Non-stationary self-similar Gaussian processes as scaling limits of power-law shot noise processes and generalizations of fractional Brownian motion

GUODONG PANG† AND MURAD S. TAQQU*

Abstract. We study shot noise processes with Poisson arrivals and non-stationary noises. The noises are conditionally independent given the arrival times, but the distribution of each noise does depend on its arrival time. We establish scaling limits for such shot noise processes in two situations: 1) the conditional variance functions of the noises have a power law and 2) the conditional noise distributions are piecewise. In both cases, the limit processes are self-similar Gaussian with non-stationary increments. Motivated by these processes, we introduce new classes of self-similar Gaussian processes with non-stationary increments, via the time-domain integral representation, which are natural generalizations of fractional Brownian motions.

1. Introduction

Fractional Brownian motion (FBM) is a Gaussian process with dependent stationary increments. Its dependence structure is best described when viewed in the context of a generalized process. Under this perspective, the spectrum of FBM is a power function which scales in the same way at all frequencies. The exponent of the power function then characterizes both the high and low frequencies. FBM has been used to model data with long-range dependence (*low frequencies*), for example, *tree ring width*, which varies from year to year and has been identified as one of the natural stationary time series data sets which exhibit long-range dependence (see, e.g., [35, 43]). FBM has also been used in the context of *high frequency* to model the behavior of the logarithm of stock prices when measured at very high frequencies (see, e.g., [14, 16]), and to study rough stochastic volatility (see, e.g., [11, 12, 24]). In this paper, we study non-stationary shot noise processes characterized by a power law and obtain variants of FBM as their scaling limits. Such self-similar Gaussian processes with non-stationary increments may be useful to study high frequency data.

A shot noise process involves the cumulative effect of the arrivals of certain noises (shots) whose influence lingers for a while according to some response function. Non-stationary shot noise processes are used to model many stochastic systems, for example, see [1, 9, 17, 39, 40, 41, 59] and references therein. A useful model for non-stationary noises is to allow the distribution of the noises to depend on the arrival times of the shots. This model has been studied in [40] for infinite-server queues with service times dependent on the arrival times, and in [41] for general non-stationary shot noise processes. In [40, 41], when the arrival process has a Brownian limit, as in the case of a renewal process, the limit process is a non-stationary Gaussian process, which is not self-similar.

With i.i.d. noises, it is shown in [28] that when the shot shape function is regularly varying, an FBM arises as the scaling limit of the shot noise process in the conventional scaling regime (both time and space are scaled), capturing the long range dependence effect. In [26], a self-similar stationary Gaussian limit is also proved for explosive shot noise processes with a Poisson arrival

2000 *Mathematics Subject Classification.* 60F17; 60G15; 60G18.

Key words and phrases. Power-law shot noise process, non-stationary increments, self-similar Gaussian process, scaling limits, functional central limit theorem, fractional Brownian motion.

process. Also, in Chapter 3.4 of [44], in the same setting as [28], the integrated shot noise process is shown to converge weakly to an FBM.

In this paper, we study a non-stationary shot noise process, with a Poisson arrival process of shot noises, where the distribution of each noise depends on its arrival time, but where these noises are conditionally independent given their arrival times. The shot shape (response) function is regularly varying as in [28, 44]. The conditional variance function of the noises is time-varying (time-inhomogeneous). As noted in the paragraph after equation (1.4) in [29], in certain cases, it is actually the integrated shot noise process that is the object of greatest interest. That is the focus of the current paper. We consider two interesting cases: (i) the conditional variance function is regularly varying and (ii) the conditional variance function is piecewise constant. In the conventional scaling regime, under the proper scalings of the space, we show that the scaled shot noise processes converge to non-stationary Gaussian processes, which are self-similar and have continuous sample paths (see the functional central limit theorems, FCLTs, Theorems 3.1 and 4.1).

The proof of weak convergence uses a new sufficient convergence criterion (Theorem 6.1 in the Appendix) developed in Pang and Zhou [41], which extends the classical convergence criterion, Theorem 13.5 in Billingsley [5]. It relies on new maximal inequalities that relax the requirement of having a finite measure as the probability or moment upper bound for the increments of the processes (Theorems 10.3 and 10.4 in [5]). It allows the upper bound to be a set function with a superadditivity property (see Definition 6.1 and Theorems 5.1 and 5.2 in [41]). To verify that the convergence criterion applies, we compute the second and fourth moments of the increments of the scaled shot noise processes (Lemmas 3.1 and 3.3), verify that they induce a finite set function that satisfies the superadditivity property (Lemma 3.2), and thus prove the probability bound for the increment of the scaled shot noise processes (Lemma 3.4). To prove the existence of the process in the space $\mathbb{D}$, the classical existence criterion Theorem 13.6 in [5] is also extended in [41] (See Theorem 5.3 therein and Theorem 6.2 in the Appendix). We apply this existence criterion to establish the existence of the limit process in the space $\mathbb{D}$. The continuity of the sample paths follows from the continuity of the process in quadratic mean, as noted in Proposition 3.1.

The study of the power-law shot noise processes introduces new classes of self-similar Gaussian processes that have non-stationary increments. We provide time-domain representations of such new self-similar processes in Section 5. They are natural generalizations of FBM to the non-stationary setting. The existence of these processes and the continuity of their sample paths can be established in the same way as was done for the limit processes of the shot noise processes. In addition, we provide their spectral representations, which are also natural generalizations of those of FBM. Many other sample path and probabilistic properties of these processes remain to be studied, for example, possible extensions of continuity properties by Marcus and Shepp [36] for certain Gaussian processes with non-stationary increments.

1.1. **Literature review.** There is a vast literature on shot noise processes with i.i.d. noises, for example, the work in [6, 10, 15, 19, 26, 27, 28, 29, 32, 37, 48] for Poisson shot noise processes, and in [46, 57, 52, 33, 45] for more general shot noise processes, as well as on their applications. Our work is in line with the scaling limits of shot noise processes in the conventional asymptotic regime, which has focused on i.i.d. noises. The classical work by Klüppelberg and Mikosch [26] and [28] has established self-similar stationary Gaussian limits. A few papers also establish stable-motion limits. Klüppelberg et al. [27] proved an FCLT for Poisson shot noise processes which has an infinite-variance stable limit process. Iksanov [20] and Iksanov et al. [21] studied renewal shot noise processes, and proved FCLTs under various conditions on the shot shape function. Iksanov et al. [22, 23] recently studied renewal shot noise processes with immigration and proved scaling limits and convergence to stationarity. In [2], a class of shot-noise fields with a marked stationary Poisson process and a power-law response function is studied and α-stable random field limit is established.

This work also contributes to the study of shot noise processes with noises that are neither independent nor identically distributed. Here we assume that the noises are non-stationary and conditionally independent given an arrival time. There is a limited literature on such shot noise processes. In [33], the noises are modulated by a finite-state Markov chain and are conditionally independent with a distribution depending on the state of the chain at the arrival time of the shot noise. In [45], a cluster shot noise model is studied where the noises depend on the same 'cluster mark' within each cluster. In [42], the noises are assumed to be weakly dependent satisfying the ρ-mixing condition, and an FCLT is established in the high intensity asymptotic regime.

Studies on FBMs are extensive, and various extensions have been developed; see, e.g., [44] for a survey. It is important that the self-similarity property is preserved; however, stationary increments can be relaxed. One well-known example is the so-called bi-fractional Brownian motion [44]. Non-stationary increments may appear in other applications, for example, in the study of fractional-colored stochastic heat equations [55], a self-similar Gaussian process solution is obtained which does not have stationary increments. The class of semi-stationary Brownian processes, introduced in [3], is Gaussian with non-stationary increments in general, but can be made self-similar if the integrands are chosen properly.

Shot noise processes and FBMs are both extensively used in financial models. We have mentioned the papers [26, 28, 27]. Earlier studies include [47] and [7], and we also refer to the recent studies in financial models in [51] and limit order books in [25]. The shot noise processes can be used to model limit-order books for very- or ultra-high frequency data. On the other hand, FBM has been used in portfolio optimization with transaction costs [8, 50] and rough stochastic volatility [4]. It would be interesting to see how one can use the generalized FBM in these studies. It may be also worth investigating whether shot noise processes could be used to model market microstructures in a way that FBM arises naturally to model the dependence structure of stochastic volatility.

It is worth mentioning the applications of these processes in other contexts. Shot noise processes are also used to study a damped harmonic oscillator subject to a random force in physics [53]. In [49], the random forces are modeled as i.i.d. symmetric α-stable random variables and in [40] they are modeled with a conditional Gaussian distribution of mean zero and time-dependent covariance matrix. FBM is also often used to model internet traffic [30, 54, 38].

1.2. **Organization of the paper.** The paper is organized as follows. In the next subsection we introduce the notation. In Section 2, we describe the model in detail and present the assumptions we make on the scaling processes. In Section 3, we establish the scaling limits in the case of power-law conditional variance functions of noises. In Section 4, the case of piecewise constant conditional variance functions of noises is studied. We introduce new classes of non-stationary self-similar Gaussian processes, generalizing FBM, in Section 5. In the Appendix we state, for completeness, the definition of set functions satisfying the superadditivity property and present the associated sufficient weak convergence criterion and the existence theorem.

1.3. **Notation.** All random variables and processes are defined in a common complete probability space $(\Omega, \mathcal{F}, P)$. Throughout the paper, $\mathbb{N}$ denotes the set of natural numbers. $\mathbb{R}^k$ ($\mathbb{R}^k_+$) denotes the space of real-valued (nonnegative) k-dimensional vectors, and we write $\mathbb{R}$ ($\mathbb{R}_+$) for $k = 1$. For $a, b \in \mathbb{R}$, we write $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. Also, $a_+ = a \vee 0$ and $a_- = -(a \wedge 0)$. Let $\mathbb{D} = \mathbb{D}(\mathbb{R}_+, \mathbb{R})$ denote $\mathbb{R}$-valued function space of all cádlág functions on $\mathbb{R}_+$. $(\mathbb{D},\ J_1)$ denotes space $\mathbb{D}$ equipped with Skorohod J_1 topology with the metric d_{J_1}, see, e.g., [5, 13, 58]. Note that the space $(\mathbb{D},\ J_1)$ is complete and separable. Let $\mathbb{C}$ be the subset of $\mathbb{D}$ for continuous functions. Notations $\to$ and $\Rightarrow$ mean convergence of real numbers and convergence in distribution, respectively. The abbreviation *a.s.* means *almost surely*. Notation $\stackrel{\mathrm{d}}{=}$ represents 'equal in distribution'. We write $f(x) \sim g(x)$ for two real-valued functions f and g (non-zero) if $\lim_{x\to\infty} f(x)/g(x) = 1$.

