_o = \theta_\iota$	Expected net reward per day				
			DP	DCP	SP	OAP	CP
v^a	7	1.25	271.98	274.71* (1)	256.11 (-5.83)	272.9* (0.34)	220.16 (-19.05)
		1.50	247.09	251.70* (2.2)	229.89 (-6.91)	248.61* (0.77)	192.72 (-22.27)
		1.75	221.51	228.47* (3.14)	202.98 (-8.37)	223.79* (1.03)	164.7 (-25.65)
	15	1.25	271.86	263.78* (-2.97)	256.11 (-5.79)	272.9* (0.38)	182.01 (-33.05)
		1.50	247.10	237.58* (-3.7)	229.38 (-6.9)	248.64* (0.88)	153.04 (-38.63)
		1.75	221.89	210.73 (-5.03)	202.53 (-8.72)	223.79* (0.86)	123.4 (-44.39)
v^b	7	1.25	254.97	244.93 (-3.94)	237.05 (-7.03)	254.53* (-0.17)	205.04 (-19.58)
		1.50	230.30	218.93 (-5.07)	210.07 (-8.84)	231.1* (0.48)	176.9 (-23.58)
		1.75	205.45	192.29 (-6.4)	182.45 (-11.2)	207.49* (1)	148.53 (-27.7)
	15	1.25	253.46	235.28 (-7.17)	235.3 (-7.17)	254.53* (0.42)	143.21 (-43.5)
		1.50	229.49	207.88 (-9.4)	209.62 (-8.63)	231.25* (1.13)	111.62 (-52.09)
		1.75	205.29	180.21 (-12.22)	183.36 (-10.69)	207.49* (1.07)	79.25 (-61.4)

$^{(\cdot)}$ The quantity in the parenthesis is the percentage gap of different policies with respect to DP
* Performance of DP is statistically different except when marked
$\theta =$ overtime and idle time cost; $v^a, v^b =$ choice parameters

Comparison of Table 4 with Table 1 shows that the objective value reduces in comparison to the case of constant arrival rate for all policies. The reduction is 5.9%, 10.2%, 3.40% and 13.04% for DP, DCP, SP and CP, respectively. This shows that the affect of the uncertainty in demand is more on CP and DCP compared to SP and DP. Similarly, the percentage gap of DP with respect to all other policies have improved. The maximum percentage gap of SP and CP with respect to DP has increased to 11.2% and 61.4%, respectively. Therefore, DP provides an additional advantage compared to the other heuristic policies in case of more general patient demand .

9. Joint Appointment Scheduling and Capacity Planning

In this section, we leverage the proposed appointment scheduling framework to build a joint appointment scheduling and physician capacity planning model. Here, we define a scenario Dynamic Joint - Current Policy (DJCP), where we assume a fixed probability CP to schedule future appointments and our objective is to find the optimum scheduling probability and the optimum physician capacity for the current decision epoch $\mathfrak{d}$. We can formulate the DJCP problem for a single period as in (9) by adding following additional constraints to the static problem discussed in section 5.

$$x_s(c) \leq \mathscr{C}_{kl}, \qquad \forall s \in \mathbb{S},\ \forall c \in \mathbb{C} \tag{21}$$

$$\sum_k \mathscr{C}_{kl} \geq 0, \qquad \forall l \in \mathbb{L} \tag{22}$$

and $\mathscr{C} \in \Omega$, where Ω is the set of constraints which can be derived from physicians' preference of which clinic locations they can go to. Constraint (21) ensures that no appointments are scheduled if physician capacity is not available and (22) is coverage constraint ensuring that all clinic locations are covered by at least one physician. From the solution of this problem we can find the optimum capacity and scheduling probability. Since the objective function of this problem is not differentiable anymore, we solve this problem by reformulating it as a Mixed Integer Nonlinear Program (MINLP)

$$\max_{h \in \mathcal{H}} \quad \mathbb{E}\left[\tilde{R} - \check{C}_o - \check{C}_\imath\right] \tag{23}$$

where

$$\check{C}_o = \theta_o \sum_{k,l} \check{\chi}^+_{k,l}, \quad \check{C}_\imath = \theta_\imath \sum_{k,l} \check{\chi}^-_{k,l}, \quad \check{\chi}_{k,l} = \sum_{c,d,t} \lambda^c \bar{\delta}^c_d x_s(c) - \gamma_{kl} \sum_{i=1}^{n} \check{\mathscr{C}}_{kl,i+1},$$

subject to

$$\mathscr{C}_{kl} = \sum_{i=0}^{n-1} \check{\mathscr{C}}_{kl,i+1},\ \forall k \in \mathbb{K},\ \forall l \in \mathbb{L} \tag{24}$$

$$\check{\mathscr{C}}_{kl,i+1} \geq \check{\mathscr{C}}_{kli},\ \forall k \in \mathbb{K}, \forall l \in \mathbb{L},\ \forall i \in \{0,1,...,n\} \tag{25}$$

$$\check{\mathscr{C}}_{kli} \in \{0,1\},\ \forall k \in \mathbb{K},\ \forall l \in \mathbb{L} \tag{26}$$

where n is the upper bound on the capacity. Based on these relationships and given $\mathscr{C}_{kl}$ we can determine $\check{\mathscr{C}}_{kli}$ and vice versa. By substituting the values $(x,u,\mathscr{C})$ and $(x,u,\check{\mathscr{C}})$ into the objective functions (23) and (9), respectively, we can show that the objective values are equal. Therefore, we can say that these two formulations are equivalent.

Corollary 1. *There exists a global optimal solution to the joint capacity planning and appointment scheduling problem* (23) *if* Ω *is a closed set.*

Since $\check{\mathscr{C}}$ belongs to a closed subset Ω and (23) is also convex and differentiable in terms of x for a fixed $\check{\mathscr{C}}$ (Lemma 1) we can say that there exists a global optimum for this MINLP problem. However, this problem is not jointly convex anymore in x and $\check{\mathscr{C}}$ due to binary variables. Therefore, we cannot use gradient descent approaches to solve this problem. Proposition 2 shows the convexity of (23) upon relaxing the binary variables. Since we can calculate the exact first and second derivatives with respect to x, we are able to solve the original problem by using Generalized

Bender's decomposition. Therefore, we can solve this problem by decomposing it into a master problem ($\check{\mathscr{C}}$-space) and a sub-problem (x-space).

Proposition 2. *The continuous relaxation of the joint mixed integer nonlinear programming problem* (23) *is a convex problem.*

Table 5 Benchmark comparison of the DJCP and DCP under joint capacity planning and scheduling

Choice	Horizon	$\theta_o = \theta_\iota$	DCP	DJCP
v^a	7	1.25	276.5	283.80* (2.64)
		1.50	254.81	263.50* (3.41)
		1.75	232.48	242.96* (4.51)
	15	1.25	267.07	271.10* (1.51)
		1.50	243.57	249.10* (2.27)
		1.75	219.47	227.44* (3.63)
v^b	7	1.25	259.81	267.00* (2.77)
		1.50	237.02	246.18* (3.86)
		1.75	214.12	224.21* (4.71)
	15	1.25	246.68	237.08* (-3.89)
		1.50	221.76	212.80* (-4.04)
		1.75	196.36	188.20* (-4.16)

(.) Percentage gap of different policies with respect to DCP
* Performance of DJCP is statistically different except items marked as *
θ = overtime and idle time cost; v^a, v^b = choice parameters

Furthermore, we conduct numerical experiments to show that a meaningful gains in the net objective can be achieved by using DJCP. From results shown in Table 5, we find that the DJCP improves upon the best scheduling policy objective in all the scenarios by comparing these results with Table 1. We can see that as the per unit overtime and idle time cost increases the improvement in the objective produced by DJCP is higher. In our numerical experiments, we see improvements of as much as 4.75%. Furthermore, we see that the benefit of using a joint planning and scheduling is higher if patient choice parameters are decreasing with scheduling horizon. Therefore, this policy might be helpful in real world HPNs where the physician capacity is limited. However, this improvement in the performance gap does not change significantly with increasing horizon. For the case of constant choice parameter v^b the performance of DJCP is worse than DCP. This is likely because we are using the random policy to assign future capacity and appointments and with longer scheduling horizon performance of CP decreases rapidly (Table 1).

