Ergodic Risk Sensitive Control of Large-Scale Parallel Server Networks with Abandonment

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Abstract. This paper studies the optimal scheduling problem for Markovian multiclass multipool queueing networks with abandonment in the Halfin–Whitt regime, under the ergodic risk sensitive cost (ERSC) criterion where the running cost penalizes the queue length and/or the idleness of the servers. Under the assumption of complete resource pooling, we show that the optimal ERSC cost associated with the diffusion-scaled queueing processes converges to the optimal ERSC cost associated with the limiting diffusion. Due to the lack of the uniform stability property of the controlled limiting diffusion, we employ the recently developed theory of ERSC of diffusions in Anugu and Pang (2024a), under a general structural risk-sensitive hypothesis with mixed conditions on the uniform stability of the controlled diffusion and near-monotonicity of the running cost over properly partitioned subspaces. In order to prove the asymptotic optimality, we make use of the variational representations for Brownian motion and Poisson process, so that the ERSC problems can be represented as classical ergodic control problems with auxiliary controls (as maximization problems), for the associated "extended" diffusion (with the auxiliary controls in the drift) and "extended" diffusion-scaled processes driven by inhomogeneous Poisson process of instantaneous rates obtained from the auxiliary controls. In particular, for these "extended" processes, we also relate them to two-person zero-sum stochastic games with ergodic pay-off criteria. A key step is then to establish the tightness of the mean empirical measures of the "extended" processes together with auxiliary controls, particularly involving the boundedness of the relative entropy term in the variational representations. The proof of the lower bound makes use of the optimal control characterizations and properties of the stochastic game for the "extended" diffusion. The proof for the upper bound uses a truncation argument in order to tackle the challenges to bound the relative entropy term, from which we obtain the desired tightness. With the tightness property, the mean empirical measures of the "extended" diffusion-scaled processes converge weakly to the ergodic occupation measure of the "extended" diffusion, which then leads us to obtain the lower and upper bounds for asymptotic optimality.

1. Introduction

In this paper, we study the ergodic risk sensitive control (ERSC) problem of Markovian multiclass multipool queueing networks with abandonment under the Halfin–Whitt regime (parametrized by n). Our objective is to minimize the ERSC cost given by

$$\limsup_{T\to\infty}\frac{1}{T}\log\mathbb{E}\Big[\exp\Big(\int_0^T \hat{r}(\hat{Q}^n(t),\hat{Y}^n(t))\mathrm{d}t\Big)\Big], \tag{1.1}$$

over all joint work conserving scheduling control policies (SCPs), where $\hat{r}$ is the running cost that penalizes the diffusion-scaled queue-length process for different classes of customers $\hat{Q}^n$ and/or the diffusion-scaled numbers of idle servers for different server pools $\hat{Y}^n$. It is well-known (from the

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works of Atar in [16, 17]) that in the Halfin–Whitt asymptotic regime, under the complete resource pooling condition, the diffusion-scaled queueing processes converges in distribution to a limiting diffusion model. The ERSC problem for the limiting diffusion X concerns with minimizing

$$\limsup_{T\to\infty}\frac{1}{T}\log\mathbb{E}\Big[\exp\Big(\int_0^T \hat{r}\big((e\cdot X(t))^+U^c(t),(e\cdot X(t))^-U^s(t)\big)\mathrm{d}t\Big)\Big], \tag{1.2}$$

over an appropriate class of controls $U(t)=(U^c(t),U^s(t))$. Here, $U^c(t)$ and $U^s(t)$ represent the limiting vectors of the proportion of queue lengths of different customer classes and the proportion of idle servers in different server pools, respectively. The main contribution of this paper is to show that the sequence of optimal values of the ERSC problems associated with the diffusion-scaled queueing processes (referred to simply as the nth model) under the Halfin–Whitt regime approaches the optimal value of the ERSC problem associated with the limiting diffusion, as $n\to\infty$. We refer to the above convergence property as the asymptotic optimality (AO). The usefulness of the AO property (in the context of ERSC) is that one can construct nearly optimal SCPs for the ERSC problem of the nth model for large n, from the optimal controls of the ERSC problem for the limiting diffusion model.

We first understand the well-posedness of the ERSC problems for both the limiting diffusion and the diffusion-scaled processes. We choose appropriate Lyapunov functions and prove Foster-Lyapunov inequalities for these processes under certain Markov controls, which then imply that the ERSC costs are finite, hence, implying the well-posedness of the ERSC problems (see Propositions 3.1 and 3.2). For the limiting diffusion, it is not clear if it is uniformly stable under all stationary Markov controls - this is currently an open problem. In addition, the running cost function is neither bounded nor satisfying the near-monotonicity condition (where the running cost is strictly greater than the optimal ERSC value). Hence, the theory of ERSC of diffusions in [3, 7] under the uniform stability and/or near-monotonicity condition (see also the relevant work in [28, 29, 33]) cannot be applied to our ERSC problem of the limiting diffusion. We therefore resort to the new theory of ERSC of diffusions recently developed in [1], which assumes an existence of a stationary Markov control with finite ERSC cost and a general structural risk-sensitive hypothesis that imposes a mixed condition of uniform stability of the controlled diffusion and the near-monotone condition on the running cost over properly partitioned subspaces. In Proposition 4.1, we choose an inf-compact function and a proper partition of the state space, in order to verify the general structural risk-sensitive hypothesis holds. As a result we are able to characterize the optimal controls for the ERSC problem of the limiting diffusion (see Theorem 4.1). In addition, we also establish the existence of the nearly optimal stationary Markov control (that is continuous) which will be used for the upper bound proof of asymptotic optimality (see Proposition 4.2).

Next, to prove the asymptotic optimality, we extend the approach in [2] that was first introduced for the multiclass 'V' network, where the uniform stability of 'V' network model (established in [9]) is used heavily. Because of this, our extension is highly non-trivial as the limiting diffusion and diffusion-scaled processes in our case are not known to be uniformly stable. We make use of variational representations of exponential functionals of Brownian motion and Poisson processes (see Lemmas 5.1 and 5.3) in order to represent the ERSC costs for the limiting diffusion and for the diffusion-scaled processes as the corresponding classical ergodic control costs with additional maximizations over certain auxiliary controls. In particular, in these reformulations, the associated running costs (which are referred to as "extended" running costs from hereon) are the differences of the original running costs and the associated relative entropy terms, and the associated processes are the so-called "extended" diffusion which inherits the similar evolution equation as the limiting diffusion, but with an additional drift obtained from the auxiliary controls and the "extended" diffusion-scaled queueing processes which inherit same evolution equations as diffusion-scaled queueing processes, but instead, driven by inhomogeneous Poisson process with the instantaneous rates obtained from the auxiliary controls. In Theorem 5.1, it is shown that the family "extended" diffusion-scaled queueing processes associated with a family of auxiliary controls (that are tight in an appropriate topology; see Section 5.3), converges in distribution sequentially, to the "extended" diffusion associated with a weak limit point of the family of auxiliary controls.

The most important consequence of these representations is that the ERSC costs become linear in the original running costs, although some additional maximization problems in terms of the auxiliary controls are introduced. Consequently, this means that the ERSC costs can be written as integrals of the "extended" running costs over the mean empirical measures (MEMs) associated with the "extended" processes. This enables us to analyze the ERSC problems using the existing techniques from the classical ergodic control (CEC) theory (see [8, Chapter 3]). On the other hand, we have to analyze the tightness of the MEMs of the "extended" processes, which is often harder to study and analyze, even when the original process is exponentially ergodic. For instance, the stability of the ground diffusion, a special case of "extended" diffusions cannot be obtained directly from the stability of the original process (see Remark 5.2 for the definition of ground diffusion and relevant discussion). Furthermore, the maximization problem arising from the variational representation and the minimization problem from the objective of the ERSC problem combined together lead to a stochastic min-max problem (a two-person zero-sum (TP-ZS) stochastic game) under an ergodic pay-off criterion with running cost that is neither bounded from above nor below. Additionally, the maximizing strategy can take values in a non-compact space - this problem is again harder to analyze. TP-ZS stochastic games in the context of diffusions are extensively studied in the literature, albeit under more restrictive conditions than what is necessary for our case; for example, see [34, 36, 40, 42, 47, 50, 51]. These conditions for instance, include bounded running costs and compactness of the space of strategies. However, the results from [34] are used in conjunction with a non-trivial perturbation and truncation argument in [1] to analyze the aforementioned TP-ZS stochastic game. Importantly, it turns out that the availability of the tools from the CEC theory and their usefulness outweigh the aforementioned difficulties.

We now explain the methodology for the proof of asymptotic optimality which involves proving the lower and upper bounds. One critical component in both the lower and upper bound proofs concerns the tightness of the MEMs of the "extended" processes with the auxiliary controls arising from the variational representations. For an arbitrary auxiliary control, this is difficult to deduce in a straightforward manner from the existing ergodicity properties of the limiting diffusion and the diffusion-scaled processes. However, if the auxiliary control is such that the relative entropy term is bounded, then it is easy to show the desired tightness of the MEMs of the associated "extended" processes (see Proposition 6.2). Hence, one of the main steps in both the proofs involves bounding the aforementioned relative entropy term.