2. Non-Stationary Power-Law Shot Noise Processes

The shot noise process $S^* := \{S^*(y) : y \in \mathbb{R}\}$ is defined as

$$S^*(y) = \sum_{j=-\infty}^{\infty} g^*(y - S_j) R_j(S_j), \quad y \in \mathbb{R}. \tag{1}$$

Here $\{S_j : j \in \mathbb{Z}\}$ is a sequence of Poisson arrival times of shots with rates λ, and R_j is the noise caused by shot j at time S_j. The effect of the noises lingers as indicated by the presence of the nonnegative shot shape function g^*. Assume that the random variables $R_j, j \in \mathbb{Z}$, are conditionally independent given $\{S_j\}$ and that the distribution of R_j depends on the time S_j, in particular,

$$P\big(R_j \le r | S_j = u\big) = F_u(r), \quad u \in \mathbb{R}, \quad r \in \mathbb{R}. \tag{2}$$

We write $R_j(S_j)$ to indicate explicitly that R_j depends on S_j. Assume that, given arrival times $\{S_j\}$, the conditional mean of R_j is zero and the conditional variance of R_j is finite, that is, for each $u \in \mathbb{R}$,

$$K_1(u) := \int_{\mathbb{R}} r dF_u(r) = 0, \quad K_2(u) := \int_{\mathbb{R}} r^2 dF_u(r) \in (0, \infty), \tag{3}$$

where the integrations are with respect to r. We also assume that $E\big[R_i(S_i)^4\big] < \infty$, that is,

$$K_4(u) := \int_{\mathbb{R}} r^4 dF_u(r) \in (0, \infty), \quad \text{for each} \quad u \in \mathbb{R}. \tag{4}$$

We study the integrated shot noise process $S := \{S(t) : t \in \mathbb{R}_+\}$ defined by

$$S(t) := \sum_{j=-\infty}^{\infty} \big(g(t - S_j) - g(-S_j)\big) R_j(S_j), \quad t \in \mathbb{R}_+, \tag{5}$$

where

$$g(t) = \int_0^t g^*(y) dy, \quad t \in \mathbb{R}_+. \tag{6}$$

Note that we are only considering the integrated effect of noises from time zero forward. The integrated process $S(t)$ is defined such that $S(0) = 0$; abusing notation, we also write $S(t) = \int_0^t S^*(y) dy$. See also Definition 3.4.1 in [44].[1]

Assume that the shot shape function g^* is heavy-tailed, namely,

$$g^*(y) = y^{-\beta} L^*(y), \quad y \ge 0, \tag{7}$$

and $g^*(y) \equiv 0$ for $y < 0$, where L^* is a positive slowly varying function at $+\infty$, and locally bounded away from 0 and ∞ in $[0, \infty)$, and

$$\beta \in (1/2, 1). \tag{8}$$

Then the function g in (6) satisfies

$$g(t) = t^{1-\beta} L(t), \quad t \ge 0, \tag{9}$$

where the slowly varying function L satisfies

$$L(t) \sim \frac{L^*(t)}{1 - \beta}, \quad \text{as} \quad t \to +\infty.$$

[1] In the literature, for example in Chapter 3.4 in [44], one does not include a star in the shot noise process as in g in (1). Since the integrated shot noise process appears frequently in the sequel, we reverse this convention in order to simplify the notation. We thus add stars in (1), define g as in (6) and omit stars in the definition (5) of the integrated shot noise process.

We focus on the integrated shot noise process $S(t)$ in (5). Observe [2] that the process $S(t)$ can be written as

$$S(t) = \int_{-\infty}^{t} \int_{-\infty}^{\infty} (g(t-u) - g(-u)) r N(du, dr), \quad t \in \mathbb{R}_+, \tag{10}$$

where N is a Poisson random measure with the intensity $F_u(dr)\lambda du$. Informally,

$$\begin{aligned} E\Big[\int\int h(u) r^p N(du, dr)\Big] &= \int h(u) \int r^p E\big[N(du, dr)\big] \\ &= \int h(u)\Big(\int r^p dF_u(r)\Big)\lambda du = \int h(u) K_p(u) \lambda du, \end{aligned}$$

where the p^{th} conditional moment is denoted $K_p(u) = \int r^p dF_u(r)$. It equals 0 if $p = 1$ by (3). In the sequal we often use the Poisson random measure $\bar{N}_a(du)$ with intensity $\lambda a du$; thus writing

$$E\Big[\int\int h(u) r^p N(du, dr)\Big] = E\Big[\int h(u) K_p(u) \bar{N}_1(du)\Big],$$

since here $a = 1$.

The auto-covariance function for the process $S(t)$ is given in the following lemma.[3]

Lemma 2.1. *The auto-covariance function of $S(t)$ is*

$$\begin{aligned} \mathrm{Cov}(S(t), S(s)) &= \lambda \int_{-\infty}^{\infty} \big(g(t-u) - g(-u)\big)\big(g(s-u) - g(-u)\big) K_2(u) du \\ &= \lambda \int_{0}^{\infty} \big(g(t+u) - g(u)\big)\big(g(s+u) - g(u)\big) K_2(-u) du \\ &\qquad + \lambda \int_0^{s \wedge t} g(t-u) g(s-u) K_2(u) du, \end{aligned} \tag{11}$$

for $t, s \in \mathbb{R}_+$, where $K_2(\cdot)$ is defined in (3).

Proof. By (10), we have

$$\begin{aligned} &\mathrm{Cov}\big(S(t), S(s)\big) \\ &= E\Big[\int_{-\infty}^{t}\int_{-\infty}^{s}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \big(g(t-u) - g(-u)\big)\big(g(s-v) - g(-v)\big) r r' N(du, dr) N(dv, dr')\Big] \\ &= \lambda \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \big(g(t-u) - g(-u)\big)\big(g(s-u) - g(-u)\big) r^2 dF_u(r) du \\ &= \lambda \int_{-\infty}^{\infty} \big(g(t-u) - g(-u)\big)\big(g(s-u) - g(-u)\big)\Big(\int_{-\infty}^{\infty} r^2 dF_u(r)\Big) du. \end{aligned}$$

Then the claim follows from the definitions of g in (6) and $K_2(u)$ in (3). □

We shall consider two classes of conditional variance functions $K_2(u)$ in (3):

- Case (i) *Power-law conditional variance functions;*
- Case (ii) *Piecewise constant conditional variance functions.*

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[2] The sum and integral notation are related as follows:

$$\sum_{i=1}^{N(t)} g(t - S_i) = \int_0^t g(t-u) N(du),$$

where $N(t)$ is Poisson with arrival times S_i's.

[3] Recall that if M is a Poisson random measure on a metric space $\mathcal{S}$ with intensity m, then $\mathrm{Cov}\big(\int_{\mathcal{S}} h_1 dM, \int_{\mathcal{S}} h_2 dM\big) = \int_{\mathcal{S}} h_1 h_2 dm$.

The specific conditions are stated in the corresponding assumptions below.

3. Power-Law Conditional Variance Functions

In this section, we prove the FCLT for the process S in (5), appropriately scaled, in Case (i). The specific assumption characterizing Case (i), is as follows:

Assumption 1 (Case (i)). Assume that $K_2(t) = t_-^{-\gamma}\tilde{L}_-(t)$ for $t<0$ and $K_2(t) = t^{-\gamma}\tilde{L}_+(t)$ for $t>0$, where $\gamma \in (0,1)$, $\tilde{L}_-(t)$ and $\tilde{L}_+(t)$ are positive slowly varying functions converging to positive constants at $-\infty$ and $+\infty$, respectively. Also assume that $L^*(y)$ converges to a positive constant as $y \to \infty$. In addition,

$$\frac{K_4(u)}{K_2(u)^2} \le \kappa \tag{12}$$

for all $u \in \mathbb{R}$, where K_4 is defined in (4) and $\kappa \in \mathbb{R}_+$ is a constant.

In this case, the conditional variance function $K_2(t)$ converges to zero as $t \to \pm\infty$. Since the conditional mean of noises is zero, the condition in (12) means that the scaled conditional kurtosis $\frac{K_4(u)}{K_2(u)^2}$ is bounded for all $u \in \mathbb{R}$. For instance, if $K_4(u) = u_\pm^{-2\gamma}\breve{L}_\pm(u)$ for some positive slowly varying functions $\breve{L}_\pm$ such that $\breve{L}_\pm(u)/\tilde{L}^2_\pm(u) \le \kappa$, then the condition (12) is satisfied.

Example 3.1. Suppose that the conditional distributions $F_u(r)$ in (2) is Gaussian with mean zero, variance $K_2(u) = u^{-\gamma}$ and thus kurtosis $K_4(u) = 3u^{-2\gamma}$ for $\gamma > 0$. Then the condition in (12) is satisfied. See also Remark 3.3 concerning the inclusion of slowly varying functions.

3.1. **The limit process.**

We first introduce the limit process, $\hat{S}$, and study its properties.

Definition 3.1. *Let $\hat{S} := \{\hat{S}(t) : t \in \mathbb{R}_+\}$ be a zero-mean Gaussian processes defined by*

$$\hat{S}(t) = c\int_{-\infty}^{t} \Big((t-u)_+^{1-\beta} - (-u)_+^{1-\beta}\Big)|u|^{-\gamma/2}\hat{B}(du), \tag{13}$$

where $c \in \mathbb{R}$ is a constant, and $\hat{B}(du)$ is a Gaussian random measure on $\mathbb{R}$ with the Lebesgue control measure du.

We state the following properties of the covariance functions and of the increment moments.

Lemma 3.1. *The process $\hat{S}$ has covariance functions*

$$\mathrm{Cov}\big(\hat{S}(t),\hat{S}(s)\big) = \int_{-\infty}^{t\wedge s} \Big((t-u)_+^{1-\beta} - (-u)_+^{1-\beta}\Big)\Big((s-u)_+^{1-\beta} - (-u)_+^{1-\beta}\Big)|u|^{-\gamma}du, \tag{14}$$

for $t,s \in \mathbb{R}_+$. For $t \ge s \ge 0$,

$$\Psi(s,t) := E\big[\big|\hat{S}(t)-\hat{S}(s)\big|^2\big] = \int_{-\infty}^{t}\Big((t-u)_+^{1-\beta} - (s-u)_+^{1-\beta}\Big)^2|u|^{-\gamma}du. \tag{15}$$

Proof. The covariance function follows as in the proof of Lemma 2.1. For Ψ, we have

$$\begin{aligned}
\Psi(s,t) &= \mathrm{Var}(\hat{S}(t)) + \mathrm{Var}(\hat{S}(s)) - 2\mathrm{Cov}(\hat{S}(t),\hat{S}(s)) \\
&= \int_{-\infty}^{t}\Big((t-u)_+^{1-\beta} - (-u)_+^{1-\beta}\Big)^2|u|^{-\gamma}du + \int_{-\infty}^{s}\Big((s-u)_+^{1-\beta} - (-u)_+^{1-\beta}\Big)^2|u|^{-\gamma}du \\
&\quad - 2\int_{-\infty}^{s}\Big((t-u)_+^{1-\beta} - (-u)_+^{1-\beta}\Big)\Big((s-u)_+^{1-\beta} - (-u)_+^{1-\beta}\Big)|u|^{-\gamma}du
\end{aligned}$$

$$
\begin{aligned}
&= \int_s^t \Big((t-u)_+^{1-\beta} - (-u)_+^{1-\beta}\Big)^2 |u|^{-\gamma} du + \int_{-\infty}^s \Big((t-u)_+^{1-\beta} - (s-u)_+^{1-\beta}\Big)^2 |u|^{-\gamma} du \\
&= \int_s^t \Big((t-u)_+^{1-\beta} - (s-u)_+^{1-\beta}\Big)^2 |u|^{-\gamma} du + \int_{-\infty}^s \Big((t-u)_+^{1-\beta} - (s-u)_+^{1-\beta}\Big)^2 |u|^{-\gamma} du \\
&= \int_{-\infty}^t \Big((t-u)_+^{1-\beta} - (s-u)_+^{1-\beta}\Big)^2 |u|^{-\gamma} du. \qquad (16)
\end{aligned}
$$

Note that we have replaced $(-u)_+^{1-\beta}$ by $(s-u)_+^{1-\beta}$ in the first integral of the third equality when $u \in (s,t)$, since both are equal to zero. □

Remark 3.1. Note that the process $\hat{S}$ does not have stationary increments if γ is not equal to zero. Recall that an FBM $B_H = \{B_H(t) : t \in \mathbb{R}\}$ has stationary increments and we get

$$E[(B_H(t) - B_H(s))^2] = E[B_H(t-s)^2] = \sigma^2 |t-s|^{2H}, \quad t, s \in \mathbb{R}, \qquad (17)$$

for some $\sigma^2 > 0$. The proofs of existence of the process in $\mathbb{D}$ and the continuity of sample paths as well as the weak convergence of scaled random walks with long range dependence to the FBM all rely on the stationary increments property. Due to the simple expression (17) for the FBM, the proofs are straightforward, because the classical maximal inequality (Theorems 10.3 and 10.4 in [5]) and as a consequence the sufficient convergence criterion (Theorem 13.5 in [5]) can be directly applied. However, this approach does not work for our model since the second moment of the increment of $\hat{S}$ does not induce a measure. If $\Psi(s,t)$ induces a measure, we would have $\Psi(r,s) + \Psi(s,t) = \Psi(r,t)$ for any $t \geq s \geq r \geq 0$, but we have this equality if and only if $t = s = r \geq 0$. The function Ψ does, however, induce a set function satisfying the following superaddivity property and therefore the generalized maximal inequalities in Theorems 5.1 and 5.2 of [41] can be applied.