10. Conclusion

Through this work, our aim is to develop an implementable appointment scheduling system by considering a setup that is more general than the ones presented in the literature. To develop and test the model we rely on three main assumptions. First, we consider the availability of physicians as the only resource needed to schedule an appointment. Second, we assume that patients with *Flexible* requests have complete flexibility and can be allocated to any clinic location. Similarly, *Urgent* and *Flexible* requests can be fulfilled by any physician. Third, we use the common assumption in appointment scheduling research of letting the demand follow a Poisson distribution.

Practical Implementation. This model is specially useful for online scheduling systems like SimplyBook and Zocdoc, where healthcare systems can offer multiple appointments to patients so that they can satisfy patient demand while maximizing profit and without losing any demand. In this paper, we have developed an automated patient scheduling model which can generate appointment sets to be offered to multi-priority patients with heterogeneous preferences, no-show and cancellation probabilities and multiple resources. Our model offers a set of appointments to patients as soon as appointment requests arrive. It does not assume a first-come-first-serve queue like traditional scheduling models.

For practical implementation, as soon as the appointment request arrives, a scheduler (or the online model) can run the chosen policy using the scheduling system. A subset of appointment sets are then shown to the patient. This is more suitable for the online scheduling system since the appointments are not shown sequentially. The single run of DP takes approximately 30 sec to run on a 2.50GHz CPU with 4 cores and 16 Gb RAM. On the other hand, OAP and SP take less than 15 seconds to run. Our results show that the proposed dynamic open access policy (OAP) with scheduling horizon of two days, works quite well for almost all scenarios. Therefore, an appointment set resulting in good performance can be shown to customers in very short time. After that customer can select the preferred appointment. In most practical settings, different patients will have varying preferences. We can use the Dynamic Policy (DP) to model this case by first classifying the patient into the 3 patient classes. This policy assumes that future appointments are scheduled with a fixed policy (SP, this can be achieved by assuming an average patient preferences). Once an appointment request arrives, the preference specific to the requesting patient can be used to solve the DP and to achieve the patient specific appointment set. Therefore, we can have as many patient classes as the number of patients without increasing the number of decision variables.

Future work. The methodology could be expanded in the future along several directions. One possible direction is to refine the allocation of resources by considering room and nurse availability in the formulation. Another direction could be to add simple linear constraints to allow *Flexible* and *Urgent* requests to be completed by a specific physician, at a location of choice. For instance, the pool of possible locations for the fulfillment of *Flexible* requests could be reduced based on the location of the patient and distance that a patient is willing to travel. The addition of linear constraints would change neither the structure of the formulation of this model, nor the solution methodology because the constraint set will remain convex. Therefore, we can still use gradient descent techniques to solve the single step optimization problem. Additionally, empirical data could be obtained regarding appointment requests so that an empirical distribution could be used to generate a dynamic scheduling policy that does not rely on an unrealistic Poisson distribution. In the development of the model, we also showed that the proposed policies are robust under a more general doubly stochastic Poisson process. This shows the practicality of our proposed scheduling policies. To our knowledge, appointment requests are not gathered in clinics and there were no readily available datasets at the moment of the development of this model to test the methodology under an empirical distribution of appointment requests. Given appropriate data, the bounds around expected demand can also be used to model the appointment scheduling problem as a robust dynamic programming problem. Reliable patient behavior data is currently a bottleneck for healthcare systems to estimate various model parameters described in here. However, online scheduling systems can easily collect data and compute these model parameters. This should further improve the performance of the model. Finally, appointments of different length can be considered by assuming different patient classes and patients needing these appointments can be constrained so that they will not be offered other appointments. Note that we do not consider the uncertainty of the length of appointment. Future work could include this feature to further improve the patient experience. The aforementioned refinements to the model impose significant efforts in data collection and relaxation of assumptions; however, it would be favorable for the model to be implemented in healthcare sites to witness its efficacy in practical situations.

Overall, the framework presented here is a significant improvement in the existing development of appointment scheduling system research. We proposed a more efficient, profitable system which maximizes patient access that could be beneficial to healthcare providers and patients. This model may serve as framework for future research that account for the heterogeneity/complexity in preferences, need, and availability.

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Appendix A: List of Notation

Parameter	Description
k	Index corresponding to physicians
l	Index corresponding to location
d	Index corresponding to day of appointment
t	Index corresponding to time slot of appointment
c	Index corresponding to class of patient
s	Index corresponding to an appointment (k, d, t, l)
K	Number of physicians
L	Number of clinic locations
D	Number of days in scheduling horizon
T	Number of slots in a given day
$\mathbb{K}$	Set of physicians
$\mathbb{L}$	Set of locations
$\mathbb{C}$	Set of patient classes
$\mathbb{D}$	Set of days in scheduling horizon
$\mathbb{T}$	Set of slots in a day
$\mathbb{S}$	Set of all appointments
$\lambda_{lk}^{\mathfrak{D}}$	Arrival rate of dedicated requests for location l and physician k
$\lambda_{l}^{\mathfrak{U}}$	Arrival rate of urgent requests
$\lambda^{\mathfrak{F}}$	Arrival rate of flexible requests
$\mathscr{S}$	Subset of appointments offered to patient requesting appointment
$\mathscr{S}^c$	Subset of appointments offered to patient of class c
v_s^c	Utility /preference weight of the appointment s for patient c
$P_s(\mathscr{S}^c)$	Probability that an appointment s is selected if a subset $\mathscr{S}^c$ is offered
$N(\mathscr{S}^c)$	Probability that a patient leaves the system without making an appointment
δ_{id}^c	Probability with which patient of class c retain an appointment booked i days ago for d days into the future from current day
r^c	Probability with which patient of class c will show up for an appointment scheduled for current day and booked i days prior
$Z(\mathfrak{D})$	Appointment schedule at the beginning of the day $\mathfrak{D}$
$Z_{i,s,c}(\mathfrak{D})$	Number of appointments requests of class c arriving on i days ago for an appointment s for d days into the future
$h^{\pi}(Z^{\pi}(\mathfrak{D}), \mathscr{S}^c)$	Probability with which a subset $\mathscr{S}^c$ of appointments are offered to the patients with class c requests for an appointment on day $\mathfrak{D}$ when the state of the system is $Z(\mathfrak{D})$ under policy π
$\mathcal{H}$	Set of the constraints of MDP
R	Revenue generated from total completed number of appointments under the single period policy
C_o	Overtime cost under the single period policy
C_i	Idle time cost under the single period policy
$\mathscr{C}_{kl}$	Number of appointments a provider k can complete in regular time at a clinic location l on any given day
γ_{kl}	Probability with which a physician k retains any scheduled appointment at a location l
β_c	Probability with which a dedicated patient will not reschedule a canceled appointment
Y_{ktcli}^{π}	The number of appointment requests booked with provider k at location l for day $\mathfrak{d}$ and slot t on day $\mathfrak{d} - i$ and showed up
θ_o	Overtime cost
θ_i	Idle time cost
$x_s(c)$	Probability with which a patient of class c schedules an appointment s
$u_l(c)$	Probability which which a patient of class c leaves a location without making an appointment
$\tilde{R}$	Revenue generated from total completed number of appointments given deterministic patient demand using single period policy for the reformulated problem (9)