In the proof of the lower bound, we choose a nearly optimal SCP and a careful choice of an auxiliary control for the nth model. This choice uses the results from [1], where the authors show that the optimal ERSC value for the limiting diffusion model can be written as the value of TP-ZS stochastic game (where, the operations minimum and maximum are interchangeable). In particular, the auxiliary control is constructed using nearly optimal compactly supported continuous maximizing strategies (Proposition 5.2). The continuity and the compactness of the support of the maximizing strategy imply the boundedness of the entropy term (in the variational representation). As mentioned above, Proposition 6.2 then implies the tightness of the MEMs of the "extended" diffusion-scaled process.

In the proof of the upper bound, we begin by constructing an SCP for the nth model from a nearly optimal stationary Markov control of the ERSC problem for the limiting diffusion model - this SCP is a priori suboptimal. The constructed SCP help us employ a truncation argument using a 'truncated' ERSC cost. Again, expressing this 'truncated' ERSC cost using the aforementioned variational representations, immediately gives the boundedness of the entropy term. This, in turn, gives us the tightness of the MEMs of the "extended" diffusion-scaled process, using Proposition 6.2.

Using the obtained tightness and the notion of ergodic occupation measures, we identify the weak limit points of MEMs to be the ergodic occupation measures of the "extended" diffusion. Consequently, this helps us in concluding the lower and upper bounds.

Literature review. Optimal control problems of multiclass multipool networks in the Halfin-Whitt regime have been studied under the finite-horizon cost criterion [38, 39, 43, 44, 45], the infinite-horizon discounted cost criterion [16, 17, 24], and the infinite-horizon long-run average (ergodic) cost criterion [6, 11, 12, 14, 15, 31, 52]. Under the ergodic risk sensitive criterion, the multiclass 'V' network is recently studied in [2].

On the other hand, our work also complements the literature on risk sensitive control of queueing networks under various asymptotic regimes. The finite-horizon risk sensitive control problems are studied for various queueing networks in [21, 22, 23, 25] under the large deviation regime, and in [18, 19, 20, 26, 30] under the moderate deviation regime. In these works, the problems are studied using the associated TP-ZS deterministic game formulations. Our work differs from these works in a fundamental way: similarly to the multiclass 'V' network in [2], our proofs make use of the properties of the associated TP-ZS stochastic games.

In the literature, various works on variational formulation of risk-sensitive control (in both finite and infinite horizons) of diffusions problems can be found. The relevant works here include [4, 5, 7, 33, 41, 48, 49]. To the best of our knowledge, using the theory of large deviations, a variational formulation in the case of finite horizon risk-sensitive control is first studied in [53]. For very extensive surveys on ERSC problems on Markov processes, we refer the reader to recent surveys like [27, 32].

1.1. **Organization of the paper.** In the rest of this section, we introduce the necessary notation used in the paper. In Section 2.1, we give a detailed introduction of the multiclass multipool large scale network models in the Halfin–Whitt regime. In Section 2.2, we state the ERSC problem for the diffusion-scaled processes. In Section 2.3, we introduce the limiting diffusion model and the ERSC problem associated with it. In Section 2.4, we state the main result (which is Theorem 2.1) of the paper and provide a sketch of its proof. In Section 3, we establish the well-posedness of the ERSC problems defined in Section 2. In Section 4, we discuss the ERSC problem for the limiting diffusion model in greater detail. In Section 5, we introduce variational formulations of the ERSC problems (in terms of the "extended" processes) for both the limiting diffusion and the diffusion-scald processes, respectively, and also present the associated properties and results that will be used for the proof of the asymptotic optimality. Finally, we conclude with the proof of the main result Theorem 2.1 in Section 6. In the Appendix, we collect some auxiliary results and supporting materials.

1.2. **Notation.** We use $(\Omega, \mathcal{F}, \mathbb{P})$ to denote the underlying abstract probability space with $\mathbb{E}$ as the associated expectation. $\mathbb{E}_x$ denotes the expectation when the underlying process starts at x. For every $k \in \mathbb{N}$, the standard Euclidean norm on $\mathbb{R}^k$ is denoted by $\|\cdot\|$, $x \cdot y$ denotes the inner product of $x, y \in \mathbb{R}^k$, and x^{T} denotes the transpose of $x \in \mathbb{R}^k$. The set of nonnegative real numbers (integers) is denoted by $\mathbb{R}_+$ ($\mathbb{Z}_+$), $\mathbb{N}$ stands for the set of natural numbers, and $\mathbb{1}_A(\cdot)$ denotes the indicator function corresponding to set A. The minimum (maximum) of two real numbers a and b is denoted by $a \wedge b$ ($a \vee b$), respectively, and $a^{\pm} \doteq (\pm a) \vee 0$. B_R denotes a Euclidean ball of radius R around the origin and τ_R denotes the first exit time of B_R by the underlying process (similarly, τ_R^n denotes the first exit time of B_R by the underlying diffusion-scaled process). For any k, we let $e^{(k)} \doteq (1, \dots, 1)^{\mathsf{T}} \in \mathbb{R}^k$ and let $e_i \in \mathbb{R}^k$ be the vector with ith entry as 1 and rest of the entries as zeros. Whenever the underlying dimension k is evident from the context, we simply write e instead of $e^{(k)}$.

For $k \in \mathbb{N}$, we let $\mathfrak{D}^k \doteq D(\mathbb{R}_+, \mathbb{R}^k)$ ($\mathfrak{C}^k \doteq C(\mathbb{R}_+, \mathbb{R}^k)$, respectively) denote the space of $\mathbb{R}^k$–valued càdlàg functions equipped with Skorohod topology (continuous functions equipped with locally uniform topology, respectively) on $\mathbb{R}_+$. Whenever the domain is $[0,T]$, we write $\mathfrak{D}^k_T$ or $\mathfrak{C}^k_T$. For any path f in either $\mathfrak{D}^k$ or $\mathfrak{C}^k$, we write $f([0,T])$ to denote the entire path on interval $[0,T]$. When $k=1$, we drop the superscript 1 in these notations. For a Polish space $\mathcal{X}$, $\mathcal{P}(\mathcal{X})$ is the set of Borel probability measures on $\mathcal{X}$ equipped with the topology of weak convergence. For a function $f:\mathbb{R}^k\to\mathbb{R}$,

$$\mathfrak{d}f(x;y) \doteq f(x+y)-f(x). \tag{1.3}$$

We say a family of probability measures $\{\mu_T : T>0\}\subset\mathcal{P}(\mathbb{R}^k)$ is the family of mean empirical measures (MEM) or simply MEMs of a $\mathbb{R}^k$–valued process Z if μ_T satisfies the following:

$$\mu_T(A) = \frac{1}{T}\int_0^T \mathbb{1}_A\big(Z(t)\big)\mathrm{d}t,$$

for every Borel set $A\subset\mathbb{R}^k$. Throughout the paper, we use many positive constants in proofs whose existence is more important and their actual values. Whenever this happens, we use C to denote these constants and this consequently means that the value of C can change from line to line. Occasionally, we use $\overline{C}$ and $\underline{C}$, in addition, if it is necessary. However when constants appear in statements of results like theorem, lemma, remark etc. and in between, we use C_0, $C_1,\ldots$, and C'_0, $C'_1,\ldots$, respectively. The values of these can also change from instance to instance. We occasionally also use M_i, $i\in\mathbb{N}$ to denote positive constants that arise from the variational formulation.

2. Model and Results

2.1. **Multi-class multi-pool Markovian queueing networks with abandonment.** We consider a sequence of multi-class multi-pool queueing networks whose associated variables, parameters and processes are indexed by n. There are I customer classes and J server pools labeled $1,\ldots,I$ and $1,\ldots,J$, respectively, which are fixed, and set $\mathcal{I}\doteq\{1,\ldots,I\}$ and $\mathcal{J}\doteq\{1,\ldots,J\}$. Customers of each class form their own queue and are served on a first-come-first-served (FCFS) basis. Queues of all the classes can accommodate infinitely many customers from their respective classes, *i.e.,* the buffer capacity is infinite. Customers can abandon/renege while waiting in queue. Class i customers can be served by a subset of server pools, denoted by $\mathcal{J}(i)$, and each server from server pool j can serve a subset of customer classes, denoted by $\mathcal{I}(j)$. Defining $\mathcal{E}\doteq\{(i,j)\in\mathcal{I}\times\mathcal{J} : j\in\mathcal{I}(i)\}$, we can construct a bipartite graph $\mathcal{G}$ with the vertex set $\mathcal{I}\cup\mathcal{J}$ and the set of edges $\mathcal{E}$. We say $i\sim j$, if $(i,j)\in\mathcal{E}$ and $i\nsim j$, otherwise. We assume that the graph $\mathcal{G}$ is a tree. Also, let $|\mathcal{E}|$ denote the cardinality of $\mathcal{E}$.