Lemma 3.2. *The function Ψ in (15) satisfies the following properties:*

(i) $\Psi(s,s) = 0$ *and* $\Psi(s,t) \geq 0$ *for each* $t \geq s \geq 0$*;*
(ii) *'non-decreasing':* $\Psi(s,t) \leq \Psi(s,t')$ *for* $t' \geq t \geq s \geq 0$*;*
(iii) *'superadditive':* $\Psi(r,s) + \Psi(s,t) \leq \Psi(r,t)$ *for* $t \geq s \geq r \geq 0$*.*

In addition, $\lim_{s \to t} \Psi(s,t) = 0$*, and* $\Psi(0,t)$ *is continuous at* t*.*

Proof. The first two properties are evident. We focus on the third. Let $t \geq s \geq r \geq 0$. We have

$$
\begin{aligned}
&\Psi(r,t) - \Psi(r,s) - \Psi(s,t) \\
&= \int_{-\infty}^t \Big((t-u)_+^{1-\beta} - (r-u)_+^{1-\beta}\Big)^2 |u|^{-\gamma} du - \int_{-\infty}^t \Big((t-u)_+^{1-\beta} - (s-u)_+^{1-\beta}\Big)^2 |u|^{-\gamma} du \\
&\quad - \int_{-\infty}^s \Big((s-u)_+^{1-\beta} - (r-u)_+^{1-\beta}\Big)^2 |u|^{-\gamma} du \\
&= \int_{-\infty}^t \Big((t-u)_+^{1-\beta} - (s-u)_+^{1-\beta} + (s-u)_+^{1-\beta} - (r-u)_+^{1-\beta}\Big)^2 |u|^{-\gamma} du \\
&\quad - \int_{-\infty}^t \Big((t-u)_+^{1-\beta} - (s-u)_+^{1-\beta}\Big)^2 |u|^{-\gamma} du \\
&\quad - \int_{-\infty}^t \Big((s-u)_+^{1-\beta} - (r-u)_+^{1-\beta}\Big)^2 |u|^{-\gamma} du + \int_s^t \Big((s-u)_+^{1-\beta} - (r-u)_+^{1-\beta}\Big)^2 |u|^{-\gamma} du \\
&= \int_{-\infty}^t 2\Big((t-u)_+^{1-\beta} - (s-u)_+^{1-\beta}\Big)\Big((s-u)_+^{1-\beta} - (r-u)_+^{1-\beta}\Big) |u|^{-\gamma} du \\
&\quad + \int_s^t \Big((s-u)_+^{1-\beta} - (r-u)_+^{1-\beta}\Big)^2 |u|^{-\gamma} du
\end{aligned}
$$

$$\geq 0. \tag{18}$$

The last claim on the continuity of Ψ follows directly from the definitions. □

Proposition 3.1. *The process $\hat{S}$ in (13) is well-defined, has continuous sample paths, and is self-similar with Hurst parameter*

$$H = \frac{3}{2} - \beta - \frac{\gamma}{2} \in (0,1). \tag{19}$$

Proof. To see that the process $\hat{S}$ in (13) is well-defined, we first check that the deterministic integrands in (14) are square integrable. First, consider the variance formula of $\hat{S}$ by letting $t = s$. At $u = t$, we consider $(t-u)_+^{2-2\beta}$, which requires that $2 - 2\beta + 1 > 0$, i.e., $\beta < 3/2$. At $u = 0$, we consider $u^{2-2\beta+1-\gamma}$, which requires that $2-2\beta+1-\gamma > 0$, i.e., $\gamma < 3-2\beta$. At $u = -\infty$, we consider $u^{2(1-\beta-1)-\gamma+1}$, which requires $2(1-\beta-1) - \gamma + 1 < 0$, that is, $\gamma > 1 - 2\beta$. Given $\beta \in (1/2, 1)$ and $\gamma \in (0,1)$, these conditions are all satisfied. This proves the well-definedness of the process $\hat{S}$.

We now show that the Gaussian process $\hat{S}$ can be viewed as a random element in the space $\mathbb{D}$ with the specified finite-dimensional distributions. We apply Theorem 6.2 in the Appendix (Theorem 5.3 in [41], which is a generalization of Theorem 13.6 in [5]). We need to verify the following three conditions:

(i) The finite dimensional distributions are consistent. This is evident from the definition.

(ii) For $0 \leq r < s < t$,

$$E\big[\big|\hat{S}(r) - \hat{S}(s)\big|^2\big|\hat{S}(s) - \hat{S}(t)\big|^2\big] \leq C\Psi(r,t)^2, \tag{20}$$

where Ψ is a real-valued set function on $\mathbb{R}_+^2$ defined in (15), and $C > 0$ is some constant.

By Hölder's inequality, it suffices to show that

$$E\big[\big|\hat{S}(s) - \hat{S}(t)\big|^4\big] \leq C\Psi(s,t)^2. \tag{21}$$

This follows from the Gaussian property of $\hat{S}$ and Lemma 3.1 by setting $C = 3$. (See also Remark 3.2 below).

(iii) For each $\epsilon > 0$,

$$\lim_{h\to 0} P\big(\big|\hat{S}(t+h) - \hat{S}(t)\big| > \epsilon\big) = 0, \quad t \in \mathbb{R}_+. \tag{22}$$

This follows from Markov's inequality, (15) and the continuity of the functions Ψ.

We show next that the process $\hat{S}$ has continuous sample paths almost surely. We apply Theorem 1 in [18], which states that if a real-valued Gaussian process with sample paths in $\mathbb{D}$ is stochastically continuous, then it has sample paths in $\mathbb{C}$ almost surely. It is well known that a real-valued Gaussian process is continuous in quadratic mean if and only if it is stochastically continuous [18]. Thus it suffices to show that the process $\hat{S}$ is continuous in quadratic mean, that is,

$$\lim_{s\to t} E\big[\big|\hat{S}(s) - \hat{S}(t)\big|^2\big] = 0 \quad \text{for all} \quad t \in \mathbb{R}. \tag{23}$$

This follows from the continuity of the function Ψ.

Finally, we verify the self-similarity property. Let $\tilde{c} > 0$. We have

$$\begin{aligned}
\hat{S}(\tilde{c}t) &= \int_{-\infty}^{\tilde{c}t} \Big((\tilde{c}t-u)_+^{1-\beta} - (-u)_+^{1-\beta}\Big)|u|^{-\gamma/2}\hat{B}(du) \\
&\stackrel{\mathrm{d}}{=} \int_{-\infty}^{t} \Big((\tilde{c}t-\tilde{c}u)_+^{1-\beta} - (-\tilde{c}u)_+^{1-\beta}\Big)\tilde{c}^{-\gamma/2}|u|^{-\gamma}c^{1/2}\hat{B}(du) \\
&= \tilde{c}^H \int_{-\infty}^{t} \Big((t-u)_+^{1-\beta} - (-u)_+^{1-\beta}\Big)|u|^{-\gamma/2}\hat{B}(du),
\end{aligned} \tag{24}$$

where $H = \frac{3}{2} - \beta - \frac{\gamma}{2}$. This completes the proof of Proposition 3.1. □

3.2. **Weak convergence of the scaled process.**

We now focus on the weak convergence, as $a \to \infty$, of the scaled process $\hat{S}_a := \{\hat{S}_a(t) : t \in \mathbb{R}_+\}$ indexed by a subscript $a \in \mathbb{R}_+$:

$$\hat{S}_a(t) := \frac{S(at)}{b(a)}, \quad t \in \mathbb{R}_+. \tag{25}$$

where the process $\{S(t) : t \in \mathbb{R}_+\}$ is defined in (5), and

$$b(a) \sim a^{3/2-\beta-\gamma/2} \quad \text{as} \quad a \to \infty. \tag{26}$$

We prove an FCLT for $\hat{S}_a$ as $a \to \infty$. Note that scaling limits for the power-law shot noise process $\hat{S}_a(t)$ have been studied when g^* satisfies (7) and the noises $\{S_j\}$ are i.i.d., see Chapter 3.4 in [44]. It was shown that the limit process is an FBM. Here, since the distributions of noises are non-stationary, the limit process turns out to be non-stationary Gaussian. The stationary-increment property of the limit will be lost but the self-similarity property is preserved.

We next study the weak convergence of the scaled processes $\hat{S}_a$ in (25). We need the following lemma on the moments of the increments of the scaled processes $\hat{S}_a$. Recall that $N(du, dr)$ is a Poisson random measure with intensity $F_u(dr)\lambda du$ in (10), and $\bar{N}_a(du)$ is a Poisson random measure with intensity $\lambda a du$.

Lemma 3.3. *For $t \geq s \geq 0$ and all n,*

$$E\big[\big(\hat{S}_a(t) - \hat{S}_a(s)\big)^2\big] = \frac{1}{b(a)^2}\lambda \int_{-\infty}^{t} (g(at-au) - g(as-au))^2 K_2(au) a du, \tag{27}$$

where K_2 is defined in (3). *Moreover,*

$$\begin{aligned} E\big[\big(\hat{S}_a(t) - \hat{S}_a(s)\big)^4\big] &= 3E\bigg[\bigg(\frac{1}{b(a)^2}\int_{-\infty}^{t} \big(g(at-au) - g(as-au)\big)^2 K_2(au)\bar{N}_a(du)\bigg)^2\bigg] \\ &\quad + \frac{1}{b(a)^4}E\bigg[\int_{-\infty}^{s} \big(g(at-au) - g(as-au)\big)^4\big(K_4(au) - 3K_2(au)^2\big)\bar{N}_a(du)\bigg] \\ &\quad + \frac{1}{b(a)^4}E\bigg[\int_{s}^{t} \big(g(at-au)\big)^4\big(K_4(au) - 3K_2(au)^2\big)\bar{N}_a(du)\bigg], \end{aligned} \tag{28}$$

where K_4 is defined in (4).