Parameter	Description
$\tilde{C}_o$	Overtime cost patient demand using single period policy for the reformulated problem (9)
$\tilde{C}_i$	Idle time cost patient demand using single period policy for the reformulated problem (9)
$\breve{C}_o$	Overtime cost given deterministic patient demand using single period policy (17)
$\breve{C}_i$	Idle time cost given deterministic patient demand using single period policy (17)
V^*	Expected net reward generated by the optimal policy which solve the MDP (7)
Z^*	Optimal objective value of the single period deterministic problem (17)
Π	Optimal objective value of the single period stochastic problem (9)
L_π	Net reward generated over next $\|D\|+1$ days under the dynamic policy π
L	Net reward generated over next $\|D\|+1$ days under the single period policy
q	Probability with which the approximate dynamic programming policy allocates appointment requests on day $\mathfrak{d}$
$\hat{R}$	Revenue generated from total completed number of appointments given doubly stochastic patient demand (27)
$\hat{C}_o$	Overtime cost of scheduled appointments given doubly stochastic patient demand (27)
$\hat{C}_i$	Idle time cost of scheduled appointments given doubly stochastic patient demand (27)
ζ_i	Probability mass function of the stochastic arrival rate of patient demand
Ω	Set of capacity constraints
$\check{C}_o$	Overtime cost of scheduled appointments in the joint appointment schedule and capacity planning problem (21)
$\check{C}_o$	Idle time cost of scheduled appointments in the joint appointment schedule and capacity planning problem (21)
$\check{\mathscr{C}}_{kli}$	Binary variable used to compute the capacity of physician k at location l in the joint appointment schedule and capacity planning problem (23)

Appendix B: Robustness with Stochastic Intensity

In this extension we assume that the demand follows a doubly stochastic Poisson process, where the mean arrival rate is a random variable following a general probability distribution with finite support such that Λ^c has a probability mass function $\zeta_i := \mathbb{P}(\Lambda^c = \lambda_i^c),\ \forall i \in 1,...,n$.

The single period scheduling problem can be written as:

$$\max_{h \in \mathcal{H}} \quad \mathbb{E}\left[\hat{R} - \hat{C}_o - \hat{C}_\imath\right] \tag{27}$$

where

$$\hat{R} = \mathbb{E}\left\{\text{Pois}\left(\sum_{c,s,\mathscr{S}^c} \Lambda^c \alpha_{kdl}^c \bar{\delta}_d^c P_s(\mathscr{S}^c) h(\mathscr{S}^c)\right)\right\},$$

$$\hat{C}_o = \theta_o \sum_{k,l} \mathbb{E}[\hat{\chi}_{k,l}^+], \quad \hat{C}_\imath = \theta_\imath \sum_{k,l} \mathbb{E}[\hat{\chi}_{k,l}^-],$$

with

$$\hat{\chi}_{k,l} = \text{Pois}\left(\sum_{c,d,t,\mathscr{S}^c} \Lambda^c \bar{\delta}_d^c P_s(\mathscr{S}^c) h(\mathscr{S}^c)\right) - \gamma_{kl} \mathscr{C}_{kl}.$$

Using conditional expectations we have

$$\mathbb{E}[\hat{R}] = \sum_{i=1}^n \zeta_i \mathbb{E}\left\{\text{Pois}\left(\sum_{c,s,\mathscr{S}^c} \lambda_i^c \alpha_{kdl}^c P_s(\mathscr{S}^c) h(\mathscr{S}^c)\right)\right\} = \sum_{c,s,\mathscr{S}^c} \alpha_{kdl}^c P_s(\mathscr{S}^c) h(\mathscr{S}^c) \mathbb{E}[\Lambda^c].$$

Similarly we obtain

$$\mathbb{E}\big[\hat{C}_o\big] = \sum_{i=1}^{n} \zeta_i \mathbb{E}\left\{\left[\text{Pois}\left(\sum_{c,d,t,\mathscr{S}^c} \lambda_i^c P_s(\mathscr{S}^c) h(\mathscr{S}^c)\right)\right]^+\right\}.$$

It is evident that the objective function (27) is differentiable and convex by following a similar logic shown in Lemma 1. Therefore, we can obtain an optimal solution to the single period problem with arrival rate following a doubly stochastic Poisson arrival when arrival intensity has a finite support. Similarly we can also rewrite the objective (19) of the approximate dynamic policy as $\mathbb{E}[G_{c,s}(z,q,h,\Lambda^c)] = \sum_{i=1}^{n} G_{c,s}(z,q,h,\lambda_i^c)\zeta_i$. We can derive results for this extension by following the similar approach in the previous sections.

Appendix C: Proofs of Technical Results

Theorem 1 *$h^* = \{h^*(\mathscr{S}^c) : \mathscr{S}^c \subset \mathbb{S}, c \in \mathbb{C}\}$ is an optimal solution to the problem* (8)*, if and only if $(x_s^*(c), u_l^*(c))$ is a feasible solution to the problem* (9)–(15)*, where*

$$x_s^*(c) = \sum_{\mathscr{S}^c} P_s(\mathscr{S}^c) h^*(\mathscr{S}^c), \quad \textit{and} \quad u_l^*(c) = \sum_{\mathscr{S}^c} N(\mathscr{S}^c) h^*(\mathscr{S}^c), \quad \forall c \in \mathbb{C}.$$

First, given that $h^* = \{h^*(\mathscr{S}^c)\}$ is an optimal solution to the original problem (8), we show that (x^*, u^*) is a feasible and optimal solution to the reformulation (9)–(15). If we substitute the value of (x^*, u^*) in (10), then we have for $c = \mathfrak{D}_k$, $\forall k \in \mathbb{K}$,

$$\sum_{d,t} x_{kdtl}(c) + u_l(c) = \sum_{\mathscr{S}^c} h(\mathscr{S}^c)\left[\sum_s P_s(\mathscr{S}^c) + N(\mathscr{S}^c)\right] = \sum_{\mathscr{S}^c} h(\mathscr{S}^c) = 1, \tag{28}$$

where the second equality follows from the definition of choice probability (1a)–(2b). Therefore, (x^*, u^*) satisfies (10) in the single period case. Similarly, we can show that (x^*, u^*) satisfies (11) by substituting the value in (11) for $c \in \{\mathfrak{F}\}$. (x^*, u^*) satisfies (12) for $c = \mathfrak{U}$, we can see that by substituting the value of (x^*, u^*) in (12),

$$\sum_{k,d,t} x_{kdtl}(c) = \sum_{\mathscr{S}^c}\left(\sum_{k,d,t} P_{kdtl}(\mathscr{S}^c)\right) h(\mathscr{S}^c) = \sum_{\mathscr{S}^c} h(\mathscr{S}^c) = 1. \tag{29}$$

Similarly using the definition of $u_l^*(c)$ we have for $c = \mathfrak{U}$, $u_l(c) = \sum_{\mathscr{S}^c} N(\mathscr{S}^c) h(\mathscr{S}^c) = 0$, which satisfies (13). Now we use indicator function $\mathbb{1}(\cdot)$ to show that (x^*, u^*) satisfies (14). Using the definition of (x^*, u^*) we have for $c \in \{\mathfrak{D}, \mathfrak{F}\}$,

$$\frac{x_s^*(c)}{v_s^c} = \sum_{\mathscr{S}^c} \frac{\mathbb{1}(s \in \mathscr{S}^c) h^*(\mathscr{S}^c)}{1 + \sum_{s' \in \mathscr{S}^c} v_{s'}^c} \le \sum_{\mathscr{S}^c} \frac{h^*(\mathscr{S}^c)}{1 + \sum_{s' \in \mathscr{S}^c} v_{s'}^c} = \sum_{\mathscr{S}^c} N(\mathscr{S}^c) h^*(\mathscr{S}^c) = u_l^*(c). \tag{30}$$

Finally, by substituting the value of (x^*, u^*) in the objective function of the reformulation single period scheduling problem (9), we can see that it has the same value as the objective of the original single period objective function (8).