For each server pool $j\in\mathcal{J}$, N^n_j is the number of servers that are statistically identical. Set $N^n=\{N^n_j : j\in\mathcal{J}\}$. Class i customers arrive according to a Poisson process with rate $\lambda^n_i>0$ and can abandon the queue at an exponential rate $\gamma^n_i>0$. Class i customers are served at the server pool $j\in\mathcal{J}(i)$ at an exponential rate $\mu^n_{ij}>0$. We assume that the arrival process, the service times, and the abandonment times of all classes are mutually independent. Let

$$\mathbb{R}^{\mathcal{G}}_+ \doteq \{\xi_{ij}\in\mathbb{R}^{I\times J}_+ : \xi_{ij}=0 \text{ for } i\nsim j\} \quad\text{and}\quad \mathbb{Z}^{\mathcal{G}}_+ \doteq \{\xi_{ij}\in\mathbb{Z}^{I\times J}_+ : \xi_{ij}=0 \text{ for } i\nsim j\}.$$

Similarly, let $\mathbb{R}^{\mathcal{G}}\doteq\{\xi_{ij}\in\mathbb{R}^{I\times J} : \xi_{ij}=0 \text{ for } i\nsim j\}$. We consider the network models in the Halfin-Whitt regime (or the Quality-and-Efficiency-Driven (QED) regime), in which the arrival rates and the numbers of servers grow indefinitely as n increases (while the service and abandonment rates are of order one) so as to make the system critically loaded. Precisely, we have the following assumptions.

Assumption 2.1 (Parameter scaling)**.** For $i\in\mathcal{I}$ and $j\in\mathcal{J}$, there exist positive constants λ_i, γ_i and ν_j, non-negative constants μ_{ij} (such that $\mu_{ij}>0$ for $i\sim j$ and $\mu_{ij}=0$ for $i\nsim j$) and, constants $\hat{\lambda}_i$ and $\hat{\mu}_{ij}$ in $\mathbb{R}$ such that as $n\to\infty$,

$$\frac{\lambda^n_i - n\lambda_i}{\sqrt{n}}\to\hat{\lambda}_i,\quad \sqrt{n}(\mu^n_{ij}-\mu_{ij})\to\hat{\mu}_{ij},\quad \frac{N^n_j-n\nu_j}{\sqrt{n}}\to 0,\quad \gamma^n_i\to\gamma_i\,. \tag{2.1}$$

Assumption 2.2. The linear program given by

$$\max_{j\in\mathcal{J}} \sum_{i\in\mathcal{I}} \xi_{ij}, \quad \text{subject to} \ \sum_{j\in\mathcal{J}} \mu_{ij}\nu_j\xi_{ij} = \lambda_i, \ \text{for } i \in \mathcal{I} \text{ and } [\xi_{ij}] \in \mathbb{R}^{\mathcal{G}}_+ ,$$

has a unique solution $\xi^* = [\xi^*_{ij}] \in \mathbb{R}^{\mathcal{G}}_+$ satisfying $\sum_{i\in\mathcal{I}(j)} \xi^*_{ij} = 1$, for $j \in \mathcal{J}$ and $\xi^*_{ij} > 0$, if $i \sim j$.

Remark 2.1. The unique existence of solution as given by Assumption 2.2 implies that the graph $\mathcal{G}$ is indeed a tree (see [17, 54]).

Define $x^* = (x^*_i : i \in \mathcal{I}) \in \mathbb{R}^I_+$ and $z^* = [z^*_{ij} : (i,j) \in \mathcal{E}] \in \mathbb{R}^{\mathcal{G}}_+$ by

$$x^*_i \doteq \sum_{j\in\mathcal{J}} \xi^*_{ij}\nu_j \quad \text{and} \quad z^*_{ij} \doteq \xi^*_{ij}\nu_j \,. \tag{2.2}$$

The vector x^* can be interpreted as the steady-state total number of customers in each class and the matrix z^* as the steady-state total number of customers in each customer class receiving service, in the fluid scale. The quantity ξ^*_{ij} can be interpreted as the steady-state fraction of service allocations of the server pool j to class i jobs in the fluid scale. Observe that $e \cdot x^* = e \cdot \nu$ with $\nu = (\nu_j : j \in \mathcal{J})$.

For $i \in \mathcal{I}$, let $X^n_i \doteq \{X^n_i(t) : t \geq 0\}$ and $Q^n_i \doteq \{Q^n_i(t) : t \geq 0\}$ be the numbers of class i customers in the system and in their respective queues, respectively. For $i \in \mathcal{I}$ and $j \in \mathcal{J}$, $Y^n_j \doteq \{Y^n_j(t) : t \geq 0\}$ denotes the number of idle servers in pool j and $Z^n_{ij} \doteq \{Z^n_{ij}(t) : t \geq 0\}$ denotes the number of class i customers being served at pool j. Let $X^n \doteq (X^n_i : i \in \mathcal{I})$, $Q^n \doteq (Q^n_i : i \in \mathcal{I})$, $Y^n \doteq (Y^n_j : j \in \mathcal{J})$ and $Z^n = \bigl(Z^n_{ij} : i \in \mathcal{I}, j \in \mathcal{J}\bigr)$. It is clear that these processes satisfy the following balance equations:

$$X^n_i(t) = Q^n_i(t) + \sum_{j\in\mathcal{J}(i)} Z^n_{ij}(t), \qquad \text{for } i \in \mathcal{I}, \tag{2.3}$$

$$N^n_j = Y^n_j(t) + \sum_{i\in\mathcal{I}(j)} Z^n_{ij}(t), \qquad \text{for } j \in \mathcal{J} \,. \tag{2.4}$$

We now introduce the state evolution equations. The process Z^n is referred to as the scheduling control policy (SCP), that is, the allocation of servers of each server pool to each class at every time. In this work, we only consider SCPs that satisfy the following properties:

(i) Jointly work conserving (JWC): As soon as a server becomes free and there are customers in queue that the server can serve, the server cannot stay idle and must decide on which customer to serve. Rigorously, this means

$$\bigl(e \cdot Q^n(t)\bigr) \wedge \bigl(e \cdot Y^n(t)\bigr) = 0, \qquad \text{for } \ t \geq 0 \,. \tag{2.5}$$

(ii) Non-anticipative: The SCP does not have any information about the future of the queueing network model. This will be made precise below.

(iii) Pre-emptive: The service of a customer can be interrupted at any time to serve some other customer from another class and be resumed at a later time.

For $(x, z) \in \mathbb{Z}^I_+ \times \mathbb{Z}^{\mathcal{G}}_+$, we define

$$\begin{aligned} q_i(x,z) &\doteq x_i - \sum_{j\in\mathcal{J}(i)} z_{ij}, \qquad \text{for } i \in \mathcal{I}, \\ y^n_j(z) &\doteq N^n_j - \sum_{i\in\mathcal{I}(j)} z_{ij}, \qquad \text{for } j \in \mathcal{J} \,. \end{aligned} \tag{2.6}$$

From these definitions, we can infer that the SCPs as chosen in (2.5) take values in the action set

$$\mathcal{Z}^n(x) \doteq \{z \in \mathbb{R}^{\mathcal{G}}_+ : \bigl(e \cdot q(x,z)\bigr) \wedge \bigl(e \cdot y^n(z)\bigr) = 0, \ q_i(x,z), \ y^n_j(z) \geq 0 \ \text{ for } \ (i,j) \in \mathcal{E}\} \,.$$

This set is the range of values that the action set is allowed to take for a given x such that (2.5) holds. On the other hand, with the constraint that $x \in \mathbb{Z}_+^I, z \in \mathbb{Z}_+^{\mathcal{G}}$, $q \in \mathbb{Z}_+^I$, $y \in \mathbb{Z}_+^J$ and (2.5), it is evident that x is not allowed to take arbitrary values in $\mathbb{Z}_+^I$. Hence, we define the following set which contains all the allowed values of x:

$$\begin{aligned}\mathfrak{X}^n \doteq \Big\{x \in \mathbb{Z}_+^I : &\big(e \cdot q(x,z)\big) \wedge \big(e \cdot y^n(z)\big) = 0,\\ &q_i(x,z),\ y_j^n(z) \ge 0 \ \text{ for } (i,j) \in \mathcal{E},\ \text{for some } z \in \mathcal{Z}^n(x)\Big\}\,. \end{aligned} \tag{2.7}$$

From (2.6), it is clear that

$$(e \cdot x) - (e \cdot N^n) = e \cdot q(x,z) - e \cdot y^n(z)\,. \tag{2.8}$$

From here, it is immediate to see that $\big(e \cdot q(x,z)\big) \wedge \big(e \cdot y^n(z)\big) = 0$ if and only if

$$e \cdot q(x,z) = \big[(e \cdot x) - (e \cdot N^n)\big]^+ \quad \text{and} \quad e \cdot y^n(z) = \big[(e \cdot x) - (e \cdot N^n)\big]^-\,.$$