Proof. We can write

$$\begin{aligned} \hat{S}_a(t) - \hat{S}_a(s) &= \frac{1}{b(a)}\int_{-\infty}^{as}\int_{-\infty}^{\infty} \big(g(at-u) - g(as-u)\big) r N(du, dr) \\ &\quad + \frac{1}{b(a)}\int_{as}^{at}\int_{-\infty}^{\infty} g(at-u) r N(du, dr). \end{aligned} \tag{29}$$

Note that the two terms on the right hand side are independent. Also, the expectations of both terms are equal to zero (since the conditional mean of the noises is zero). Thus we have

$$\begin{aligned} E\big[\big(\hat{S}_a(t) - \hat{S}_a(s)\big)^2\big] &= \frac{1}{b(a)^2}\lambda \int_{as}^{at} g(at-u)^2 K_2(u) du \\ &\quad + \frac{1}{b(a)^2}\lambda \int_{-\infty}^{as} (g(at-u) - g(as-u))^2 K_2(u) du, \end{aligned} \tag{30}$$

where $K_2(u)$ is defined in (3) and $g(t) = 0$ for $t < 0$. Since

$$\frac{1}{b(a)^2}\lambda \int_{as}^{at} g(at-u)^2 K_2(u) du = \frac{1}{b(a)^2}\lambda \int_{as}^{at} (g(at-u) - g(as-u))^2 K_2(u) du,$$

we can combine the two integrals and make a change of variables to obtain (27).

For the fourth moment (27), we again use the integral representation in (29). By conditioning and since the conditional mean of noises is zero (see (3)), we have that

$$\begin{aligned}
&E\big[\big(\hat{S}_a(t)-\hat{S}_a(s)\big)^4\big] \\
&= \frac{1}{b(a)^4}E\bigg[\bigg(\int_{as}^{at}\int_{-\infty}^{\infty} g(at-u)rN(du,dr)\bigg)^4\bigg] && \text{(31a)}\\
&\quad + \frac{1}{b(a)^4}E\bigg[\bigg(\int_{-\infty}^{as}\int_{-\infty}^{\infty} \big(g(at-u)-g(as-u)\big)rN(du,dr)\bigg)^4\bigg] && \text{(31b)}\\
&\quad + \frac{6}{b(a)^4}E\bigg[\bigg(\int_{as}^{at}\int_{-\infty}^{\infty} g(at-u)rN(du,dr)\bigg)^2 \\
&\qquad\qquad \times \bigg(\int_{-\infty}^{as}\int_{-\infty}^{\infty} \big(g(at-u)-g(as-u)\big)rN(du,dr)\bigg)^2\bigg]. && \text{(31c)}
\end{aligned}$$

We obtain this expression because expanding $(a+b)^4$, we get $a^4+b^4+6a^2b^2$ when the terms with a or b raised to the power 1 vanish, due to the fact that the conditional mean of the noises is zero. Similarly, for the first term (31a), by conditioning, we obtain that it is equal to

$$\begin{aligned}
&\frac{1}{b(a)^4}E\bigg[\int_s^t g(at-au)^4K_4(au)\bar{N}_a(du)\bigg] \\
&\quad + \frac{6}{b(a)^4}E\bigg[\int_s^t\int_s^t \Big(g(at-au)^2K_2(au)\Big)\Big(g(at-av)^2K_2(av)\Big)\mathbf{1}(u\neq v)\bar{N}_a(du)\bar{N}_a(dv)\bigg]. \qquad (32)
\end{aligned}$$

Indeed, by expanding (31a), we get the product $N(du_1,dr_1)N(du_2,dr_2)N(du_3,dr_3)N(du_4,dr_4)$. The first term in (32) results from the case $u_1=u_2=u_3=u_4$ and in the second term in (32) the u's are pairwise equal. There are $\binom{4}{2}=6$ such pairs.

We need to evaluate the double integral in (32). We do so through the relation

$$\begin{aligned}
&\frac{1}{b(a)^4}E\bigg[\bigg(\int_s^t g(at-au)^2K_2(au)\bar{N}_a(du)\bigg)^2\bigg] && \text{(33a)}\\
&= \frac{1}{b(a)^4}E\bigg[\int_s^t g(at-au)^4K_2(au)^2\bar{N}_a(du)\bigg] && \text{(33b)}\\
&\quad + \frac{2}{b(a)^4}E\bigg[\int_s^t\int_s^t \Big(g(at-au)^2K_2(au)\Big)\Big(g(at-av)^2K_2(av)\Big)\mathbf{1}(u\neq v)\bar{N}_a(du)\bar{N}_a(dv)\bigg]. && \text{(33c)}
\end{aligned}$$

By replacing the double integral in (32) by the two terms (33a) and (33b) given in this equation, we obtain that the first term (31a) is equal to

$$\begin{aligned}
&\frac{1}{b(a)^4}E\bigg[\int_s^t g(at-au)^4K_4(au)\bar{N}_a(du)\bigg] \\
&\quad + \frac{3}{b(a)^4}E\bigg[\bigg(\int_s^t g(at-au)^2K_2(au)\bar{N}_a(du)\bigg)^2\bigg] - \frac{3}{b(a)^4}E\bigg[\int_s^t g(at-au)^4K_2(au)^2\bar{N}_a(du)\bigg]. \qquad (34)
\end{aligned}$$

Similarly, the second term (31b) is equal to

$$\frac{1}{b(a)^4}E\bigg[\int_{-\infty}^{s}\big(g(at-au)-g(as-au)\big)^4K_4(au)\bar{N}_a(du)\bigg]$$

$$+\frac{3}{b(a)^4}E\Big[\Big(\int_{-\infty}^{s}\big(g(at-au)-g(as-au)\big)^2K_2(au)\bar{N}_a(du)\Big)^2\Big]$$
$$-\frac{3}{b(a)^4}E\Big[\int_{-\infty}^{s}\big(g(at-au)-g(as-au)\big)^4K_2(au)^2\bar{N}_a(du)\Big]. \tag{35}$$

The third term (31c) is equal to

$$\frac{6}{b(a)^4}E\Big[\Big(\int_{s}^{t}g(at-au)^2K_2(au)\bar{N}_a(du)\Big)\Big(\int_{-\infty}^{s}\big(g(at-au)-g(as-au)\big)^2K_2(au)\bar{N}_a(du)\Big)\Big]. \tag{36}$$

Thus, combining (34)–(36), we obtain the expression in (28). □

Remark 3.2. Under the conditions on $b(a)$ in Theorem 3.1, the second moment of the increment of $\hat{S}_a$ in Lemma 3.3 converges to that of the limit $\hat{S}$ given in Lemma 3.1. Since the limit process is mean-zero Gaussian, we have $E[(\hat{S}(t)-\hat{S}(s))^4]=3\big(E[(\hat{S}(t)-\hat{S}(s))^2]\big)^2$, while this property does not hold in general for the scaled process $\hat{S}_a$. We obtain, however, an expression which converges to the same limit as $a\to\infty$.

As a consequence of Lemma 3.1, it is also easy to show that for $0\le r\le s\le t$ and any $\epsilon>0$, we have

$$P\Big(\big|\hat{S}(r)-\hat{S}(s)\big|\vee\big|\hat{S}(s)-\hat{S}(t)\big|\ge\epsilon\Big)\le\frac{3}{\epsilon^4}\Psi(r,t)^2.$$

Indeed, we have

$$\begin{aligned}&P\Big(\big|\hat{S}(r)-\hat{S}(s)\big|\vee\big|\hat{S}(s)-\hat{S}(t)\big|\ge\epsilon\Big)\\&\le\frac{1}{\epsilon^4}E\Big[\big|\hat{S}(r)-\hat{S}(s)\big|^2\big|\hat{S}(s)-\hat{S}(t)\big|^2\Big]\\&\le\frac{1}{\epsilon^4}\Big(E\Big[\big|\hat{S}(r)-\hat{S}(s)\big|^4\Big]\Big)^{1/2}\Big(E\Big[\big|\hat{S}(s)-\hat{S}(t)\big|^4\Big]\Big)^{1/2}\\&\le\frac{3}{\epsilon^4}\Psi(r,t)^2,\end{aligned}$$

using the increment moment formulas in Lemma 3.1 and the monotonicity property of Ψ. In the lemma below we prove the analogous property for the scaled process $\hat{S}_a$, which in general is not Gaussian.

Lemma 3.4. *For $0\le r\le s\le t\le T<\infty$, all $a>0$ and any $\epsilon>0$,*

$$P\Big(\big|\hat{S}_a(r)-\hat{S}_a(s)\big|\vee\big|\hat{S}_a(s)-\hat{S}_a(t)\big|\ge\epsilon\Big)\le\frac{C}{\epsilon^4}\Psi(r,t)^2, \tag{37}$$

where $C>0$ is a constant, and $\Psi(s,t)=E\big[\big|\hat{S}(t)-\hat{S}(s)\big|^2\big]$ as given in Lemma 3.1.

Proof. We have as above,

$$\begin{aligned}&P\Big(\big|\hat{S}_a(r)-\hat{S}_a(s)\big|\vee\big|\hat{S}_a(s)-\hat{S}_a(t)\big|\ge\epsilon\Big)\\&\le\frac{1}{\epsilon^4}\Big(E\Big[\big|\hat{S}_a(r)-\hat{S}_a(s)\big|^4\Big]\Big)^{1/2}\Big(E\Big[\big|\hat{S}_a(s)-\hat{S}_a(t)\big|^4\Big]\Big)^{1/2}.\end{aligned} \tag{38}$$

Note that the nondecreasing property of the set function Ψ implies that $\Psi(r,s)\Psi(s,t)\le\Psi(r,t)^2$. By (28) in Lemma 3.3, it suffices to show that

$$E\Big[\Big(\frac{1}{b(a)^2}\int_{-\infty}^{t}\big(g(at-au)-g(as-au)\big)^2K_2(au)\bar{N}_a(du)\Big)^2\Big]\to C_1\Psi(s,t)^2\quad\text{as}\quad a\to\infty, \tag{39}$$

for some constant $C_1 > 0$, independent of s, t, and

$$\frac{1}{b(a)^4} E\bigg[\int_{-\infty}^{s} \big(g(at-au)-g(as-au)\big)^4 \big(K_4(au)-3K_2(au)^2\big)\bar{N}_a(du)\bigg] \to 0 \quad \text{as} \quad a \to \infty, \tag{40}$$

$$\frac{1}{b(a)^4} E\bigg[\int_{s}^{t} \big(g(at-au)\big)^4 \big(K_4(au)-3K_2(au)^2\big)\bar{N}_a(du)\bigg] \to 0 \quad \text{as} \quad a \to \infty, \tag{41}$$

where K_4 is defined in (4).

We first prove (39). To show the convergence of the expectation, we proceed in two steps: i) convergence in distribution of the random process inside the expectation for each fixed $s, t \in \mathbb{R}_+$ as $a \to \infty$, that is,

$$\frac{1}{b(a)^2} \int_{-\infty}^{t} \big(g(at-au)-g(as-au)\big)^2 K_2(au)\bar{N}_a(du) \Rightarrow C^{1/2}\Psi(s,t), \quad \text{as} \quad a \to \infty, \tag{42}$$

for each $s, t \in \mathbb{R}_+$ and constant $C > 0$ (which implies the convergence in distribution for their squares),

and

ii) the uniform integrability of the sequence

$$\bigg\{\bigg(\frac{1}{b(a)^2} \int_{-\infty}^{t} \big(g(at-au)-g(as-au)\big)^2 K_2(au)\bar{N}_a(du)\bigg)^2 : a \in \mathbb{R}_+\bigg\}. \tag{43}$$

Then the convergence in (39) follows from Theorem 3.5 in [5].