Now we show the second part of the proof by assuming that (x^*, u^*) is an optimal solution to the problem (9)–(15). Before that, we construct an optimal solution h^* to the original single period problem using (16). We reorder the appointments such that $x_{s_1}^*(c)/v_{s_1}^c \ge \cdots \ge x_{s_n}^*(c)/v_{s_n}^c$, where n is the maximum number of available appointments. Let the appointment request by patient $s_j = (k,d,t,l)$ be the j^{th} appointment in the above sequence. Now, we can define $\mathscr{S}_j^c$ with the reordered and redefined sequence as $\{s_1,...,s_j\}$. Therefore, we can write

$$\begin{aligned}\sum_{\mathscr{S}^c\subset\mathbb{S}} P_{s_j}(\mathscr{S}^c)h^*(\mathscr{S}^c) = \sum_{i=1}^{n} P_{s_j}(\mathscr{S}_i^c)h^*(\mathscr{S}_i^c) &= \sum_{i=j}^{n} P_{s_j}(\mathscr{S}_i^c)h^*(\mathscr{S}_i^c)\\ &= \sum_{i=j}^{n} v_{s_j}^c\left[\frac{x_{s_i}^*(c)}{v_{s_i}^c} - \frac{x_{s_{i+1}}^*(c)}{v_{s_{i+1}}^c}\right] = x_s^*(c). \end{aligned} \tag{31}$$

In above relationship, the first equality follows from the fact the ϕ (in the case of dedicated and flexible requests) and $\mathscr{S}_i^c$, $i=1,...,n$, is the list of all available appointments and h^* is zero for any other appointment by design. The second equality follows from the fact that $s_j\in\mathscr{S}_i^c$ only if $i\geq j$ and the third equality follows from the definition of choice probability (1a)–(1b) and h^* in (16). The final equality follows from noting that $x_{n+1}^*(c)=0$ and s_j is the appointment requested by the patient s.

Now we show that h^* is a feasible solution the problem by substituting the value in the single period constraint of (3) as below

$$\begin{aligned}\sum_{\mathscr{S}^c\subset\mathbb{S}} h^*(\mathscr{S}^c) &= \begin{cases} u_l^*(c) - \frac{x_{s_1}^*(c)}{v_{s_1}^c} + \sum_{i=1}^n\left[1+\sum_{s\in\mathscr{S}_i^c} v_{s_i}^c\right]\left[\frac{x_{s_i}^*(c)}{v_{s_i}^c} - \frac{x_{s_{i+1}}^*(c)}{v_{s_{i+1}}^c}\right], c\in\mathfrak{D}_k, \forall k\in\mathbb{K}, \forall l\in\mathbb{L}\\ \sum_l u_l^*(c) - \frac{x_{s_1}^*(c)}{v_{s_1}^c} + \sum_{i=1}^n\left[1+\sum_{s\in\mathscr{S}_i^c} v_{s_i}^c\right]\left[\frac{x_{s_i}^*(c)}{v_{s_i}^c} - \frac{x_{s_{i+1}}^*(c)}{v_{s_{i+1}}^c}\right], c\in\mathfrak{F}\\ \sum_{i=1}^n\left[\sum_{s\in\mathscr{S}_i^c} v_{s_i}^c\right]\left[\frac{x_{s_i}^*(c)}{v_{s_i}^c} - \frac{x_{s_{i+1}}^*(c)}{v_{s_{i+1}}^c}\right], c=\mathfrak{U}\end{cases}\\ &= \begin{cases} u_l^*(c)+\sum_{i=1}^n x_{s_i} = u^*(c)+\sum_s x_s(c) = 1, & c\in\mathfrak{D}_k, \forall k\in\mathbb{K}, l\in\mathbb{L}\\ \sum_l u_l^*(c)+\sum_{i=1}^n x_{s_i} = u^*(c)+\sum_s x_s(c) = 1, & c\in\mathfrak{F}\\ \sum_{k,d=0,t} x_{k,d,t} = 1, & c=\mathfrak{U}.\end{cases}\end{aligned} \tag{32}$$

The second equality follows by expanding and simplifying the summation expression. The third equality follows by the assumption that (x^*,u^*) is a feasible solution to the reformulation (9)–(15). Finally, by substituting the value of h^* in the objective function (8), we can see that it has the same value as we obtain from (9) at the optimal solution (x^*,u^*). This completes the proof. □

Lemma 1 $\theta_o\mathbb{E}[\tilde{\chi}_{k,l}^+]+\theta_\imath\mathbb{E}[\tilde{\chi}_{k,l}^-]$ *is convex and differentiable with respect to* $x_s(c)$.

Let $F(\psi)=\mathbb{E}\{\theta_o\,[\mathrm{Pois}(\psi)-\mathscr{C}]^+ + \theta_\imath\,[\mathrm{Pois}(\psi)-\mathscr{C}]^-\}$. By using the probability mass function of the Poisson distribution, we can write

$$\begin{aligned} F(\psi) &= \theta_o\sum_{\lfloor\mathscr{C}\rfloor+1}^{\infty}\frac{e^{-\psi}\psi^i}{i!}(i-\mathscr{C}) + \theta_\imath\sum_{i=0}^{\lceil\mathscr{C}\rceil-1}\frac{e^{-\psi}\psi^i}{i!}(\mathscr{C}-i)\\ &= \theta_o\left(\sum_{\lfloor\mathscr{C}\rfloor+1}^{\infty}\frac{e^{-\psi}\psi^i}{(i-1)!} - \sum_{\lfloor\mathscr{C}\rfloor+1}^{\infty}\frac{e^{-\psi}\psi^i}{i!}\mathscr{C}\right) + \theta_\imath\left(\sum_{i=0}^{\lceil\mathscr{C}\rceil-1}\frac{e^{-\psi}\psi^i}{i!}\mathscr{C} - \sum_{i=0}^{\lceil\mathscr{C}\rceil-1}\frac{e^{-\psi}\psi^i}{(i-1)!}\right)\\ &= \theta_o\left(\psi\sum_{\lfloor\mathscr{C}\rfloor}^{\infty}\frac{e^{-\psi}\psi^i}{i!} - \mathscr{C}\sum_{\lfloor\mathscr{C}\rfloor+1}^{\infty}\frac{e^{-\psi}\psi^i}{i!}\right) + \theta_\imath\left(\sum_{i=0}^{\lceil\mathscr{C}\rceil-1}\frac{e^{-\psi}\psi^i}{i!} - \psi\sum_{i=0}^{\lceil\mathscr{C}\rceil-2}\frac{e^{-\psi}\psi^i}{i!}\right)\\ &= \theta_o\left(\psi\mathbb{P}\{\mathrm{Pois}(\psi)\geq\mathscr{C}\} - \mathscr{C}\mathbb{P}\{\mathrm{Pois}(\psi)\geq\mathscr{C}+1\}\right)\\ &\qquad + \theta_\imath\left(\mathscr{C}\mathbb{P}\{\mathrm{Pois}(\psi)\leq\mathscr{C}\text{-}1\} - \psi\mathbb{P}\{\mathrm{Pois}(\psi)\leq\mathscr{C}\text{-}2\}\right). \end{aligned} \tag{33}$$

The differentiability of the above expression follows by the differentiability of Poisson cumulative distribution function (cdf) with respect to its mean. We can calculate the derivative of the Poisson cdf as

$$\begin{aligned}\frac{d\mathbb{P}\left\{\text{Pois}(\psi) \geq \mathscr{C}\right\}}{d\psi} &= -\frac{d\mathbb{P}\left\{\text{Pois}(\psi) \leq \mathscr{C}-1\right\}}{d\psi} = -\sum_{i=0}^{\lfloor \mathscr{C} \rfloor -1} \frac{d(\frac{e^{-\psi}\psi^i}{i!})}{d\psi} \\ &= \sum_{i=0}^{\lfloor \mathscr{C} \rfloor -1} \frac{e^{-\psi}\psi^i}{i!} - \sum_{i=1}^{\lfloor \mathscr{C} \rfloor -1} \frac{e^{-\psi}\psi^{(i-1)}}{(i-1)!} = \mathbb{P}\{\text{Pois}(\psi) = \mathscr{C}-1\}. \qquad (34)\end{aligned}$$