Thus, considering this constraint, we define the set of all possible pairs of $\big(q(x,z), y^n(z)\big)$ for a given $x \in \mathbb{Z}_+^I$ by

$$\Theta^n(x) \doteq \Big\{(q,y) \in \mathbb{Z}_+^I \times \mathbb{Z}_+^J : e \cdot q(x,z) = \big[(e\cdot x) - (e \cdot N^n)\big]^+ \quad \text{and} \quad e \cdot y^n(z) = \big[(e \cdot x) - (e \cdot N^n)\big]^-\Big\}\,.$$

Recall (2.6). It is clear that

$$\sum_{j \in \mathcal{J}(i)} z_{ij} = x_i - q_i(x,z), \qquad \text{for } i \in \mathcal{I}, \tag{2.9}$$

$$\sum_{i \in \mathcal{I}(j)} z_{ij} = N_j^n - y_j^n(z), \qquad \text{for } j \in \mathcal{J}\,. \tag{2.10}$$

It is desirable to define the map $z = z(x - q, N^n - y)$, if it exists (uniquely), for given $q \in \mathbb{Z}_+^I$, $y \in \mathbb{Z}_+^J$ and $x \in \mathbb{Z}_+^I$. In fact, the following stronger result (which is [16, Proposition A.2]) holds.

Proposition 2.1. *Set $\mathcal{D} \doteq \{(\alpha,\beta) \in \mathbb{R}^I \times \mathbb{R}^J : e \cdot \alpha = e \cdot \beta\}$. Under Assumption 2.2, there exists a unique linear map $\Upsilon : \mathcal{D} \to \mathbb{R}^{I \times J}$ that satisfies*

$$\sum_{j \in \mathcal{J}(i)} \Upsilon_{ij}(\alpha,\beta) = \alpha_i, \quad \textit{for} \quad i \in \mathcal{I} \quad \textit{and} \quad \sum_{i \in \mathcal{I}(j)} \Upsilon_{ij}(\alpha,\beta) = \beta_j, \quad \textit{for} \quad j \in \mathcal{J},$$

and $\Upsilon_{ij}(\alpha,\beta) = 0$, whenever $i \nsim j$.

As an immediate corollary, we have the following.

Corollary 2.1. *For $q \in \mathbb{Z}_+^I$, $y \in \mathbb{Z}_+^J$ and $x \in \mathbb{Z}_+^I$, suppose that $(e \cdot q) \wedge (e \cdot y) = 0$. The family of equations*

$$\sum_{j \in \mathcal{J}(i)} z_{ij} = x_i - q_i, \quad \textit{for} \;\; i \in \mathcal{I} \quad \textit{and} \quad \sum_{i \in \mathcal{I}(j)} z_{ij} = N_j^n - y_j, \quad \textit{for} \;\; j \in \mathcal{J}$$

has a unique solution given by $\mathbb{Z}_+^{\mathcal{G}} \ni z = \Upsilon(x - q, N^n - y)$. In particular, $z = z(x - q, N^n - y)$ depends linearly on q, y and x.

For $(i,j) \in \mathcal{E}$, let A_i^n, R_i^n, S_{ij}^n be mutually independent rate–1 Poisson processes. For $t \ge 0$, define the σ–algebras

$$\mathfrak{F}_t^n \doteq \sigma\Big\{\widetilde{A}_i^n(s), \widetilde{S}_{ij}^n(s), \widetilde{R}_i^n(s) : \text{ for } i \in \mathcal{I},\ j \in \mathcal{J}(i),\ 0 \le s \le t\Big\} \vee \mathcal{N},$$

$$\mathfrak{G}_t^n \doteq \sigma\Big\{\delta\widetilde{A}_i^n(t,r), \delta\widetilde{S}_{ij}^n(t,r), \delta\widetilde{R}_i^n(t,r) : \text{ for } i \in \mathcal{I},\ j \in \mathcal{J}(i),\ r \ge 0\Big\},$$

where $\mathcal{N}$ is the collection of all $\mathbb{P}$–null sets and

$$\widetilde{A}_i^n(t) \doteq A_i^n(\lambda_i^n t), \qquad\qquad \delta\widetilde{A}_i^n(t,r) \doteq \widetilde{A}_i^n(t+r) - \widetilde{A}_i^n(t),$$

$$\widetilde{S}^n_{ij}(t) \doteq S^n_{ij}\Big(\mu^n_{ij}\int_0^t Z^n_{ij}(s)\mathrm{d}s\Big) \qquad \delta\widetilde{S}^n_{ij}(t,r) \doteq S^n_{ij}\Big(\mu^n_{ij}\int_0^t Z^n_{ij}(s)\mathrm{d}s + \mu^n_{ij}r\Big) - \widetilde{S}^n_{ij}(t),$$
$$\widetilde{R}^n_i(t) \doteq R^n_i\Big(\gamma^n_i\int_0^t Q^n_i(s)\mathrm{d}s\Big) \qquad \delta\widetilde{R}^n_i(t,r) \doteq R^n_i\Big(\gamma^n_i\int_0^t Q^n_i(s)\mathrm{d}s + \gamma^n_i r\Big) - \widetilde{R}^n_i(t)\,.$$

It is evident that the σ–algebra $\mathfrak{F}^n_t$ represents the information up to time t and the σ–algebra $\mathfrak{G}^n_t$ represents the information about the future increments of the processes after time t.

Definition 2.1. An SCP $Z^n = \{Z^n_{ij} : (i,j) \in \mathcal{E}, t \ge 0\}$ is said to be admissible if the following conditions hold:

- (i) $Z^n(t) \in \mathcal{Z}^n(X^n(t))$, for $t \ge 0$;
- (ii) $Z^n(t)$ is $\mathfrak{F}^n_t$-adapted;
- (iii) $\mathfrak{F}^n_t$ is independent of $\mathfrak{G}^n_t$ at each time $t \ge 0$;
- (iv) For each $(i,j) \in \mathcal{E}$ and $t \ge 0$, the process $\delta\widetilde{S}^n_{ij}(t,\cdot)$ agrees in law with $S^n_{ij}(\mu^n_{ij}\cdot)$ and the process $\delta\widetilde{R}^n_i(t,\cdot)$ agrees in law with $R^n_i(\gamma^n_i\cdot)$.

We denote the set of admissible SCPs by $\mathfrak{Z}^n$.

Remark 2.2. The sets $\mathcal{Z}^n(\cdot)$, $\mathfrak{X}^n$ and $\Theta^n(\cdot)$ can be understood as follows: assuming that the JWC condition holds, the process X^n under an admissible SCP Z^n always takes values in the set $\mathfrak{X}^n$. On the other hand, given $X^n(t)$ at time t, the SCP Z^n can take any values in the intersection of the set $\Theta^n\big(X^n(t)\big)$ and the range of the map $\mathcal{Z}^n\big(X^n(t)\big) \ni z \mapsto \big(q(X^n(t),z), y^n(z)\big)$.

For an admissible SCP Z^n, the evolution equation for X^n becomes

$$X^n_i(t) = X^n_i(0) + A^n_i(\lambda^n_i t) - \sum_{j\in\mathcal{J}(i)} S^n_{ij}\Big(\mu^n_{ij}\int_0^t Z^n_{ij}(s)\mathrm{d}s\Big) - R^n_i\Big(\gamma^n_i\int_0^t Q^n_i(s)\mathrm{d}s\Big), \tag{2.11}$$

for $i \in \mathcal{I}$ and $t \ge 0$. We assume that $X^n(0)$ is deterministic from hereon.