To prove (42), it suffices to show the convergence of the corresponding characteristic functions, that is,

$$E\bigg[\exp\bigg(i\theta \frac{1}{b(a)^2} \int_{-\infty}^{t} \big(g(at-au)-g(as-au)\big)^2 K_2(au)\bar{N}_a(du)\bigg)\bigg] \to \exp(i\theta C\Psi(s,t)) \quad \text{as} \quad a \to \infty. \tag{44}$$

The left hand side is equal to

$$\exp\bigg(\int_{-\infty}^{t} \bigg[\exp\bigg(i\theta \frac{1}{b(a)^2}\big(g(at-au)-g(as-au)\big)^2 K_2(au)\bigg) - 1\bigg]\lambda a du\bigg).$$

We have

$$\begin{aligned}
&\frac{1}{b(a)^2} \int_{-\infty}^{t} \big(g(at-au)-g(as-au)\big)^2 K_2(au)\ a\lambda du \\
&= \frac{a}{b(a)^2}\lambda \int_{-\infty}^{0} \big((at-au)_+^{1-\beta}L((at-au)_+) - (as-au)_+^{1-\beta}L((as-au)_+)\big)^2 (au)_-^{-\gamma}\tilde{L}_-(au)du \\
&\qquad + \frac{a}{b(a)^2}\lambda \int_{0}^{t} \big((at-au)_+^{1-\beta}L((at-au)_+) - (as-au)_+^{1-\beta}L((as-au)_+)\big)^2 (au)_+^{-\gamma}\tilde{L}_+(au)du \\
&= \frac{a^{3-2\beta-\gamma}}{b(a)^2}\lambda\bigg(\int_{-\infty}^{0} \big((t-u)_+^{1-\beta}L((at-au)_+) - (s-u)_+^{1-\beta}L((as-au)_+)\big)^2 (u)_-^{-\gamma}\tilde{L}_-(au)du \\
&\qquad + \int_{0}^{t} \big((t-u)_+^{1-\beta}L((at-au)_+) - (s-u)_+^{1-\beta}L((as-au)_+)\big)^2 (u)_+^{-\gamma}\tilde{L}_+(au)du\bigg).
\end{aligned} \tag{45}$$

Using the asymptotic behavior of $b(a)$ in (26), it follows that the right hand side of (45) converges to $C\Psi(s,t)^2$ for some constant $C > 0$. Similarly, we obtain

$$\frac{1}{b(a)^4} \int_{-\infty}^{t} \big(g(at-au)-g(as-au)\big)^4 K_2(au)^2\ a\lambda du \to 0 \quad \text{as} \quad a \to \infty. \tag{46}$$

Thus by Taylor remainder theorem, we have shown (44).

Now to prove the uniform integrability of the sequence in (43), it suffices to show that

$$\sup_a E\Big[\Big(\frac{1}{b(a)^2}\int_{-\infty}^t \big(g(at-au)-g(as-au)\big)^2 K_2(au)\bar{N}_a(du)\Big)^3\Big] < \infty. \tag{47}$$

Similar to the calculations leading to (32), since the conditional expectation of noises is zero, we obtain that the expectation in (47) is equal to

$$\begin{aligned}
&E\Big[\frac{1}{b(a)^6}\int_{-\infty}^t \big(g(at-au)-g(as-au)\big)^6 K_2(au)^3\bar{N}_a(du)\Big]\\
&= \frac{1}{b(a)^6}\int_{-\infty}^t \big(g(at-au)-g(as-au)\big)^6 K_2(au)^3\ a\lambda du\\
&= \frac{a}{b(a)^6}\lambda\int_{-\infty}^0 \big((at-au)_+^{1-\beta}L((at-au)_+)-(as-au)_+^{1-\beta}L((as-au)_+)\big)^6(au)_-^{-3\gamma}\tilde{L}_-(au)^3du\\
&\quad + \frac{a}{b(a)^2}\lambda\int_0^t \big((at-au)_+^{1-\beta}L((at-au)_+)-(as-au)_+^{1-\beta}L((as-au)_+)\big)^6(au)_+^{-3\gamma}\tilde{L}_+(au)^3du\\
&= \frac{a^{6\times(3/2-\beta-\gamma/2)}}{a^2 b(a)^2}\lambda\Big(\int_{-\infty}^0 \big((t-u)_+^{1-\beta}L((at-au)_+)-(s-u)_+^{1-\beta}L((as-au)_+)\big)^6(u)_-^{-3\gamma}\tilde{L}_-(au)^3du\\
&\qquad + \int_0^t \big((t-u)_+^{1-\beta}L((at-au)_+)-(s-u)_+^{1-\beta}L((as-au)_+)\big)^6(u)_+^{-3\gamma}\tilde{L}_+(au)^3du\Big).
\end{aligned}$$

By the assumption in (26), we obtain the claim in (47).

Next we prove (41) since (40) follows similarly. We have

$$\begin{aligned}
&\frac{1}{b(a)^4}\int_s^t g(at-au)^4\big(K_4(au)-3K_2(au)^2\big)\ a\lambda du\\
&= \frac{a}{b(a)^4}\lambda\int_s^t (at-au)^{4-4\beta}L(at-au)^4\Big(\frac{K_4(au)}{3K_2(au)^2}-1\Big)3(au)^{-2\gamma}\tilde{L}(au)^2\big)du\\
&= \frac{1}{a}\times\frac{a^{4(3/2-\beta-\gamma/2)}}{b(a)^4}\lambda\int_s^t (t-u)^{4-4\beta}L(at-au)^4\Big(\frac{K_4(au)}{3K_2(au)^2}-1\Big)3u^{-2\gamma}\tilde{L}(au)^2\big)du\\
&\to 0 \quad \text{as} \quad a\to\infty,
\end{aligned} \tag{48}$$

where the convergence follows from the assumption (12). This completes the proof. □

We now prove the weak convergence of $\hat{S}_a$ in Case (i).

Theorem 3.1. *Under the assumptions in Case (i), we have the weak convergence*

$$\hat{S}_a \Rightarrow \hat{S} \quad \text{in} \quad (\mathbb{D}, J_1) \quad \text{as} \quad a\to\infty, \tag{49}$$

where the scaled process $\hat{S}_a$ *is defined in* (25) *and the limit process* $\hat{S}$ *is given in Definition 13.*

Proof. We apply Theorem 6.1 in the Appendix below to prove the weak convergence. We first prove the convergence of finite-dimensional distributions, that is,

$$\big(\hat{S}_a(t_1),\dots,\hat{S}_a(t_k)\big) \Rightarrow \big(\hat{S}(t_1),\dots,\hat{S}(t_k)\big) \quad \text{in} \quad \mathbb{R}^k \quad \text{as} \quad a\to\infty, \tag{50}$$

for $t_i \geq 0$, $i=1,\dots,k$. We start with $k=1$ (removing the subscript 1 for brevity). We have

$$\begin{aligned}
E\Big[\exp\Big(i\theta\hat{S}_a(t)\Big)\Big] &= E\Big[\exp\Big(i\theta\frac{1}{b(a)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\big(g(at-u)-g(-u)\big)rN(du,dr)\Big)\Big]\\
&= \exp\Big(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big[\exp\Big(i\theta\frac{1}{b(a)}(g(at-u)-g(-u))r\Big)
\end{aligned}$$

$$- 1 - i\theta \frac{1}{b(a)} (g(at-u) - g(-u))r\Big] F_u(dr)\lambda du\Big)$$

$$= \exp\Big(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big[\exp\Big(i\theta\frac{1}{b(a)}(g(at-au)-g(-au))r\Big) - 1 - i\theta\frac{1}{b(a)}(g(at-au)-g(-au))r\Big]F_{au}(dr)\lambda a du\Big).$$

Recall that $e^{ix} - 1 - ix \sim -x^2/2$ as $x \to 0$. Thus, we obtain that

$$\begin{aligned}
I_a &:= \log E\Big[\exp\Big(i\theta\hat{S}_a(t)\Big)\Big] \\
&\sim \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big[-\theta^2\frac{1}{2b(a)^2}\big(g(at-au)-g(-au)\big)^2 r^2\Big]F_{au}(dr)\lambda a du \\
&= \int_{-\infty}^{\infty}\Big[-\theta^2\frac{1}{2b(a)^2}\big(g(at-au)-g(-au)\big)^2 K_2(au)\lambda a\Big]du \\
&= \int_{-\infty}^{t}\Big[-\theta^2\frac{1}{2b(a)^2}\big(g(at-au)-g(-au)\big)^2 K_2(au)\lambda a\Big]du =: \int_{-\infty}^{t}\Big(-\frac{\theta^2}{2}\lambda I_a(t,u)\Big)du, \qquad (51)
\end{aligned}$$

where the last equality follows since $g(t) \equiv 0$ for $t < 0$. By the assumptions on $b(a)$ in (26), g in (9) and K_2 in Assumption in Case (i), we have

$$\begin{aligned}
I_a(t,u) &= \frac{a}{b(a)^2}\big(g(at-au)-g(-au)\big)^2 K_2(au) \\
&\to I(t,u) := C'\big((t-u)_+^{1-\beta} - (-u)_+^{1-\beta}\big)^2 u^{-\gamma}, \qquad (52)
\end{aligned}$$

as $a \to \infty$, for some positive $C' > 0$. To prove the convergence

$$E\big[e^{-i\theta\hat{S}_a(t)}\big] \to E\big[e^{-i\theta\hat{S}(t)}\big] \quad \text{as} \quad a \to \infty,$$

it suffices to show that $I_a(t,u)$ is dominated by an integrable function. As in the proof of Theorem 3.4.5 in [44], it can be shown that

$$\frac{|g(at-au)-g(-au)|}{a^{1-\beta}L(a)} \le \frac{C''}{1-\beta\pm\delta}\big((t-u)_+^{1-\beta\pm\delta} - (-u)_+^{1-\beta\pm\delta}\big) \qquad (53)$$

where $\delta > 0$ is small and C'' is some positive constant. Since $G(u) = |u|^{-\gamma}\tilde{L}(u)$, it is clear that $I_a(t,u)$ can be dominated by an integrable function. Thus we have shown that

$$\hat{S}_a(t) \Rightarrow \hat{S}(t) \quad \text{in} \quad \mathbb{R} \quad \text{as} \quad a \to \infty.$$

Now for the convergence of finite dimensional distributions, we show that

$$\big(\hat{S}_a(t_1),\dots,\hat{S}_a(t_k)\big) \Rightarrow \big(\hat{S}(t_1),\dots,\hat{S}(t_k)\big) \quad \text{in} \quad \mathbb{R}^k \quad \text{as} \quad a \to \infty. \qquad (54)$$

Let θ_i, $i = 1, ..., k$, be positive constants. Consider the joint characteristic function

$$E\Big[e^{i\sum_{j=1}^k \theta_j \hat{S}_a(t_j)}\Big] = E\Big[\exp\Big(i\frac{1}{b}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big(\sum_{j=1}^k \theta_j\big(g(at_j-s)-g(-s)\big)\Big)rN(ds,dr)\Big)\Big]$$

The same calculations above apply to the above equation by considering the function $\sum_{j=1}^k \theta_j\big(g(at_j - s) - g(-s)\big)$, and thus (54) holds.