By differentiating both sides of (33) and substituting the differentiation of poisson cdf from (34), we obtain

$$\begin{aligned}\frac{dF(\psi)}{d\psi} &= \theta_o\left(\mathbb{P}\left\{\text{Pois}(\psi \geq \mathscr{C})\right\} + \psi\mathbb{P}\{\text{Pois}(\psi) = \mathscr{C}-1\} - \mathscr{C}\mathbb{P}\{\text{Pois}(\psi) = \mathscr{C}\}\right) \\ &\quad - \theta_\imath\left(\mathscr{C}\mathbb{P}\left\{\text{Pois}(\psi) = \mathscr{C}\text{-}1\right\} + \mathbb{P}\left\{\text{Pois}(\psi) \leq \mathscr{C}\text{-}2\right\} - \psi\mathbb{P}\left\{\text{Pois}(\psi) = \mathscr{C}\text{-}2\right\}\right) \\ &= \theta_o\left(\mathbb{P}\{\text{Pois}(\psi) \geq \mathscr{C}\} + \psi\frac{e^{-\psi}\psi^{\mathscr{C}-1}}{(\mathscr{C}-1)!} - \mathscr{C}\frac{e^{-\psi}\psi^{\mathscr{C}-1}}{\mathscr{C}!}\right) \\ &\quad - \theta_\imath\left(\mathbb{P}\{\text{Pois}(\psi) \leq \mathscr{C}-2\} + \mathscr{C}\frac{e^{-\psi}\psi^{\mathscr{C}-1}}{(\mathscr{C}-1)!} - \psi\frac{e^{-\psi}\psi^{\mathscr{C}-2}}{(\mathscr{C}-2)!}\right) \\ &= \theta_o\mathbb{P}\{\text{Pois}(\psi) \geq \mathscr{C}\} - \theta_\imath\mathbb{P}\{\text{Pois}(\psi) \leq \mathscr{C}-1\}. \qquad (35)\end{aligned}$$

To determine the convexity of $F(\psi)$, we can calculate the second order derivate of $F(\psi)$

$$\frac{d^2F(\psi)}{d\psi} = \frac{d(\mathbb{P}\{\text{Pois}(\psi) \geq \mathscr{C}\} - \mathbb{P}\{\text{Pois}(\psi) \leq \mathscr{C}-1\})}{d\psi} = \theta_o\mathbb{P}\{\text{Pois}(\psi) = \mathscr{C}-1\} + \theta_\imath\mathbb{P}\{\text{Pois}(\psi) = \mathscr{C}-1\}. \quad (36)$$

Since Poisson probability mass function is always positive, we can conclude that the function $F(\psi)$ and therefore, $\mathbb{E}\left\{\left|\text{Pois}\left(\sum_{c,d,t}\lambda^c\bar{\delta}_d^c x_{kdt}(c)\right) - \mathscr{C}\right|\right\}$ is convex and differentiable. □

Lemma 2 *Let Z^* be the optimal objective of the problem* (17). *Then $Z^* \geq V^*$ holds.*

Given that we use the optimal policy, let $\pi^*(\mathscr{S}^c)$ be the steady state probabilities with which we offer a subset $\mathscr{S}$ of appointments to patients with class c requests. Therefore, we represent the total number of patients who requested an appointment s and were scheduled under the optimal policy by a Poisson random variable A_s^* with mean $\sum_{c,\mathscr{S}^c}\lambda^c P_s(\mathscr{S}^c)\pi^*(\mathscr{S}^c)$. Similarly, we let R_s^* be the total number of patients who requested an appointment s and were scheduled under the optimal policy and were retained until the day of the appointment. Finally, we denote the total number of patients who requested an appointment s and were scheduled under the optimal policy and showed up as a random variable S_s^*. Furthermore, the show-up and retaining probabilities are independent of the number of patients scheduled. Therefore, we have $\mathbb{E}\{S_s^*\} = \alpha_{kl}^c\bar{\delta}_d^c\mathbb{E}\{A_s^*\}$ and $\mathbb{E}\{R_s^*\} = \bar{\delta}_d^c\mathbb{E}\{A_s^*\}$. We can calculate the expected profit per day using the optimal state-dependent policy as

$$\begin{aligned}V^* &= \mathbb{E}\left\{\textstyle\sum_s S_s^*\right\} - \textstyle\sum_{k,l}\mathbb{E}\left\{\theta_o\left[\textstyle\sum_{d,t}R_{kdtl}^* - \gamma_{kl}\mathscr{C}_{kl}\right]^+ + \theta_\imath\left[\textstyle\sum_{d,t}R_{kdtl}^* - \gamma_{kl}\mathscr{C}_{kl}\right]^-\right\} \\ &\leq \textstyle\sum_s\mathbb{E}\{S_s^*\} - \textstyle\sum_{k,l}\left(\theta_o\left[\textstyle\sum_{d,t}\mathbb{E}\{R_{kdtl}^*\} - \gamma_{kl}\mathscr{C}_{kl}\right]^+ + \theta_\imath\left[\textstyle\sum_{d,t}\mathbb{E}\{R_{kdtl}^*\} - \gamma_{kl}\mathscr{C}_{kl}\right]^-\right) \\ &= \sum_{c,s,\mathscr{S}}\lambda^c\alpha_{kl}^c\bar{\delta}_d^c P_s(\mathscr{S}^c)\pi^*(\mathscr{S}^c) - \sum_{k,l}\left\{\theta_o\left[\sum_{c,d,t,\mathscr{S}}\lambda^c\bar{\delta}_d^c P_{kdt}(\mathscr{S}^c)\pi^*(\mathscr{S}^c) - \gamma_{kl}\mathscr{C}_{kl}\right]^+\right.\end{aligned}$$

$$+\theta_\imath \left[\sum_{c,d,t,\mathscr{S}} \lambda^c \bar{\delta}^c_d P_{kdt}(\mathscr{S}^c)\pi^*(\mathscr{S}^c) - \gamma_{kl}\mathscr{C}_{kl} \right]^- \Bigg\} \le Z^*. \tag{37}$$

Here, the first inequality is by the Jensen's inequality. The second equality is by the definition of $\mathbb{E}\{S^*_s\}$ and $\mathbb{E}\{R^*_s\}$. For the second inequality, we note that $\pi^*(\mathscr{S}^c)$ is a feasible but not necessarily optimal solution to the problem

$$\begin{aligned}
\max \sum_{c,s,\mathscr{S}^c} \lambda^c \alpha^c_{kl}\bar{\delta}^c_d P_s(\mathscr{S}^c)\omega(\mathscr{S}^c) - \sum_{k,l} \Bigg\{ \theta_o \left[\sum_{c,d,t,\mathscr{S}^c} \lambda^c \bar{\delta}^c_d P_{kdt}(\mathscr{S}^c)\omega(\mathscr{S}^c) - \gamma_{kl}\mathscr{C}_{kl} \right]^+ \\
+\theta_\imath \left[\sum_{c,d,t,\mathscr{S}^c} \lambda^c \bar{\delta}^c_d P_{kdt}(\mathscr{S}^c)\omega(\mathscr{S}^c) - \gamma_{kl}\mathscr{C}_{kl} \right]^- \Bigg\}
\end{aligned}$$

$$\begin{aligned}
\text{s.t.} \qquad & \sum_{\mathscr{S}^c} \omega(\mathscr{S}^c) = 1 \qquad && \forall c \in \mathbb{C} \\
& \omega(\mathscr{S}^c) \ge 0 && \forall c \in \mathbb{C}.
\end{aligned}$$