An admissible SCP is called stationary Markov if $Z^n(t) = z\big(X^n(t)\big)$ for some function $z : \mathbb{Z}^I_+ \to \mathbb{Z}^{\mathcal{G}}_+$. In this case, we simply denote the SCP by z. For a stationary Markov SCP $Z^n(t) = z\big(X^n(t)\big)$, for $t \ge 0$, X^n is a $\mathbb{Z}^I_+$–valued Markov process with the infinitesimal generator given as follows: for $f \in \mathcal{C}^2(\mathbb{R}^I)$ and for $x \in \mathbb{Z}^I_+$,

$$\mathcal{L}^z_n f(x) \doteq \sum_{i\in\mathcal{I}} \lambda^n_i \mathfrak{d}f(x;e_i) + \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}(i)} \mu^n_{ij} z_{ij}(x)\mathfrak{d}f(x;-e_i) + \sum_{i\in\mathcal{I}} \gamma^n_i q_i\big(x,z(x)\big)\mathfrak{d}f(x;-e_i)\,. \tag{2.12}$$

2.2. **ERSC problem for the diffusion-scaled processes.** For $(i,j) \in \mathcal{E}$, define the diffusion-scaled processes

$$\begin{aligned}\hat{X}^n_i(t) &\doteq \frac{X^n_i(t) - nx^*_i}{\sqrt{n}}, & \hat{Z}^n_{ij}(t) &\doteq \frac{Z^n_{ij}(t) - nz^*_{ij}}{\sqrt{n}},\\ \hat{Q}^n_i(t) &\doteq \frac{Q^n_i(t)}{\sqrt{n}}, & \hat{Y}^n_j(t) &\doteq \frac{Y^n_j(t)}{\sqrt{n}}\,.\end{aligned} \tag{2.13}$$

Recall that x^* and z^* are defined in (2.2). For $(i,j) \in \mathcal{E}$, let

$$\begin{aligned}\hat{M}^n_{A,i}(t) &\doteq \frac{A^n_i(\lambda^n_i t) - \lambda^n_i t}{\sqrt{n}},\\ \hat{M}_{R,i}(t) &\doteq \frac{1}{\sqrt{n}}\Big(R^n_i\Big(\gamma^n_i\int_0^t Q^n_i(s)\mathrm{d}s\Big) - \gamma^n_i\int_0^t Q^n_i(s)\mathrm{d}s\Big),\\ \hat{M}_{S,ij}(t) &\doteq \frac{1}{\sqrt{n}}\Big(S^n_{ij}\Big(\mu^n_{ij}\int_0^t Z^n_{ij}(s)\mathrm{d}s\Big) - \mu^n_{ij}\int_0^t Z^n_{ij}(s)\mathrm{d}s\Big)\,.\end{aligned}$$

Rewriting (2.11) in terms of the processes defined above, for $1 \le i \le I$, we get

$$\hat{X}^n_i(t) = \hat{X}^n_i(0) + l^n_i t - \sum_{j\in\mathcal{J}(i)} \mu^n_{ij} \int_0^t \hat{Z}^n_{ij}(s)\mathrm{d}s - \gamma^n_i \int_0^t \hat{Q}^n_i(s)\mathrm{d}s + \hat{M}^n_i(t), \tag{2.14}$$

where

$$\ell^n_i \doteq \frac{1}{\sqrt{n}}\Big(\lambda^n_i - \sum_{j\in\mathcal{J}(i)} n\mu^n_{ij} z^*_{ij}\Big) \to \ell_i \doteq \hat{\lambda}_i - \sum_{j\in\mathcal{J}(i)} \hat{\mu}_{ij} z^*_{ij}, \text{ as } n\to\infty \tag{2.15}$$

and $\hat{M}^n_i(t) \doteq \hat{M}^n_{A,i}(t) + \hat{M}^n_{R,i}(t) + \sum_{j\in\mathcal{J}(i)} \hat{M}^n_{S,ij}(t)$, for $t \ge 0$. To get (2.15), we use Assumption 2.1. Using the above processes, (2.3) and (2.4) become

$$\hat{X}^n_i(t) = \hat{Q}^n_i(t) + \sum_{j\in\mathcal{J}(i)} \hat{Z}^n_{ij}(t) \quad \text{for } i\in\mathcal{I}, \qquad \hat{Y}^n_j(t) + \sum_{i\in\mathcal{I}(j)} \hat{Z}^n_{ij}(t) = \frac{N^n_j - n\nu_j}{\sqrt{n}} \quad \text{for } j\in\mathcal{J}.$$

Here, we use the fact that $\sum_{i\in\mathcal{I}(j)} z^*_{ij} = \nu_j \sum_{i\in\mathcal{I}(j)} \xi^*_{ij} = \nu_j$.

For $x \in \mathbb{Z}^I_+$ and $z\in\mathcal{Z}^n(x)$, set $\tilde{x}^n \doteq \tilde{x}^n(x) \doteq x - nx^*$ and $\hat{x} = \hat{x}^n(x) \doteq \frac{\tilde{x}^n(x)}{\sqrt{n}}$. Also, set

$$\hat{q}^n(x,z) \doteq \frac{q(x,z)}{\sqrt{n}}, \quad \hat{y}^n(z) \doteq \frac{y^n(z)}{\sqrt{n}} \quad \text{and} \quad \hat{z}^n(z) \doteq \frac{z - nz^*}{\sqrt{n}}\,.$$

With abuse of notation, we simply write

$$q^n_i(\hat{x},\hat{z}) = q^n_i(\hat{x}^n,\hat{z}^n) = \hat{x}^n_i - \sum_{j\in\mathcal{J}(i)} \hat{z}^n_{ij}, \quad \text{for} \quad i\in\mathcal{I},$$

$$\hat{y}^n_j(\hat{z}) = \hat{y}^n_j(\hat{z}^n) = \frac{N^n_j - n\sum_{i\in\mathcal{I}(j)} z^*_{ij}}{\sqrt{n}} - \sum_{i\in\mathcal{I}(j)} \hat{z}^n_{ij}, \quad \text{for} \quad j\in\mathcal{J}\,.$$

Since $\hat{x}^n(x)$ is no longer an integer for $x\in\mathfrak{X}^n$, we define the set of all possible values of $\hat{x}(x)$ under the JWC condition by $\mathcal{S}^n \doteq \{\hat{x}(x) : x\in\mathfrak{X}^n\}$. Similarly, $\hat{z}^n(z)$ is no longer an integer and hence we define $\hat{\mathcal{Z}}^n(\hat{x}) \doteq \{\hat{z}^n(z) : z\in\mathcal{Z}^n(\sqrt{n}\hat{x} + nx^*)\}$ for $\hat{x}\in\mathcal{S}^n$, which is the set of all possible values of $\hat{z}(z)$. The following useful bound can be found in [12, Lemma 2.2]: there exists a constant $C'_0 > 0$ such that for any $n\in\mathbb{N}$, $z\in\mathcal{Z}^n(x)$ and $x\in\mathbb{Z}^I_+$,

$$\max\Big\{\max_{(i,j)\in\mathcal{E}} |\hat{z}^n_{ij}|, \|\hat{q}^n(x,z)\|, \|\hat{y}^n(z)\|, \big(e\cdot\hat{q}^n(x,z)\big)\wedge\big(e\cdot\hat{y}^n(z)\big)\Big\} \le C'_0\|\hat{x}^n(x)\|\,. \tag{2.16}$$

Under a stationary Markov SCP Z^n, $\hat{X}^n$ is also a Markov process with the infinitesimal generator given by

$$\widehat{\mathcal{L}}^z_n f(\hat{x}) \doteq \mathcal{L}^z_n f\big(\hat{x}^n(x)\big), \quad \text{for} \quad f\in\mathcal{C}^2(\mathbb{R}^I) \text{ and } x\in\mathbb{Z}^I_+. \tag{2.17}$$

We next define the control parametrization which will help us relate the ERSC problem for the nth model to the ERSC problem for the limiting diffusion. For $i\in\mathcal{I}$ and $j\in\mathcal{J}$, define

$$u^c_i(\hat{x},\hat{z}) = u^{c,n}_i(\hat{x},\hat{z}) \doteq \begin{cases} \frac{\hat{q}^n_i(\hat{x},\hat{z})}{e\cdot\hat{q}^n(\hat{x},\hat{z})}, & \text{if } e\cdot\hat{q}^n(\hat{x},\hat{z}) > 0, \\ e_I, & \text{otherwise}, \end{cases} \tag{2.18}$$

$$u^s_j(\hat{x},\hat{z}) = u^{s,n}_j(\hat{x},\hat{z}) \doteq \begin{cases} \frac{\hat{y}^n_j(\hat{z})}{e\cdot\hat{y}^n(\hat{z})}, & \text{if } e\cdot\hat{y}^n(\hat{z}) > 0, \\ e_J, & \text{otherwise}. \end{cases} \tag{2.19}$$

Then, $u(\hat{x},\hat{z}) \doteq (u^c(\hat{x},\hat{z}), u^s(\hat{z}))$ lies in the set $\mathbb{U}$ given by

$$\mathbb{U} \doteq \{u = (u^c,u^s)\in\mathbb{R}^I_+\times\mathbb{R}^J_+ : e\cdot u^c = e\cdot u^s = 1\}\,.$$

Finally, we define the control process $U^n(t) = \big((U_i^{c,n}(t), U_j^{s,n}(t)) : (i,j) \in \mathcal{E}\big)$ as

$$U_i^{c,n}(t) \doteq u_i^c\big(\hat{X}^n(t), \hat{Z}^n(t)\big), \qquad U_j^{s,n}(t) \doteq u_j^s\big(\hat{X}^n(t), \hat{Z}^n(t)\big).$$

From the above definitions, it is evident that the process $U_i^{c,n}$ represents the fraction of the total queue length at the queue i and the process $U_j^{s,n}$ represents the fraction of the total idle servers at station j, at each time.