We now verify that the conditions (ii)–(iv) of Theorem 6.1 for the weak convergence of $\hat{S}_a$ to $\hat{S}$ are satisfied. We have shown (i) the convergence of the finite-dimensional distributions (50). Condition (ii), namely the continuity of the processes at each t, follows from the proof of Proposition 3.1. The condition (iii) on the probability bound for the increments of $\hat{S}_a$ is verified by Lemma 3.4. Finally, condition (iv) holds because $\Psi(0,t)$ is continuous at t. This completes the proof of Theorem 3.1. □

Remark 3.3. We have assumed that the slowly varying functions $L(t)$ and $\tilde{L}_+(t)$ (*resp.* $\tilde{L}_-(t)$) converge to positive constants as $t \to \infty$ (*resp.* $t \to -\infty$)) and assumed that $b(a)$ satisfies (26) to prove the weak convergence in Theorem 3.1. If $K_2(t) = |t|^{-\gamma}\tilde{L}(t)$ for $\gamma \in (0,1)$, L^* is a positive slowly varying function at ∞ and $\tilde{L}$ is a positive slowly varying function at both $\pm\infty$, then the weak convergence in Theorem 3.1 holds provided that

$$b(a) \sim a^{3/2-\beta-\gamma/2} L(a)^2 \tilde{L}(a) \quad \text{as} \quad a \to \infty.$$

instead of the condition on $b(a)$ in (26).

4. Piecewise Noise Functions

In this section, we study Case (ii) where the conditional variance function $K_2(s)$ of noises are piecewise constant. The specific assumptions are as follows.

Assumption 2 (Case (ii)). Let $F^{(k)}$ be a c.d.f. with mean zero, finite second moment σ_k^2 and finite fourth moment ς_k, for $k = 0, \dots, \ell$. Assume that there exist finite time points $-\infty = u_0 < u_1 < u_2 < \cdots < u_\ell < u_{\ell+1} = \infty$ such that the distribution function $F_u(\cdot) = F^{(k)}(\cdot)$ if $u \in [u_k, u_{k+1})$. In this case, $K_2(u)$ and $K_4(u)$ are piecewise constant, taking values σ_k^2 and ς_k in the interval $u \in [u_k, u_{k+1})$ for $k = 0, \dots, \ell$, respectively. Let k_0 be the integer such that $0 \in [u_{k_0}, u_{k_0+1})$.

We prove the weak convergence of the scaled process $\hat{S}_a(t) = S(at)/b(a)$ defined in (25) with

$$b(a) \sim a^{3/2-\beta} L(a) \quad \text{as} \quad a \to \infty, \tag{55}$$

where $\beta \in (1/2, 1)$ is assumed in (8).

Theorem 4.1. *Under the assumptions in Case (ii), we have the weak convergence*

$$\hat{S}_a \Rightarrow \tilde{S} \quad \textit{in} \quad (\mathbb{D}, J_1) \quad \textit{as} \quad n \to \infty, \tag{56}$$

where the limit process $\tilde{S} := \{\tilde{S}(t) : t \in \mathbb{R}\}$ is a non-stationary self-similar Gaussian process $\tilde{S}$, with $\tilde{S} = \{\tilde{S}(t) : t \geq 0\}$ defined by

$$\tilde{S}(t) \stackrel{\mathrm{d}}{=} \lambda^{1/2} \int_{-\infty}^{t} \Big((t-u)_+^{1-\beta} - (-u)_+^{1-\beta}\Big) \widetilde{B}(du) \tag{57}$$

where $\widetilde{B}(du)$ is a Gaussian random measure on $\mathbb{R}$ with the control measure

$$\sigma(du) = \sigma_0 \mathbf{1}(u<0)du + \sigma_{k_0}\delta_0(du) + \sigma_\ell \mathbf{1}(u>0)du$$

and $\delta_0(du)$ is the Dirac measure at 0. The self-similarity parameter of $\tilde{S}$ is $H = 3/2 - \beta \in (1/2, 1)$.

Proof. First, we obtain the covariance function of $\tilde{S}(t)$:

$$\mathrm{Cov}\big(\tilde{S}(t), \tilde{S}(s)\big) = \lambda \int_{-\infty}^{t\wedge s} \big((t+u)^{1-\beta} - u^{1-\beta}\big)\big((s+u)^{1-\beta} - u^{1-\beta}\big)\sigma_2(du), \tag{58}$$

for $t, s \in \mathbb{R}_+$, where

$$\sigma_2(du) := \sigma_0^2 \mathbf{1}(u<0)du + \sigma_{k_0}^2 \delta_0(du) + \sigma_\ell^2 \mathbf{1}(u>0)du.$$

The second moments of the increments of $\tilde{S}$ is

$$E\big[\big(\tilde{S}(t) - \tilde{S}(s)\big)^2\big] = \lambda \int_{-\infty}^{t} \big((t-u)_+^{1-\beta} - (s-u)_+^{1-\beta}\big)^2 \sigma_2(du) =: \tilde{\Psi}(s,t), \tag{59}$$

for $t \geq s \geq 0$. By (58), we have

$$E\big[\big(\tilde{S}(t-s)\big)^2\big] = \lambda \int_{-\infty}^{t-s} \Big((t-s-u)_+^{1-\beta} - (-u)_+^{1-\beta}\Big)^2 \sigma_2(du),$$

It is evident that $E\big[\big(\tilde{S}(t)-\tilde{S}(s)\big)^2\big]$ is not equal to $E\big[\big(\tilde{S}(t-s)\big)^2\big]$. However, we observe that as in the proof of Lemma 3.2, the function $\tilde{\Psi}(s,t)$ satisfies all the properties in Lemma 3.2. Thus we can follow a similar argument in the proof of Theorem 3.1.

We first check the convergence of finite-dimensional distributions. Recall (51). Under the assumptions of g and K_2, we have

$$\begin{aligned}&\int_{-\infty}^t \Big[\frac{1}{b(a)^2}\big(g(at-au)-g(-au)\big)^2 K_2(au)\lambda a\Big]du\\ &=\frac{a^{3-2\beta}}{b(a)^2}\lambda\int_{-\infty}^t \big((t-u)_+^{1-\beta}L((at-au)_+)-(s-u)_+^{1-\beta}L((as-au)_+)\big)^2K_2(au)du\\ &\sim\lambda\int_{-\infty}^t\big((t-u)_+^{1-\beta}-(s-u)_+^{1-\beta}\big)^2\sigma_2(du).\end{aligned}$$

Then we verify the condition on the probability bound for the increments of $\hat{S}_a$ as in Lemma 3.4, for which we use the moment formulas for the increments in Lemma 3.3. The proof follows essentially the same argument, but using the pieceiwsie constant conditional variance function $K_2(t)$ and fourth moment $K_4(t)$ in the assumptions of Case (ii). In particular, in the proof of the convergence of (41), we have

$$\begin{aligned}&\frac{a}{b(a)^4}\lambda\int_s^t g(at-au)^4\big(K_4(au)-3K_2(au)^2\big)du\\ &=\frac{1}{a}\times\frac{a^{4(3/2-\beta)}}{b(a)^4}\lambda\int_s^t(t-u)^{4-4\beta}L(at-au)^4\big(K_4(au)-3K_2(au)^2\big)du\\ &\to 0 \quad \text{as} \quad a\to\infty.\end{aligned}$$

Here the convergence follows from the assumption on $b(a)$ in (55) and the boundedness of the second and fourth moments, $K_p(\cdot), p=2,4$, under the assumptions in Case (ii). The rest of the proof can be carried out similarly as in the proof of Theorem 3.1. □

5. New Classes of Non-Stationary Self-Similar Gaussian Processes

Motivated by the scaling limits of the shot noise processes in Sections 3 and 4, we introduce three new classes of self-similar Gaussian processes with non-stationary increments. They reduce to FBM in special cases. They are characterized via the integral representations, generalizing that of FBM by Mandelbrot and Van Ness [34].

5.1. **A first class.**

Consider a process $X:=\{X(t):t\in\mathbb{R}\}$ defined by the following (time-domain) integral representation

$$\{X(t)\}_{t\in\mathbb{R}}\overset{\text{d}}{=}\left\{\int_{\mathbb{R}}\Big(a_1\left((t-u)_+^\alpha-(-u)_+^\alpha\right)+a_2\left((t-u)_-^\alpha-(-u)_-^\alpha\right)\Big)|u|^{-\gamma/2}B(du)\right\}_{t\in\mathbb{R}},\tag{60}$$

where

$$a_i\in\mathbb{R},\quad i=1,2,\quad \gamma>0,\quad \alpha\in\Big(-\frac{1}{2}+\frac{\gamma}{2},\ \frac{1}{2}+\frac{\gamma}{2}\Big),\tag{61}$$

and $B(du)$ is a Gaussian random measure on $\mathbb{R}$ with the Lebesgue control measure du. When $\alpha=0$, the integrand $\big((t-u)_+^\alpha-(-u)_+^\alpha\big)$ is interpreted as $\mathbf{1}([0,t))$ for $t\ge0$ and $\mathbf{1}((t,0])$ for $t<0$. In this case, when $a_1=1$ and $a_2=0$, we get $\int_0^t u^{-\gamma/2}B(du)$ for $t\ge0$, which is a driftless Brownian motion with variance function $\int_0^t u^{-\gamma/2}du$.

To see that the process is well-defined, we check if the square of the integrand in (60) is integrable. At $u=t$, we consider $(t-u)_+^{2\alpha}$ and $(t-u)_-^{2\alpha}$, which requires that $2\alpha+1>0$, that is, $\alpha>-1/2$. At $u=0$, we consider $(-u)_+^{2\alpha-\gamma+1}$, which requires that $2\alpha-\gamma+1>0$, that is, $\alpha>(\gamma-1)/2$. At

$u = -\infty$, we consider $(-u)_+^{2(\alpha-1)-\gamma+1}$, which requires that $2(\alpha-1)-\gamma+1 < 0$, that is, $\alpha < (\gamma+1)/2$. Similarly, at $u = +\infty$, we also require $\alpha < (\gamma+1)/2$. Therefore, given the conditions on γ and α in (61), the process $X(t)$ in (60) is well defined.

Observe that when $\gamma = 0$ and $\alpha = H - 1/2$ for $H \in (0,1)$, the process X in (60) is an FBM B_H, which can be expressed through its time-domain representation (see, e.g., Section 2.6, Remark 2.6.8 in [44]). The definition of the process $X(t)$ in (60) is a natural way to generalize FBMs to be a Gaussian process with non-stationary increments which preserves the self-similarity property.

Note also that like FBMs, if $a_1 \neq 0$ and $a_2 = 0$, then we can call the process $X(t)$ *causal* (non-anticipative) since the integration is over $(-\infty, t]$ for $t > 0$. And when $a_1 = a_2$, the process $X(t)$ can be called *well-balanced*. When $a_1 = 1$ and $a_2 = 0$, the limit process $\hat{S}$ in Definition 13 can be regarded as a special case of the process $X(t)$.

We summarize some basic properties of this process.

Proposition 5.1. *The process X has the following properties.*

(i) *X is a continuous self-similar Gaussian process with Hurst parameter $H = \alpha - \frac{\gamma}{2} + \frac{1}{2} \in (0,1)$.*

(ii) *The covariance function of X is*

$$\begin{aligned}\mathrm{Cov}(X(t), X(s)) = \int_{\mathbb{R}} \Big(a_1 \left((t-u)_+^\alpha - (-u)_+^\alpha \right) + a_2 \left((t-u)_-^\alpha - (-u)_-^\alpha \right) \Big) \\ \times \Big(a_1 \left((s-u)_+^\alpha - (-u)_+^\alpha \right) + a_2 \left((s-u)_-^\alpha - (-u)_-^\alpha \right) \Big) |u|^{-\gamma} du,\end{aligned}$$

and the second moment of its increment is

$$\begin{aligned}\Phi(s,t) &:= E[(X(t) - X(s))^2] \\ &= \int_{\mathbb{R}} \Big(a_1 \left((t-u)_+^\alpha - (s-u)_+^\alpha \right) + a_2 \left((t-u)_-^\alpha - (s-u)_-^\alpha \right) \Big)^2 |u|^{-\gamma} du,\end{aligned}$$

for $t, s \in \mathbb{R}$. The set function $\Phi(s,t)$ satisfies the properties in Lemma 3.2.