By using the arguments in proof of Theorem 1, the optimal objective values of this problem and problem (17) are equal. This completes the proof. □

Proposition 1 *If* Π^* *is optimal objective to the problem* (8),

$$\frac{\Pi^*}{V^*} \ge 1 - \frac{\theta_o + \theta_\imath}{\sqrt{2\pi}} \max\left(\frac{\sum_{kl}(\sum_{c,d,t}\lambda^c\bar{\delta}^c_d v^c_s/\nu^c)/\sqrt{\gamma_{kl}\mathscr{C}_{kl}-1}}{\sum_{c,s}\lambda^c\bar{\delta}^c_d(\alpha^c_{kdl}+\theta_\imath)v^c_s/\nu^c - \theta_\imath\sum_{kl}\gamma_{kl}\mathscr{C}_{kl}}, \frac{1}{\alpha^*_{kdl}\sqrt{(\gamma_{kl}\mathscr{C}_{kl})^*-1}} \right) \tag{38}$$

Let $(\hat{x},\hat{u})$ be an optimal solution to the problem (17). Then we have $\Pi(x^*,u^*) \ge \Pi(\hat{x},\hat{u})$ because $(\hat{x},\hat{u})$ is a feasible but not necessarily optimal solution to the problem (9)–(15). Now, using Lemma 2 we have $\Pi(x^*,u^*)/V^* \ge \Pi(\hat{x},\hat{u})/V^* \ge \Pi(\hat{x},\hat{u})/Z^*$. Therefore, it is enough to show that the expression on the right hand side of (38), provides a lower bound on $\Pi(\hat{x},\hat{u})/Z^*$. For notation brevity we let $\phi = \sum_{c,s}\lambda^c\alpha^c_{kl}\bar{\delta}^c_d\hat{x}_s(c) - \sum_{c,l}\lambda^c u_l(c)$ and $\psi_{kl} = \sum_{c,d,t}\lambda^c\bar{\delta}^c_d\hat{x}_{kdtl}(c)$. We derive an upper bound on the expectation $\mathbb{E}\{\theta_o\,[\mathrm{Pois}(\psi_{kl}) - \gamma_{kl}\mathscr{C}_{kl}]^+ + \theta_\imath\,[\mathrm{Pois}(\psi_{kl}) - \gamma_{kl}\mathscr{C}_{kl}]^-\}$ as below:

$$\begin{aligned}
&\theta_o \sum_{j=\lfloor\gamma_{kl}\mathscr{C}_{kl}\rfloor+1}^{\infty} [j-\gamma_{kl}\mathscr{C}_{kl}]^+ \frac{e^{-\psi_{kl}}\psi^j_{kl}}{j!} + \theta_\imath \sum_{j=0}^{\lceil\gamma_{kl}\mathscr{C}_{kl}\rceil-1} [j-\gamma_{kl}\mathscr{C}_{kl}]^- \frac{e^{-\psi_{kl}}\psi^j_{kl}}{j!} \\
&\le \theta_o\,[\psi_{kl}-\gamma_{kl}\mathscr{C}_{kl}]^+ + \theta_o \sum_{j=\lfloor\gamma_{kl}\mathscr{C}_{kl}\rfloor+1}^{\infty} \left[[j-\gamma_{kl}\mathscr{C}_{kl}]^+ - [\psi_{kl}-\gamma_{kl}\mathscr{C}_{kl}]^+ \right] \frac{e^{-\psi_{kl}}\psi^j_{kl}}{j!} \\
&\quad + \theta_\imath\,[\psi_{kl}-\gamma_{kl}\mathscr{C}_{kl}]^- + \theta_\imath \sum_{j=0}^{\lceil\gamma_{kl}\mathscr{C}_{kl}\rceil-1} \left[[j-\gamma_{kl}\mathscr{C}_{kl}]^- - [\psi_{kl}-\gamma_{kl}\mathscr{C}_{kl}]^- \right] \frac{e^{-\psi_{kl}}\psi^j_{kl}}{j!} \\
&\le \theta_o\,[\psi_{kl}-\gamma_{kl}\mathscr{C}_{kl}]^+ + \theta_o \sum_{j=\lfloor\gamma_{kl}\mathscr{C}_{kl}\rfloor+1}^{\infty} [j-\psi_{kl}] \frac{e^{-\psi_{kl}}\psi^j_{kl}}{j!} \\
&\quad + \theta_\imath\,[\psi_{kl}-\gamma_{kl}\mathscr{C}_{kl}]^- - \theta_\imath \sum_{j=0}^{\lceil\gamma_{kl}\mathscr{C}_{kl}\rceil-1} [j-\psi_{kl}] \frac{e^{-\psi_{kl}}\psi^j_{kl}}{j!} \\
&= \theta_o\,[\psi_{kl}-\gamma_{kl}\mathscr{C}_{kl}]^+ + \theta_o\psi_{kl} \frac{e^{-\psi_{kl}}\psi^{\lfloor\gamma_{kl}\mathscr{C}_{kl}\rfloor}_{kl}}{\Gamma(\lfloor\gamma_{kl}\mathscr{C}_{kl}\rfloor+1)} + \theta_\imath\,[\psi_{kl}-\gamma_{kl}\mathscr{C}_{kl}]^- + \theta_\imath\psi_{kl} \frac{e^{-\psi_{kl}}\psi^{\lceil\gamma_{kl}\mathscr{C}_{kl}\rceil-1}_{kl}}{\Gamma(\lceil\gamma_{kl}\mathscr{C}_{kl}\rceil)} \\
&\le \theta_o\,[\psi_{kl}-\gamma_{kl}\mathscr{C}_{kl}]^+ + \theta_o\psi_{kl} \frac{e^{-\lfloor\gamma_{kl}\mathscr{C}_{kl}\rfloor}(\lfloor\gamma_{kl}\mathscr{C}_{kl}\rfloor)^{\lfloor\gamma_{kl}\mathscr{C}_{kl}\rfloor}}{\Gamma(\lfloor\gamma_{kl}\mathscr{C}_{kl}\rfloor+1)} + \theta_\imath\,[\psi_{kl}-\gamma_{kl}\mathscr{C}_{kl}]^-
\end{aligned}$$