Let $\hat{r} : \mathbb{R}_+^I \times \mathbb{R}_+^J \to \mathbb{R}_+$ be the running cost defined as

$$\hat{r}(\hat{q}, \hat{y}) \doteq \sum_{i \in \mathcal{I}} \xi_i \hat{q}_i + \sum_{j \in \mathcal{J}} \zeta_j \hat{y}_j, \tag{2.20}$$

for some positive vector $\xi = (\xi_i : i \in \mathcal{I}) \in \mathbb{R}_+^I$ and positive vector $\zeta = (\zeta_j : j \in \mathcal{J}) \in \mathbb{R}_+^J$. It is easy to see that there exist $r_1, r_2 > 0$ and a sequence $\{\epsilon_n : n \in \mathbb{N}\}$ such that $\epsilon_n \downarrow 0$ as $n \to \infty$ and

$$r_1(|e \cdot \hat{x}| - \epsilon_n) \le \hat{r}(\hat{q}, \hat{y}) \le r_2(|e \cdot \hat{x}| + \epsilon_n), \tag{2.21}$$

for every $\hat{x} \in \mathcal{S}^n$.

For any admissible SCP Z^n, the associated ERSC cost is given by

$$J^n\big(\hat{X}^n(0), Z^n\big) \doteq \limsup_{T \to \infty} \frac{1}{T} \log \mathbb{E}^{Z^n}\left[\exp\Big(\int_0^T \hat{r}(\hat{Q}^n(t), \hat{Y}^n(t))\mathrm{d}t\Big)\right]. \tag{2.22}$$

Here and in what follows, $\mathbb{E}^{Z^n}$ is used to emphasize that the underlying SCP is Z^n and we suppress the dependence on $\hat{X}^n(0)$. The ERSC problem for the nth model is now defined as

$$\hat{\Lambda}^n\big(\hat{X}^n(0)\big) \doteq \inf_{Z^n \in \mathfrak{Z}^n} J^n\big(\hat{X}^n(0), Z^n\big). \tag{2.23}$$

2.3. **ERSC problem for the limiting diffusion model.** Under the assumption that $\hat{X}^n(0) \to x \in \mathbb{R}^I$, as $n \to \infty$, the process $\hat{X}^n$ converges weakly to an I-dimensional process X (see [17, Page 2618 and Proposition 1]) which is given as the unique strong solution to the controlled diffusion

$$X(t) = x + \int_0^t b\big(X(s), U(s)\big)\mathrm{d}s + \Sigma W(t), \tag{2.24}$$

where W is an I-dimensional Brownian motion, $U(\cdot)$ is a $\mathbb{U}$–valued process that is admissible (see below for definition), $\Sigma = \mathrm{diag}(\sqrt{2\lambda_1}, \dots, \sqrt{2\lambda_I})$ and $b : \mathbb{R}^I \times \mathbb{U} \to \mathbb{R}^I$ takes the form

$$b_i(x,u) \doteq b_i\big(x, (u^c, u^s)\big) \doteq \ell_i - \sum_{j \in \mathcal{J}(i)} \mu_{ij} \widehat{\Upsilon}_{ij}[u](x) - \gamma_i (e \cdot x)^+ u_i^c, \quad \text{for } \ i \in \mathcal{I}. \tag{2.25}$$

Here, for $u = (u^c, u^s) \in \mathbb{U}$, $\widehat{\Upsilon}[u] : \mathbb{R}^I \to \mathbb{R}^{\mathcal{G}}$ is defined as $\widehat{\Upsilon}[u](x) = \Upsilon\big(x - (e \cdot x)^+ u^c, -(e \cdot x)^- u^s\big)$ with Υ being the unique linear map from Proposition 2.1. The drift has the following more explicit representation with a piecewise-linear structure (as obtained in Lemma 4.3 of [10]).

Lemma 2.1. *Under Assumptions 2.1 and 2.2, the drift $b : \mathbb{R}^I \times \mathbb{U} \to \mathbb{R}^I$ given by (2.25) has the following representation:*

$$b(x,u) = b\big(x, (u^c, u^s)\big) = \ell - B\big(x - (e \cdot x)^+ u^c\big) + (e \cdot x)^- M u^s - (e \cdot x)^+ \Gamma u^c, \tag{2.26}$$

where B is a lower diagonal $I \times I$ matrix with positive diagonal elements, M is an $I \times J$ matrix and $\Gamma = diag\{\gamma_1, \dots, \gamma_I\}$.

Definition 2.2. A $\mathbb{U}$–valued process U is said to be admissible if it satisfies the following: if $U(t) = U(t, \omega)$ is jointly measurable in $(t, \omega) \in \mathbb{R}^+ \times \Omega$ and for every $0 \le s < t$, $W(t) - W(s)$ is independent of the completed filtration (with respect to $(\Omega, \mathcal{F}, \mathbb{P})$) generated by $\{X(0), U(r), W(r) : r \le s\}$. The set of all such controls is denoted by $\mathfrak{U}$.

Now let us introduce the ERSC problem for the limiting diffusion X. The running cost $r : \mathbb{R}^I \times \mathbb{U} \to \mathbb{R}_+$ is given by

$$r(x,u) \doteq r\big(x,(u^c,u^s)\big) \doteq \hat{r}\big((e\cdot x)^+ u^c, (e\cdot x)^- u^s\big),$$

where $\hat{r}$ is defined in (2.20). Analogous to (2.21), we have

$$r_1|e\cdot x| \le r(x,u) \le r_2|e\cdot x|, \tag{2.27}$$

for every $(x,u) \in \mathbb{R}^I \times \mathbb{U}$. Here, $r_1, r_2 > 0$ are constants from (2.21).

Let $\mathfrak{U}_{\mathrm{SM}} \subset \mathfrak{U}$ denote the set of stationary Markov controls. In order to study the convergence of stationary Markov controls or existence of optimal stationary Markov controls, it is useful to consider a weaker notion of a stationary Markov control, *viz.,* relaxed control - the control is defined in the sense of distribution. To be more precise, a control v is said to be a relaxed Markov control if $v = v(\cdot)$ is a Borel measurable map from $\mathbb{R}^I$ to $\mathcal{P}(\mathbb{U})$. In this case, we write $v(\mathrm{d}u|x)$ to distinguish the relaxed Markov control v from other Markov controls which are referred to as precise Markov controls. Clearly, the set of relaxed Markov controls contains $\mathfrak{U}_{\mathrm{SM}}$. But with a slight abuse of notation, we represent the set of relaxed Markov controls also by $\mathfrak{U}_{\mathrm{SM}}$. Under $v \in \mathfrak{U}_{\mathrm{SM}}$, the controlled diffusion X in (2.24) has a unique solution [8, Theorem 2.2.4]. For every $u \in \mathbb{U}$, the generator $\mathcal{L}^u : \mathcal{C}^2(\mathbb{R}^I) \mapsto \mathcal{C}(\mathbb{R}^I)$ of the controlled diffusion X is given by

$$\mathcal{L}^u f(x) \doteq \sum_{i=1}^{I} b_i(x,u)\frac{\partial}{\partial x_i} f(x) + \sum_{j=1}^{I} \lambda_i \frac{\partial^2}{\partial x_i^2} f(x)\,. \tag{2.28}$$

To keep the expressions concise, we let

$$r^v(x) \doteq r(x, v(x)) \quad \text{and} \quad \mathcal{L}^v f(x) \doteq \sum_{i=1}^{I} b_i(x,v(x))\frac{\partial}{\partial x_i} f(x) + \sum_{i=1}^{I} \lambda_i \frac{\partial^2}{\partial x_i^2} f(x)\,,$$

whenever the underlying control is $v \in \mathfrak{U}_{\mathrm{SM}}$. In the following, we denote a generic admissible control by U and a generic stationary Markov control by v.