(iii) *The spectral representation of X is*

$$\{X(t)\}_{t\in\mathbb{R}} \stackrel{\mathrm{d}}{=} \Big\{ \frac{1}{\sqrt{2\pi}} \Gamma(\alpha+1) \int_{\mathbb{R}} \left(a_1(-ix)^{-\alpha} + a_2(ix)^{-\alpha} \right) \frac{e^{itx}-1}{ix} \widehat{\varphi}(x) \widehat{B}(dx) \Big\}_{t\in\mathbb{R}}, \tag{62}$$

where $\Gamma(\cdot)$ is the Gamma function,

$$\widehat{\varphi}(x) := 2|x|^{\gamma/2-1} \Gamma(1-\gamma/2) \cos\left((1-\gamma/2)\pi/2 \right), \quad x \in \mathbb{R},$$

and $\widehat{B}$ is a Hermitian Gaussian measure on $\mathbb{R}$ with the Lebesgue control measure dx.

Proof. The existence of the process, the continuity of the sample path and the self-similarity property follow from similar argument as Proposition 3.1. The proof of (ii) follows from similar arguments in the proof of Lemmas 3.1 and 3.2.

For the spectral representation, first by (2.6.6) in [44], we have for $x > 0$,

$$\begin{aligned}\int_{\mathbb{R}} e^{ixu} |u|^{-\gamma/2} du &= \int_0^\infty e^{iv} v^{-\gamma/2} x^{\gamma/2-1} dv + \int_0^\infty e^{-iv} v^{-\gamma/2} x^{\gamma/2-1} dv \\ &= x^{\gamma/2-1} \Gamma(1-\gamma/2) \left(e^{i(1-\gamma/2)\pi/2} + e^{-i(1-\gamma/2)\pi/2} \right) \\ &= 2x^{\gamma/2-1} \Gamma(1-\gamma/2) \cos((1-\gamma/2)\pi/2),\end{aligned}$$

and for $x < 0$,

$$\begin{aligned}\int_{\mathbb{R}} e^{ixu} |u|^{-\gamma/2} du &= \int_0^\infty e^{-iv} v^{-\gamma/2} (-x)^{\gamma/2-1} dv + \int_0^\infty e^{iv} v^{-\gamma/2} (-x)^{\gamma/2-1} dv \\ &= 2(-x)^{\gamma/2-1} \Gamma(1-\gamma/2) \cos((1-\gamma/2)\pi/2).\end{aligned}$$

Thus, by (B.1.14) in [44], we have

$$\int_{\mathbb{R}} |u|^{-\gamma/2} B(du) \stackrel{\mathrm{d}}{=} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{\varphi}(x) d\widehat{B}(dx),$$

and then the spectral representation follows from Proposition 2.6.10 as in the first proof of Proposition 2.6.11 for FBMs in [44]. □

Remark 5.1. One obtains a further variation by considering

$$c\int_0^t (t-u)^{\alpha} u^{-\gamma/2} B(du), \quad t \geq 0, \tag{63}$$

where $B(du)$ is a Gaussian random measure on $\mathbb{R}$ with the Lebesgue control measure du and

$$c \in \mathbb{R}, \quad \gamma > 0, \quad \alpha \in \Big(-\frac{1}{2} + \frac{\gamma}{2}, \frac{1}{2} + \frac{\gamma}{2}\Big).$$

Such a process is also a continuous self-similar Gaussian process with Hurst parameter $H = \alpha - \frac{\gamma}{2} + \frac{1}{2} \in (0,1)$. For $0 \leq s \leq t$,

$$E[(X(t)-X(s))^2] = \int_s^t (t-u)^{2\alpha} u^{-\gamma} du + \int_0^s |(t-u)^{\alpha} - (s-u)^{\alpha}|^2 u^{-\gamma} du.$$

Note that when $\gamma = 0$, it reduces to the Riemann-Liouville (R-L) FBM introduced by Lévy ([31], see also Chapter 6 in [44]; modulo some constant scaling). Recall that the increments of R-L FBM are also non-stationary. The R-L FBM has recently been used to define the rough Bergomi model to study stochastic volatility in [4]. □

5.2. **A second class.** An alternative generalization is to consider a process Y defined by the (time-domain) integral representation as follows:

$$\{Y(t)\}_{t\in\mathbb{R}} \stackrel{\mathrm{d}}{=} \Big\{\int_{\mathbb{R}} \Big(a_1\left((t-u)_+^{\alpha} - (-u)_+^{\alpha}\right)(-u)_+^{-\gamma/2} + a_2\left((t-u)_-^{\alpha} - (-u)_-^{\alpha}\right)(-u)_-^{-\gamma/2}\Big) B(du)\Big\}_{t\in\mathbb{R}}, \tag{64}$$

where

$$a_i \in \mathbb{R}, \quad i = 1,2, \quad \gamma > 0, \quad \alpha \in \Big(-\frac{1}{2} + \frac{\gamma}{2}, \frac{1}{2} + \frac{\gamma}{2}\Big),$$

and $B(du)$ is a Gaussian random measure on $\mathbb{R}$ with the Lebesgue control measure du. It can be similarly verified that the process Y is well defined in the same way as X. Analogous to Proposition 5.1, we can show that the process $Y(t)$ is a continuous self-similar Gaussian process with Hurst parameter $H = \alpha - \frac{\gamma}{2} + \frac{1}{2} \in (0,1)$. It has covariance function

$$\begin{aligned}\mathrm{Cov}(Y(t), Y(s)) = \int_{\mathbb{R}} &\Big(a_1\left((t-u)_+^{\alpha} - (-u)_+^{\alpha}\right)(-u)_+^{-\gamma/2} + a_2\left((t-u)_-^{\alpha} - (-u)_-^{\alpha}\right)(-u)_-^{-\gamma/2}\Big) \\ &\times \Big(a_1\left((s-u)_+^{\alpha} - (-u)_+^{\alpha}\right)(-u)_+^{-\gamma/2} + a_2\left(s-u\right)_-^{\alpha} - (-u)_-^{\alpha}\big)(-u)_-^{-\gamma/2}\Big) du,\end{aligned}$$

and the second moment of its increment

$$\begin{aligned}&E[(Y(t)-Y(s))^2] \\ &= \int_{\mathbb{R}} \Big(a_1\left((t-u)_+^{\alpha} - (s-u)_+^{\alpha}\right)(-u)_+^{-\gamma/2} + a_2\left((t-u)_-^{\alpha} - (s-u)_-^{\alpha}\right)(-u)_-^{-\gamma/2}\Big)^2 du,\end{aligned}$$

for $s, t \in \mathbb{R}$.

Remark 5.2. Note that the process $X(t)$ is equivalent in distribution to the process $Y(t)$ only when $a_1 = 0$. Indeed, by direct calculations, we obtain that the variances of $X(t)$ and $Y(t)$ are

$$\begin{aligned}\mathrm{Var}(X(t)) &= \int_{\mathbb{R}} \Big(a_1\left((t-u)_+^{\alpha} - (-u)_+^{\alpha}\right) + a_2\left((t-u)_-^{\alpha} - (-u)_-^{\alpha}\right)\Big)^2 |u|^{-\gamma} du \\ &= \int_{-\infty}^{0} \Big(a_1\left((t-u)_+^{\alpha} - (-u)_+^{\alpha}\right)\Big)^2 (-u)_+^{-\gamma} du + \int_0^t \Big(a_1 (t-u)_+^{\alpha}\Big)^2 u^{-\gamma} du \\ &\quad + \int_0^{\infty} \Big(a_2\left((t-u)_-^{\alpha} - (-u)_-^{\alpha}\right)\Big)^2 (-u)_-^{-\gamma} du,\end{aligned}$$

and

$$\begin{aligned}\mathrm{Var}(Y(t)) &= \int_{\mathbb{R}} \Big(a_1\left((t-u)_+^{\alpha} - (-u)_+^{\alpha}\right)(-u)_+^{-\gamma/2} + a_2\left((t-u)_-^{\alpha} - (-u)_-^{\alpha}\right)(-u)_-^{-\gamma/2}\Big)^2 du \\ &= \int_{-\infty}^{0} \Big(a_1\left((t-u)_+^{\alpha} - (-u)_+^{\alpha}\right)(-u)_+^{-\gamma/2}\Big)^2 du \\ &\quad + \int_0^{\infty} \Big(a_2\left((t-u)_-^{\alpha} - (-u)_-^{\alpha}\right)(-u)_-^{-\gamma/2}\Big)^2 du.\end{aligned}$$

It is then evident that $Var(X(t)) = Var(Y(t))$ only when $a_1 = 0$.

Also, when $a_2 = 0$, $X(t)$ and $Y(t)$ are represented (in distribution) as

$$X(t) \stackrel{\mathrm{d}}{=} \int_{\mathbb{R}} \Big(a_1\left((t-u)_+^{\alpha} - (-u)_+^{\alpha}\right)\Big)|u|^{-\gamma/2} B(du),$$

and

$$Y(t) \stackrel{\mathrm{d}}{=} \int_{\mathbb{R}} \Big(a_1\left((t-u)_+^{\alpha} - (-u)_+^{\alpha}\right)(-u)_+^{-\gamma/2}\Big) B(du),$$

respectively, where the integral for $X(t)$ goes from $-\infty$ to t, while that for $Y(t)$ goes from $-\infty$ only up to 0.

The spectral representation of Y is

$$\{Y(t)\}_{t\in\mathbb{R}} \stackrel{\mathrm{d}}{=} \Big\{\frac{1}{\sqrt{2\pi}}\Gamma(\alpha+1)\int_{\mathbb{R}} \frac{e^{itx}-1}{ix}\big(a_1(-ix)^{-\alpha}\widehat{\varphi}_+(x) + a_2(ix)^{-\alpha}\widehat{\varphi}_-(x)\big)\widehat{B}(dx)\Big\}_{t\in\mathbb{R}}, \tag{65}$$

where

$$\begin{aligned}\widehat{\varphi}_+(x) &:= |x|^{\gamma/2-1}\Gamma(1-\gamma/2)e^{\mathrm{sgn}(x)i(1-\gamma/2)\pi/2}, \quad x\in\mathbb{R}, \\ \widehat{\varphi}_-(x) &= |x|^{\gamma/2-1}\Gamma(1-\gamma/2)e^{-\mathrm{sgn}(x)i(1-\gamma/2)\pi/2}, \quad x\in\mathbb{R},\end{aligned}$$

$\mathrm{sgn}(x) = 1$ if $x > 0$ and $\mathrm{sgn}(x) = -1$ if $x < 0$, and $\widehat{B}$ is a Hermitian Gaussian measure on $\mathbb{R}$ with the Lebesgue control measure dx. Note that by (2.6.6) in [44], we have

$$\begin{aligned}\widehat{\varphi}_+(x) &= \int_{\mathbb{R}} e^{ixu}(-u)_+^{-\gamma/2} du = \int_0^{\infty} e^{-ixu}u^{-\gamma/2} du \\ &= x_+^{\gamma/2-1}\Gamma(1-\gamma/2)e^{i(1-\gamma/2)\pi/2} + x_-^{\gamma/2-1}\Gamma(1-\gamma/2)e^{-i(1-\gamma/2)\pi/2} \\ &= |x|^{\gamma/2-1}\Gamma(1-\gamma/2)e^{\mathrm{sgn}(x)i(1-\gamma/2)\pi/2},\end{aligned} \tag{66}$$

and $\widehat{\varphi}_-(x)$ follows from similar calculations. The spectral representation follows from (B.1.14) and Proposition 2.6.10 in [44].