$$
\begin{aligned}
&+\theta_\imath \psi_{kl} \frac{e^{-\lceil\gamma_{kl}\mathscr{C}_{kl}\rceil}(\lceil\gamma_{kl}\mathscr{C}_{kl}\rceil)^{\lceil\gamma_{kl}\mathscr{C}_{kl}\rceil-1}}{\Gamma(\lceil\gamma_{kl}\mathscr{C}_{kl}\rceil)} \\
&\le \theta_o\left[\psi_{kl}-\gamma_{kl}\mathscr{C}_{kl}\right]^+ + \theta_\imath\left[\psi_{kl}-\gamma_{kl}\mathscr{C}_{kl}\right]^- + \theta_o\psi_{kl}\frac{e^{-\lfloor\gamma_{kl}\mathscr{C}_{kl}\rfloor}(\lfloor\gamma_{kl}\mathscr{C}_{kl}\rfloor)^{\lfloor\gamma_{kl}\mathscr{C}_{kl}\rfloor}}{\sqrt{2\pi\lfloor\gamma_{kl}\mathscr{C}_{kl}\rfloor}(\lfloor\gamma_{kl}\mathscr{C}_{kl}\rfloor/e)^{\lfloor\gamma_{kl}\mathscr{C}_{kl}\rfloor}} \\
&+\theta_\imath\psi_{kl}\frac{e^{-\lceil\gamma_{kl}\mathscr{C}_{kl}\rceil-1}(\lceil\gamma_{kl}\mathscr{C}_{kl}\rceil-1)^{\lceil\gamma_{kl}\mathscr{C}_{kl}\rceil-1}}{\sqrt{2\pi(\lceil\gamma_{kl}\mathscr{C}_{kl}\rceil-1)}(\lceil\gamma_{kl}\mathscr{C}_{kl}\rceil-1)^{(\lceil\gamma_{kl}\mathscr{C}_{kl}\rceil-1)}e^{-\lceil\gamma_{kl}\mathscr{C}_{kl}\rceil-1}} \\
&= \theta_o\left[\psi_{kl}-\gamma_{kl}\mathscr{C}_{kl}\right]^+ + \theta_\imath\left[\psi_{kl}-\gamma_{kl}\mathscr{C}_{kl}\right]^- + \theta_o\frac{\psi_{kl}}{\sqrt{2\pi\lfloor\gamma_{kl}\mathscr{C}_{kl}\rfloor}} + \theta_\imath\frac{\psi_{kl}}{\sqrt{2\pi(\lceil\gamma_{kl}\mathscr{C}_{kl}\rceil-1)}} \\
&\le \theta_o\left[\psi_{kl}-\gamma_{kl}\mathscr{C}_{kl}\right]^+ + \theta_\imath\left[\psi_{kl}-\gamma_{kl}\mathscr{C}_{kl}\right]^- + \frac{\theta_o+\theta_\imath}{\sqrt{2\pi}}\frac{\psi_{kl}}{\sqrt{\gamma_{kl}\mathscr{C}_{kl}-1}}.
\end{aligned} \tag{39}
$$

The first equality follows by the probability mass function of Poisson distribution. The first inequality follows by adding and subtracting $[\psi_{kl}-\gamma_{kl}\mathscr{C}_{kl}]^+$ and $[\psi_{kl}-\gamma_{kl}\mathscr{C}_{kl}]^-$ on the left side of this inequality. The second inequality follows by noting two cases. First, for $j\ge\gamma_{kl}\mathscr{C}_{kl}$, $\left[[j-\gamma_{kl}\mathscr{C}_{kl}]^+ - [\psi_{kl}-\gamma_{kl}\mathscr{C}_{kl}]^+\right] \le j-\psi_{kl}$. We can show this by using conditions $j,\psi_{kl}\ge\lfloor\gamma_{kl}\mathscr{C}_{kl}\rfloor+1$ and $j\ge\lfloor\gamma_{kl}\mathscr{C}_{kl}\rfloor+1\ge\psi_{kl}$. Similarly, we can show that for $j\le\gamma_{kl}\mathscr{C}_{kl}$, $\left[[j-\gamma_{kl}\mathscr{C}_{kl}]^- - [\psi_{kl}-\gamma_{kl}\mathscr{C}_{kl}]^-\right]\le\psi_{kl}-j$. The second inequality follows by arranging the terms in the summation on the left side of the equality and the third inequality follows because the function $f(\psi_{kl})=e^{-\psi_{kl}}\psi_{kl}^{\gamma_{kl}\mathscr{C}_{kl}}$ attains its maximum at $\psi_{kl}=\gamma_{kl}\mathscr{C}_{kl}$. The fourth inequality follows by noting that $\Gamma(\lfloor\gamma_{kl}\mathscr{C}_{kl}\rfloor+1)\ge\sqrt{2\pi\lfloor\gamma_{kl}\mathscr{C}_{kl}\rfloor}(\lfloor\gamma_{kl}\mathscr{C}_{kl}\rfloor/e)^{\lfloor\gamma_{kl}\mathscr{C}_{kl}\rfloor}$ by Stirling's formula.

Now we construct a lower bound on Z^*. To that end, we find a feasible solution (x_s^c,u^c) to Z for two cases. For the first case $\sum_{c,d,t}\lambda^c\bar{\delta}_d^c v_s^c/\nu^c<\gamma_{kl}\mathscr{C}_{kl}$, we find a feasible solution by setting $x_s^c=v_s^c/\nu^c$ and $u^c=1/\nu^c$ as a feasible solution, where ν^c is $1+\sum_{s\in\mathscr{S}_j^c}v_s^c$ and $\sum_{s\in\mathscr{S}_j^c}v_s^c$ for $c\in\{\mathfrak{D},\mathfrak{F}\}$ and $c=\mathfrak{U}$, respectively. Using this, we can bound Z^* as $Z^*\ge\sum_{c,d,k,l,t}\lambda^c\alpha_{kl}^c\bar{\delta}_d^c v_s^c/\nu^c-\theta_\imath\sum_{kl}(\gamma_{kl}\mathscr{C}_{kl}-\sum_{c,d,t}\lambda^c\bar{\delta}_d^c v_s^c/\nu^c)$. Similarly for the second case $\sum_{c,d,t}\lambda^c\bar{\delta}_d^c v_s^c/\nu^c\ge\gamma_{kl}\mathscr{C}_{kl}$, we can bound Π/Z^* by the following:

$$
\begin{aligned}
&1-\frac{\begin{matrix}[\phi-\sum_{k,l}(\theta_o[\psi_{kl}-\gamma_{kl}\mathscr{C}_{kl}]^+ + \theta_\imath[\psi_{kl}-\gamma_{kl}\mathscr{C}_{kl}]^-)]- \\ [\phi-\sum_{k,l}\mathbb{E}\{\theta_o[\mathrm{Pois}(\psi_{kl})-\gamma_{kl}\mathscr{C}_{kl}]^+ + \theta_\imath[\mathrm{Pois}(\psi_{kl})-\gamma_{kl}\mathscr{C}_{kl}]^-\}]\end{matrix}}{\mathbb{Z}^*} \\
&\ge 1-\frac{\theta_o+\theta_\imath}{\sqrt{2\pi}}\frac{\sum_{k,l}\frac{\psi_{kl}}{\sqrt{\gamma_{kl}\mathscr{C}_{kl}-1}}}{Z^*} \\
&\ge 1-\frac{\theta_o+\theta_\imath}{\sqrt{2\pi}}\frac{\sum_{k,l}\frac{\psi_{kl}}{\sqrt{\gamma_{kl}\mathscr{C}_{kl}-1}}}{(\alpha_{kl}^*-\theta_o)\sum_{kl}\psi_{kl}+\theta_o\sum_{kl}\gamma_{kl}\mathscr{C}_{kl}} \\
&\ge 1-\frac{\theta_o+\theta_\imath}{\sqrt{2\pi}\sqrt{(\gamma_{kl}\mathscr{C}_{kl})^*-1}}\frac{1}{(\alpha_{kl}^*-\theta_o)+\theta_o\frac{\sum_{kl}\gamma_{kl}\mathscr{C}_{kl}}{\sum_{kl}\psi_{kl}}} \\
&\ge 1-\frac{1}{\sqrt{2\pi}}\frac{\theta_o+\theta_\imath}{\alpha_{kl}^*\sqrt{(\gamma_{kl}\mathscr{C}_{kl})^*-1}}
\end{aligned}
$$

where $\alpha_{kl}^*=\max\alpha_{kl}$ and $(\gamma_{kl}\mathscr{C}_{kl})^*=\max\gamma_{kl}\mathscr{C}_{kl}$. The first inequality follows from (39). The second inequality follows from substituting the value of Z^* and α_{kl}^*. The third and the fourth inequalities are from substituting the value of $(\gamma_{kl}\mathscr{C}_{kl})^*$ and from the case that $\psi_{kl}\ge\mathscr{C}_{kl}$. We can use both of these cases to generate lower bound on Π/Z^* as follows

$$
\begin{aligned}
&1-\frac{\theta_o+\theta_\imath}{\sqrt{2\pi}}\frac{\sum_{k,l}\frac{\psi_{kl}}{\sqrt{\gamma_{kl}\mathscr{C}_{kl}-1}}}{Z^*} \\
&\ge 1-\frac{\theta_o+\theta_\imath}{\sqrt{2\pi}}\max\left(\frac{\sum_{kl}(\sum_{c,d,t}\lambda^c\bar{\delta}_d^c v_s^c/\nu^c)/\sqrt{\gamma_{kl}\mathscr{C}_{kl}-1}}{\sum_{c,s}\lambda^c\bar{\delta}_d^c(\alpha_{kl}^c+\theta_\imath)v_s^c/\nu^c-\theta_\imath\sum_{kl}\gamma_{kl}\mathscr{C}_{kl}},\frac{1}{\alpha_{kl}^*\sqrt{(\gamma_{kl}\mathscr{C}_{kl})^*-1}}\right).
\end{aligned} \tag{40}
$$