The ERSC cost, for any $U \in \mathfrak{U}$, is given by

$$J(x,U) \doteq \limsup_{T\to\infty} \frac{1}{T} \log \mathbb{E}_x^U \left[\exp\left(\int_0^T r(X(t),U(t))\mathrm{d}t\right)\right]. \tag{2.29}$$

In the above and in what follows, we emphasize that the underlying controls are $U \in \mathfrak{U}$ and $v \in \mathfrak{U}_{\mathrm{SM}}$ by writing $\mathbb{E}_x^U$ and $\mathbb{E}_x^v$, respectively. The associated ERSC problem is given by

$$\Lambda(x) \doteq \inf_{U\in\mathfrak{U}} J(x,U) \qquad \text{and} \qquad \Lambda \doteq \inf_{x\in\mathbb{R}^I} \Lambda(x)\,. \tag{2.30}$$

Let

$$\Lambda_{\mathrm{SM}}(x) \doteq \inf_{v\in\mathfrak{U}_{\mathrm{SM}}} J(x,v), \qquad \Lambda_{\mathrm{SM}} \doteq \inf_{x\in\mathbb{R}^I} \Lambda_{\mathrm{SM}}(x)\,.$$

It is not a priori clear if $J(x,U)$ is finite for every $U \in \mathfrak{U}$ and hence, we work with the following set:

$$\begin{aligned} \mathfrak{U}^* &\doteq \{U \in \mathfrak{U} : J(x,U) < \infty, \text{ for some } x \in \mathbb{R}^I\}, \\ \mathfrak{U}^*_{\mathrm{SM}} &\doteq \{v \in \mathfrak{U}_{\mathrm{SM}} : J(x,v) < \infty, \text{ for some } x \in \mathbb{R}^I\} \subset \mathfrak{U}^*\,. \end{aligned} \tag{2.31}$$

In particular, $\mathfrak{U}^*_{\mathrm{SM}}$ can be a priori empty and so can $\mathfrak{U}^*$. In Proposition 3.1, we show that there exists a constant control $v \in \mathfrak{U}_{\mathrm{SM}}$ such that $J(x,v) < \infty$, for every $x \in \mathbb{R}^I$. This shows that $\mathfrak{U}^*_{\mathrm{SM}} \neq \emptyset$ and hence, $\mathfrak{U}^* \neq \emptyset$. In fact, we give in Theorem 4.1, the characterization of optimal stationary Markov controls, *i.e.,* $v \in \mathfrak{U}^*_{\mathrm{SM}}$ such that $\inf_{x\in\mathbb{R}^I} J(x,v) = \Lambda$.

2.4. **The main result: asymptotic optimality.** In this section, we state the main result of the paper which is the asymptotic optimality and also provide a sketch of its proof. Without further mention in the rest of the paper, we assume that $\hat{X}^n(0) \to x \in \mathbb{R}^I$ as $n \to \infty$, for some $x \in \mathbb{R}^I$.

Theorem 2.1. *Under Assumptions 2.1 and 2.2, the following hold*

(i) *(Lower bound)* $\liminf_{n\to\infty} \hat{\Lambda}^n\big(\hat{X}^n(0)\big) \ge \Lambda$.

(ii) *(Upper bound)* $\limsup_{n\to\infty} \hat{\Lambda}^n\big(\hat{X}^n(0)\big) \le \Lambda$.

The proof of this theorem is deferred to Section 6. Below, we briefly discuss the main methodology. The cornerstones of the proof are the following representations.

(I) The ERSC cost for the nth model: for $Z^n \in \mathfrak{Z}^n$,

$$J^n\big(\hat{X}^n(0), Z^n\big) = \limsup_{T\to\infty} \sup_{\uppsi} \mathbb{E}\Big[\frac{1}{T}\int_0^T \hat{r}\big(\hat{Q}^{n,\uppsi}(t), \hat{Y}^{n,\uppsi}(t)\big)\mathrm{d}t - \mathfrak{K}^n(\uppsi, T)\Big], \tag{2.32}$$

where $\uppsi = \big(\phi_i, \psi_{ij}, \varphi_i : 1 \le i \le I, j \in \mathcal{J}(i)\big)$ is varied over an appropriate class of $\mathbb{R}_+^{2I+|\mathcal{E}|}$–valued processes and the definition of $\mathfrak{K}^n(\uppsi, T)$ can be found in (5.14). The definitions of processes $\hat{Q}^{n,\uppsi}$, $\hat{Y}^{n,\uppsi}$ (associated with the "extended" diffusion-scaled processes $\hat{X}^{n,\uppsi}$ in (5.12)) can be found in (5.12) and (5.13).

(II) The ERSC cost for the limiting model: for $v \in \mathfrak{U}_{\mathrm{SM}}$,

$$J(x,v) = \limsup_{T\to\infty} \sup_{w} \mathbb{E}\Big[\frac{1}{T}\int_0^T \Big(\hat{r}\big(\big(e\cdot X^*(t)\big)^+ v^c\big(X^*(t)\big), \big(e\cdot X^*(t)\big)^- v^s\big(X^*(t)\big)\big) - \frac{1}{2}\|w(t)\|^2\Big)\mathrm{d}t\Big],$$

where w in the above supremum is varied over an appropriate class of $\mathbb{R}^I$–valued processes and the process X^*, referred to as the "extended" diffusion process, is defined in (5.2).

The proof of the lower bound begins with the choice of the family of SCPs Z^n that are nearly optimal to $\hat{\Lambda}^n\big(\hat{X}^n(0)\big)$, for every n. We apply (2.32) to the ERSC cost associated with Z^n and choose $\uppsi = \uppsi^n = \big(\phi_i^n, \psi_{ij}^n, \varphi_i^n : 1 \le i \le I, j \in \mathcal{J}(i)\big)$ that depends on n and on a particular w^* that is a $\mathbb{R}^I$–valued stationary Markov control and is nearly optimal for the following two-person zero-sum stochastic game:

$$\sup_{w} \inf_{v} \limsup_{T\to\infty} \mathbb{E}\Big[\frac{1}{T}\int_0^T \Big(r\big(X^*(t), v\big(X^*(t)\big)\big) - \frac{1}{2}\|w\big(X^*(t)\big)\|^2\Big)\mathrm{d}t\Big] = \Lambda\,.$$

From Proposition 5.2, we can ensure that w^* is continuous and compactly supported. As a consequence, $\mathfrak{K}^n(\uppsi^n, T)$ is uniformly bounded in T and n which implies that the family of MEMs of $\hat{X}^{n,\uppsi^n}$ is tight by Proposition 6.2. Upon taking $n \to \infty$, using the definition of $\mathfrak{K}^n$ and the concept of ergodic occupation measure, the lower bound is obtained.

The proof of the upper bound is more involved than that of the lower bound. Here, we begin by choosing a nearly optimal stationary Markov control (for Λ) v that is continuous and fixed outside a compact set (see Proposition 4.2). From this, we construct an SCP $Z^n = Z^n[v]$, for every n which is a priori only sub-optimal for $\hat{\Lambda}^n\big(\hat{X}^n(0)\big)$ and using Proposition 3.2, we show that $J^n\big(\hat{X}^n(0), Z^n\big)$ is uniformly bounded in n. Again using (2.32) and choosing $\uppsi = \uppsi_T^n$ that is nearly optimal to the supremum in (2.32), we have for small $\delta > 0$,

$$\hat{\Lambda}^n\big(\hat{X}^n(0)\big) \le J^n\big(\hat{X}^n(0), Z^n\big) \le \limsup_{T\to\infty} \mathbb{E}\Big[\frac{1}{T}\int_0^T \hat{r}\big(\hat{Q}^{n,\uppsi^n}(t), \hat{Y}^{n,\uppsi^n}(t)\big)\mathrm{d}t - \mathfrak{K}^n(\uppsi^n, T)\Big] + \delta\,. \tag{2.33}$$

The main difficulty in proving the upper bound lies in proving the tightness of the family of MEMs of the "extended" processes $\hat{X}^{n,\uppsi^n}$ and identifying the limiting behavior of $\hat{X}^{n,\uppsi^n}$ (associated with nearly optimal $\uppsi^n$) and $\liminf_{T\to\infty} T^{-1}\int_0^T \mathfrak{K}^n(\uppsi^n, T)\mathrm{d}t$ as $n \to \infty$. This stems from the fact that it is a priori unclear if $T^{-1}\mathbb{E}\big[\int_0^T \mathfrak{K}^n(\psi^n, T)\mathrm{d}t\big]$ is uniformly bounded in both T and n. We overcome this difficulty by employing a truncation argument using a 'truncated' ERSC cost. We then apply the analog of (2.32) to this 'truncated' ERSC cost and obtain a nearly optimal control $\psi = \widetilde{\psi}^n$ for which the aforementioned boundedness is immediate. Again using the notion of the ergodic occupation measure, we obtain the desired upper bound.

In the coming sections, we prove various key results concerning the ERSC problems for the nth model and the limiting diffusion model that will eventually be used in the proof of Theorem 2.1.

3. Well-posedness of the ERSC problems

In this section, we prove that the ERSC problems introduced in Sections 2.2 and 2.3 are well posed, that is, $\hat{\Lambda}^n\big(\hat{X}^n(0)\big)$ defined in (2.23) is finite for every n and Λ defined in (2.30) is finite.

3.1. Well-posedness of the ERSC problem for the limiting diffusion.

We begin by proving a Foster-Lyapunov inequality for the limiting diffusion under a constant control, which will facilitate to prove the finiteness of Λ defined in (2.30). We define a constant Markov control

$$v^*(x) = v^* \doteq \big(u^{*,c}, u^{*,s}\big), \tag{3.1}$$

where $u^{*,c} = e_I$ and $u^{*,s} = e_J$. In the following, we write $x \in \mathbb{R}^I$ as $(\widetilde{x}, x_I)$. In this case, following the leaf-elimination algorithm introduced in [10, Section 4.1], we can express the drift $b(x, v^*)$ in the following form:

$$b_i(x, v^*) = \begin{cases} -(Bx)_i + \ell_i, & \text{for } i < I, \\ -F(\widetilde{x}) - \mu_{IJ}x_I - (\gamma_I - \mu_{IJ})(e\cdot x)^+ + \ell_I, & \text{for } i = I\,. \end{cases}$$

Here, $F(\widetilde{x})$ is the difference of the Ith element of vector Bx and $\mu_{IJ}x_I$ - in particular, $F(\cdot)$ is linear.