5.3. **A third class.** Motivated by the shot noise process in Case (ii), we introduce a third generalization of standard FBMs. Consider a process Z defined by the integral representation:

$$\{Z(t)\}_{t\in\mathbb{R}} \stackrel{\mathrm{d}}{=} \left\{ \int_{\mathbb{R}} \Big(a_1\left((t-u)_+^\alpha - (-u)_+^\alpha\right) + a_2\left((t-u)_-^\alpha - (-u)_-^\alpha\right) \Big) \widetilde{B}(du) \right\}_{t\in\mathbb{R}}, \tag{67}$$

where $a_i \in \mathbb{R}$, $i=1,2$, $\alpha \in (-1/2,1/2)\setminus\{0\}$, $\widetilde{B}(du)$ is a Gaussian random measure on $\mathbb{R}$ with the control measure

$$\sigma(du) = \sigma_-\mathbf{1}(u<0)du + \sigma_0\delta_0(du) + \sigma_+\mathbf{1}(u>0)du$$

for $\sigma_\pm > 0$ and $\sigma_0 > 0$ and $\delta_0(du)$ is the Dirac measure at 0.

It is easy to verify that the process Z is well defined, and it is continuous self-similar with Hurst parameter $H = \alpha + 1/2 \in (0,1)\setminus\{1/2\}$. The covariance function and second moment of its increments can be also calculated similarly. In particular, with $\sigma_0 = 0$, we have the variance function

$$\begin{aligned}\mathrm{Var}(Z(t)) = &\int_{-\infty}^0 \Big(a_1^2\left((t-u)_+^\alpha - (-u)_+^\alpha\right)^2 \Big)\sigma_-^2 du \\ &+ \int_0^\infty \Big(a_1\left((t-u)_+^\alpha\right) + a_2\left((t-u)_-^\alpha - (-u)_-^\alpha\right) \Big)^2 \sigma_+^2 du.\end{aligned} \tag{68}$$

However, an FBM B_H with the representation

$$\int_{\mathbb{R}} \Big(a_1\left((t-u)_+^\alpha - (-u)_+^\alpha\right) + a_2\left((t-u)_-^\alpha - (-u)_-^\alpha\right) \Big) B(du)$$

for $\alpha = H - 1/2$, has variance

$$\begin{aligned}\mathrm{Var}(B_H(t)) &= \int_{\mathbb{R}} \Big(a_1\left((t-u)_+^\alpha - (-u)_+^\alpha\right) + a_2\left((t-u)_-^\alpha - (-u)_-^\alpha\right) \Big)^2 du \\ &= \int_{-\infty}^0 \Big(a_1^2\left((t-u)_+^\alpha - (-u)_+^\alpha\right)^2 \Big) du \\ &\quad + \int_0^\infty \Big(a_1\left((t-u)_+^\alpha\right) + a_2\left((t-u)_-^\alpha - (-u)_-^\alpha\right) \Big)^2 du.\end{aligned} \tag{69}$$

Thus it can be seen that if $\sigma_- \neq \sigma_+$, the process $Z(t)$ cannot be an FBM.

The spectral representation of Z is

$$\{Z(t)\}_{t\in\mathbb{R}} \stackrel{\mathrm{d}}{=} \left\{ \frac{1}{\sqrt{2\pi}}\Gamma(\alpha+1) \int_{\mathbb{R}} \frac{e^{itx}-1}{ix} \big(a_1(-ix)^{-\alpha} + a_2(ix)^{-\alpha}\big)\widehat{\sigma}(x)\widehat{B}(dx) \right\}_{t\in\mathbb{R}}, \tag{70}$$

where

$$\widehat{\sigma}(x) = \int_{\mathbb{R}} e^{ixu}\sigma(du), \quad x \in \mathbb{R},$$

and $\widehat{B}$ is a Hermitian Gaussian measure on $\mathbb{R}$ with the Lebesgue control measure dx.

Note that by convention, when $\alpha = 0$, for $a_1 = 1$ and $a_2 = 0$, we have

$$\int_{[0,t)} \widetilde{B}(du) \stackrel{\mathrm{d}}{=} \sigma_0 Z_0 + \sigma_+ B(t)$$

where $Z_0 \sim N(0,1)$ and $B(t)$ is the standard Brownian motion, independent of Z_0. The process is not self-similar unless $\sigma_0 = 0$, in which case the process $Z(t)$ reduces to a Brownian motion with variance coefficient σ_+. In general, when $\alpha = 0$, assuming $\sigma_0 = 0$, the process $Z(t)$ in (67) is self-similar with non-stationary increments.

6. Appendix: A sufficient condition for weak convergence

A classical sufficient condition for a sequence of stochastic processes X_n converging weakly to X in space $(\mathbb{D}, J_1)$ is stated in Theorem 13.5 in [5]. The condition involves probability bounds for the increments of the processes, which turns out to be straightforward to verify in many applications (see, e.g., Example 2.2.12 in [56] and the proof of Donsker's theorem in Section 14 of [5]). The underlying proof for that condition relies on two maximal inequalities (Theorems 10.3 and 10.4 in [5]). A critical assumption in those maximal inequalities is the presence of a *finite measure* in the probability bounds. However, such a finite measure may not be always attainable, but instead one can use a set function with a superadditivity property. In [41], the maximal inequalities are extended for such scenarios in the study of shot noise processes with non-stationary noise distributions. As a consequence, we obtain a sufficient condition to prove weak convergence of stochastic processes with sample paths in $\mathbb{D}$ which generalizes the classical sufficient criterion, Theorem 13.5 in [5]. The criterion was implicitly used in [41]. We state it as a theorem and prove it here.

We need the following definition of a set function with a superadditivity property.

Definition 6.1. *Let μ be a set function from the Borel subset of $\mathbb{R}_+$ into $\mathbb{R}_+ \cup \{\infty\}$ such that*

(i) *μ is nonnegative and $\mu(\emptyset) = 0$;*
(ii) *μ is monotone, that is, if $A \subseteq B \subset \mathbb{R}_+$, then $\mu(A) \le \mu(B)$;*
(iii) *μ is superadditive, that is, for any disjoint Borel sets A and B, $\mu(A) + \mu(B) \le \mu(A \cup B)$.*

Let $\{X_n : n \in \mathbb{N}\}$ and X be stochastic processes with sample paths in $\mathbb{D}$, and consider the interval $[0, T]$ for some $T > 0$.

Theorem 6.1. *Suppose that the following hold:*

(i) *The finite dimensional distributions of X_n converge to those of X;*
(ii) *For any $\epsilon > 0$,*
$$\lim_{\delta \to 0} P\left(|X(T) - X(T-\delta)| \ge \epsilon\right) = 0;$$
(iii) *There exists a constant $C > 0$ such that for any $0 \le r \le s \le t \le T$ with $\delta > 0$, $t - r < 2\delta$, and $\epsilon > 0$, and for all $n \ge 1$,*
$$P\big(|X_n(r) - X_n(s)| \wedge |X_n(s) - X_n(t)| \ge \epsilon\big) \le \frac{C}{\epsilon^{4\beta}} \big(\mu(r,t]\big)^{2\alpha}, \tag{71}$$
where μ is a finite set function as in Definition 6.1, $\alpha > 1/2$ and $\beta \ge 0$;
(iv) *$\mu(0,t]$ is continuous in t.*

Then $X_n \Rightarrow X$ in $(\mathbb{D}[0,T], J_1)$ as $n \to \infty$.

Proof. By Theorem 13.3 in [5], it suffices to show that
$$\lim_{\delta \to 0} \limsup_n P\left(\sup_{\substack{0 \le r < s < t \le T, \\ t - r \le 2\delta}} |X_n(r) - X_n(s)| \wedge |X_n(s) - X_n(t)| \ge \epsilon \right) = 0. \tag{72}$$
By condition (iii) and Theorem 5.2 in [40], condition (72) reduces to
$$\lim_{\delta \to 0} \sup_{0 \le t \le T - 2\delta} \big(\mu(t, t+2\delta]\big)^{2\alpha - 1} = 0.$$
Since μ is superadditive (Definition 6.1 (iii)), we obtain that for $0 \le t \le T - 2\delta$ and $\delta > 0$,
$$\mu(t, t+2\delta] \le \mu(0, t+2\delta] - \mu(0,t].$$
Since $\alpha > 1/2$ and $\mu(0,t]$ is continuous on the interval $[0,T]$, we have
$$\lim_{\delta \to 0} \sup_{0 \le t \le T - 2\delta} \big(\mu(t, t+2\delta]\big)^{2\alpha - 1}$$

$$\leq \lim_{\delta\to 0} \sup_{0\leq t\leq T-2\delta} \big(\mu(0,t+2\delta]-\mu(0,t]\big)^{2\alpha-1} = 0.$$

This completes the proof. □

In addition, the new maximal inequalities result in the following new criterion (Theorem 5.3 in [41]) to prove the existence of a stochastic process with sample paths in $\mathbb{D}$ given its finite dimensional distributions, extending the classical criterion in Theorem 13.6 in [5].

Theorem 6.2. *There exists a random element X in $\mathbb{D}([0,T],\mathbb{R})$ with finite-dimensional distributions $\pi_{t_1,\cdots,t_k}$ for any $0\leq t_1<\cdots<t_k\leq T$, that is, $\pi_{t_1,\cdots,t_k}(x_1,\dots,x_k)=P(X(t_1)\leq x_1,\dots,X(t_k)\leq x_k)$ for $x_i\in\mathbb{R}$, $i=1,...,k$, if the following conditions are satisfied:*

(i) *the finite dimensional distributions $\pi_{t_1,\cdots,t_k}$ are consistent, satisfying the conditions of Kolmogorov's existence theorem;*

(ii) *for any $0\leq r\leq s\leq t\leq T$, $\beta\geq 0$, $\alpha>1/2$ and $\epsilon>0$,*

$$P\big(|X(r)-X(s)|\wedge|X(s)-X(t)|\geq\epsilon\big)\leq\frac{C}{\epsilon^{4\beta}}\big(\mu(r,t]\big)^{2\alpha}, \tag{73}$$

where C is a positive constant, μ is a finite set function in Definition 6.1 and $\mu(0,t]$ is continuous in t;

(iii) *for any $\epsilon>0$ and $t\in[0,T)$,*

$$\lim_{\delta\downarrow 0} P\big(|X(t)-X(t+\delta)|>\epsilon\big)=0. \tag{74}$$

Acknowledgement

This work was initiated and conducted when both authors were visiting Columbia University. We thank the university for providing us the opportunity and hospitality. G. Pang is partly supported by NSF grants CMMI-1538149, CMMI-1635410, and DMS/CMMI-1715875 and the Army Research Office Grant W911NF-17-1-0019. M. S. Taqqu is partly supported by the Simons grant 569118.

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† The Harold and Inge Marcus Department of Industrial and Manufacturing Engineering, Pennsylvania State University, University Park, PA 16802
Email address: gup3@psu.edu

* Department of Mathematics and Statistics, Boston University, Boston, MA 02215
Email address: murad@math.bu.edu