This completes the proof. □

Proposition 2 *The continuous relaxation of the joint mixed integer nonlinear programming problem* (23) *is a convex problem.*

Since the revenue $\tilde{R}$ is linear in x and independent of $\underset{\frown}{\mathscr{C}}$, we only need to show that $\underset{\frown}{C}_o + \underset{\frown}{C}_\imath$ is convex in x and $\underset{\frown}{\mathscr{C}}$ assuming that $\underset{\frown}{\mathscr{C}} \in [0,1]$. For notational brevity we can write $\underset{\frown}{C}_o + \underset{\frown}{C}_\imath$ as

$$\begin{aligned} F(\psi, \underset{\frown}{\mathscr{C}}) = \theta_o & \left(\psi \left(1 - e^{-\psi} \sum_{i=0}^{n-1} \underset{\frown}{\mathscr{C}}_{i+1} \frac{\psi^i}{i!} \right) - \sum_{i=0}^{n-1} \underset{\frown}{\mathscr{C}}_{i+1} \left(1 - e^{-\psi} \sum_{i=0}^{n} \underset{\frown}{\mathscr{C}}_i \frac{\psi^i}{i!} \right) \right) \\ & + \theta_\imath \left(\sum_{i=0}^{n-1} \underset{\frown}{\mathscr{C}}_{i+1} \left(1 - e^{-\psi} \sum_{i=0}^{n-1} \underset{\frown}{\mathscr{C}}_{i+1} \frac{\psi^i}{i!} \right) - \psi \left(1 - e^{-\psi} \sum_{i=0}^{n-2} \underset{\frown}{\mathscr{C}}_{i+2} \frac{\psi^i}{i!} \right) \right). \end{aligned} \tag{41}$$

We can calculate the second derivative of the above function with respect to $(\psi, \underset{\frown}{\mathscr{C}}_j)$:

$$\begin{aligned} \frac{\partial^2 F(\psi, \underset{\frown}{\mathscr{C}}_j)}{\partial \psi^2} = e^{-\psi} & \left(\theta_\imath \sum_{i=0}^{n-1} \underset{\frown}{\mathscr{C}}_{i+1} \left(\sum_{i=0}^{n-1} \frac{(i-1) i \psi^{i-2} \underset{\frown}{\mathscr{C}}_{i+1}}{i!} \right) - 2\theta_\imath \sum_{i=0}^{n-1} \underset{\frown}{\mathscr{C}}_{i+1} \left(\sum_{i=0}^{n-1} \frac{i \psi^{i-1} \underset{\frown}{\mathscr{C}}_{i+1}}{i!} \right) \right. \\ & - \theta_\imath \psi \sum_{i=0}^{n-2} \frac{(i-1) i \psi^{i-2} \underset{\frown}{\mathscr{C}}_{i+2}}{i!} + 2\theta_\imath \psi \sum_{i=0}^{n-2} \frac{i \psi^{i-1} \underset{\frown}{\mathscr{C}}_{i+2}}{i!} - 2\theta_\imath \sum_{i=0}^{n-2} \frac{i \psi^{i-1} \underset{\frown}{\mathscr{C}}_{i+2}}{i!} \\ & + \theta_o \sum_{i=0}^{n-1} \underset{\frown}{\mathscr{C}}_{i+1} \left(\sum_{i=0}^{n} \frac{(i-1) i \psi^{i-2} \underset{\frown}{\mathscr{C}}_i}{i!} \right) - 2\theta_o \sum_{i=0}^{n-1} \underset{\frown}{\mathscr{C}}_{i+1} \left(\sum_{i=0}^{n} \frac{i \psi^{i-1} \underset{\frown}{\mathscr{C}}_i}{i!} \right) - \theta_o \psi \sum_{i=0}^{n-1} \frac{(i-1) i \psi^{i-2} \underset{\frown}{\mathscr{C}}_{i+1}}{i!} \\ & + 2\theta_o \psi \sum_{i=0}^{n-1} \frac{i \psi^{i-1} \underset{\frown}{\mathscr{C}}_{i+1}}{i!} - 2\theta_o \sum_{i=0}^{n-1} \frac{i \psi^{i-1} \underset{\frown}{\mathscr{C}}_{i+1}}{i!} + \theta_\imath \sum_{i=0}^{n-1} \underset{\frown}{\mathscr{C}}_{i+1} \left(\sum_{i=0}^{n-1} \frac{\psi^i \underset{\frown}{\mathscr{C}}_{i+1}}{i!} \right) - \theta_\imath (\psi - 2) \sum_{i=0}^{n-2} \frac{\psi^i \underset{\frown}{\mathscr{C}}_{i+2}}{i!} \\ & \left. + \theta_o \sum_{i=0}^{n-1} \underset{\frown}{\mathscr{C}}_{i+1} \left(\sum_{i=0}^{n} \frac{\psi^i \underset{\frown}{\mathscr{C}}_i}{i!} \right) - \theta_o \psi \sum_{i=0}^{n-1} \frac{\psi^i \underset{\frown}{\mathscr{C}}_{i+1}}{i!} + 2\theta_o \sum_{i=0}^{n-1} \frac{\psi^i \underset{\frown}{\mathscr{C}}_{i+1}}{i!} \right), \end{aligned} \tag{42}$$

$$\frac{\partial^2 F(\psi, \underset{\frown}{\mathscr{C}}_j)}{\partial \underset{\frown}{\mathscr{C}}_j \partial \underset{\frown}{\mathscr{C}}_l} = \frac{2 e^{-\psi} \psi^{n-1} (\theta_\imath n + \theta_o \psi)}{\Gamma(n+1)} \tag{43}$$

$$\begin{aligned} \frac{\partial^2 F(\psi, \underset{\frown}{\mathscr{C}}_j)}{\partial \underset{\frown}{\mathscr{C}}_j \partial \psi} = e^{-\psi} & \left(\theta_o \sum_{i=0}^{n} \frac{i \psi^{i-1} \underset{\frown}{\mathscr{C}}_i}{i!} - \theta_o \sum_{i=0}^{n} \frac{\psi^i \underset{\frown}{\mathscr{C}}_i}{i!} + \theta_\imath \left(\sum_{i=0}^{n-1} \frac{i \psi^{i-1} \underset{\frown}{\mathscr{C}}_{i+1}}{i!} - \sum_{i=0}^{n-1} \frac{\psi^i \underset{\frown}{\mathscr{C}}_{i+1}}{i!} \right) - \right. \\ & \left. \frac{\psi^{n-2} \left((\theta_\imath n(-n + \psi + 1) + \theta_o \psi(\psi - n)) \sum_{i=0}^{n-1} \underset{\frown}{\mathscr{C}}_{i+1} + \theta_\imath (n-1) n (n - \psi - 1) + \theta_o n \psi (n - \psi) \right)}{\Gamma(n+1)} \right). \end{aligned} \tag{44}$$

Since we can calculate the Cholskey decomposition of the Hessian matrix resulting from the second derivatives (42)-(44) corresponding to the function (41), we can conclude that the function (41)is convex. □