For $\eta > 0$, define a function $\mathcal{V}_\eta : \mathbb{R}^I \to [1, \infty)$ as

$$\mathcal{V}_\eta(x) \doteq \exp\Big(\frac{\eta}{2}\widetilde{x}^{\mathsf{T}}\widetilde{Q}\widetilde{x} + \frac{\eta^2}{2}x_I^2\Big), \tag{3.2}$$

where the $\mathbb{R}^{I-1}\times\mathbb{R}^{I-1}$ matrix $\widetilde{Q}$ is positive diagonal and satisfies

$$\widetilde{x}^{\mathsf{T}}(\widetilde{Q}\widetilde{B} + \widetilde{B}^{\mathsf{T}}\widetilde{Q})\widetilde{x} \geq 16\|\widetilde{x}\|^2\,. \tag{3.3}$$

Here, $\widetilde{B}$ is the $(I-1)\times(I-1)$ sub-matrix of B obtained by removing the Ith column and the Ith row. From the lower diagonal structure of B, $\widetilde{B}$ is also lower diagonal which ensures the existence of $\widetilde{Q}$. This is a consequence of the well-known Lyapunov's theorem; see [46, Corollary 2.2.4].

Lemma 3.1. *Suppose Assumptions 2.1 and 2.2 hold. For $v^* \in \mathfrak{U}_{\mathrm{SM}}$ as defined in (3.1), there exists a constant $\eta_0 > 0$ such that for $\eta < \eta_0$, the following holds: there exist constants $C_i = C_i(\eta) > 0$ for $i = 0, 1, 2$, such that*

$$\mathcal{L}^{v^*}\mathcal{V}_\eta(x) \leq \Big(C_0 + C_1\|x\| - C_2\|x\|^2\Big)\mathcal{V}_\eta(x), \quad \textit{for} \quad x \in \mathbb{R}^I. \tag{3.4}$$

Remark 3.1. In [10, Theorem 4.2], the result analogous to the above lemma is shown. More precisely, consider a function $\mathcal{V}^m(x) \doteq (x^{\mathsf{T}}Qx)^{\frac{m}{2}}$ with an $I\times I$ positive diagonal matrix Q that satisfies

$$x^{\mathsf{T}}(QB + B^{\mathsf{T}}Q)x \geq 16\|x\|^2\,. \tag{3.5}$$

From the lower diagonal structure of B, the existence of Q is ensured; see the discussion below (3.3). Then, for v^* as defined in (3.1), it is shown that there exist $\delta > 0$ and a large enough constant $C_0' = C_0'(\eta) > 0$ such that

$$\mathcal{L}^{v^*}\mathcal{V}^m(x) \leq C_0' - \delta\mathcal{V}^m(x), \quad \text{for } x \in \mathbb{R}^I. \tag{3.6}$$

Upon an application of the Itô's formula, this gives us the following: for an open ball $B \subset \mathbb{R}^I$, $v \in \mathfrak{U}_{\mathrm{SM}}$ and $x \notin B$,

$$\mathbb{E}_x^v\Big[\exp\big(\delta\breve{\tau}_B\big)\Big] < \infty\,.$$

Here, $\breve{\tau}_B$ is the first hitting time of the set B. In contrast to this, by applying the Itô's formula, (3.4) implies the following:

$$\mathbb{E}_x^v\Big[\exp\Big(\int_0^{\breve{\tau}_B}\big(C_2\|X(t)\|^2 - C_1\|X(t)\|\big)\mathrm{d}t\Big)\Big] < \infty\,.$$

From the above, we can infer that (3.4) is a stronger condition than (3.6). This is needed to prove the finiteness of Λ defined in (2.30) due to the exponential nature of the ERSC cost.

The proof of Lemma 3.1 is deferred to the appendix.

Remark 3.2. In addition to $\mathcal{V}_\eta$ defined in (3.2), there also exists a family of Lyapunov functions that grow exponentially in $\|x\|$: for any $\eta > 0$ and small enough $\kappa > 0$, choose a $\mathcal{C}^2(\mathbb{R}^I)$ function $\widetilde{\mathcal{V}}_\eta : \mathbb{R}^I \to [1,\infty)$ given by

$$\widetilde{\mathcal{V}}_\eta(x) \doteq \exp\Big(\frac{\eta}{2}(\widetilde{x}^{\mathsf{T}}\widetilde{Q}\widetilde{x})(1+\widetilde{x}^{\mathsf{T}}\widetilde{Q}\widetilde{x})^{-\frac{1}{2}} + \frac{\eta\kappa}{2}x_I^2(1+x_I^2)^{-\frac{1}{2}}\Big), \tag{3.7}$$

with the positive diagonal matrix $\widetilde{Q}$ which is the same as the one used in the definition of $\mathcal{V}_\eta$ in (3.2). Under Assumptions 2.1 and 2.2, there exist positive constants C_0' and C_1' such that for every $\eta > 0$,

$$\mathcal{L}^{v^*}\widetilde{\mathcal{V}}_\eta(x) \le \Big(C_0'\eta - C_1'\eta\|x\|\Big)\widetilde{\mathcal{V}}_\eta(x), \qquad \text{for} \quad x \in \mathbb{R}^I. \tag{3.8}$$

The proof of this inequality is given in the appendix, for completeness.

Using Lemma 3.1, we prove the well-posedness of the ERSC problem for the limiting diffusion.

Proposition 3.1. *Under Assumptions 2.1 and 2.2, $v^* \in \mathfrak{U}_{\mathrm{SM}}$ defined in (3.1) satisfies $J(x,v^*) < \infty$. In particular, this implies that (i) $\Lambda < \infty$, (ii) $\mathfrak{U}_{\mathrm{SM}}^* \neq \emptyset$ and (iii) $\mathfrak{U}^* \neq \emptyset$.*

Proof. Fix $x \in \mathbb{R}^I$ and consider $v^* \in \mathfrak{U}_{\mathrm{SM}}$ defined in (3.1). It is clear that with C_1 and C_2 obtained from Lemma 3.1, the following function is inf-compact:

$$C_2\|x\|^2 - C_1\|x\| - r\big(x,v^*\big)\,. \tag{3.9}$$

Recall the definition of $\mathcal{V}_\eta$ from (3.2). Using Lemma 3.1 and applying Itô's formula to

$$\exp\big(\int_0^t\big(C_2\|X(s)\|^2 - C_1\|X(s)\| + C_0\big)\mathrm{d}s\big)\mathcal{V}_\eta\big(X(t)\big)$$

up to the stopping time $T \wedge \tau_R$ (recall that τ_R is the first exit time of the R–radius ball around origin, by the process X), gives us

$$\mathbb{E}_x^{v^*}\Big[\exp\Big(\int_0^{T\wedge\tau_R}\big(C_2\|X(t)\|^2 - C_1\|X(t)\| + C_0\big)\mathrm{d}t\Big)\mathcal{V}_\eta(X(T\wedge\tau_R))\Big] \le \mathcal{V}_\eta(x)\,.$$

From here, using the fact that $\mathcal{V}_\eta \ge 1$ and then applying Fatou's Lemma while taking $R \to \infty$, we get

$$\mathbb{E}_x^{v^*}\Big[\exp\Big(\int_0^T\big(C_2\|X(t)\|^2 - C_1\|X(t)\| + C_0\big)\mathrm{d}t\Big)\Big] \le \mathcal{V}_\eta(x)\,.$$

$$\implies \limsup_{T\to\infty}\frac{1}{T}\log\mathbb{E}_x^{v^*}\Big[\exp\Big(\int_0^T\big(C_2\|X(t)\|^2 - C_1\|X(t)\| + C_0\big)\mathrm{d}t\Big)\Big] \le 0\,.$$

Using the inf-compactness of the function in (3.9), we have

$$\int_0^T r(X(t),v^*)\mathrm{d}t \le \int_0^T \big(C_1\|X(t)\|^2 - C_0\|X(t)\|\big)\mathrm{d}t + (C_0+C)T,$$

for some large enough $C > 0$. Using the above two displays, we have $J(x,v^*) \le C_0 + C$. This proves the result. □

3.2. **Well-posedness of the ERSC problem for the diffusion-scaled queueing processes.** To analyze the well-posedness of the ERSC problem for the nth model, one cannot directly work with $v \in \mathfrak{U}_{\mathrm{SM}}$ and needs to define a mapping from $v \in \mathfrak{U}_{\mathrm{SM}}$ to the integer-valued SCP Z^n. The following construction of an SCP from a given stationary Markov control v is taken from [12, Pg. 313].

Definition 3.1. Let $\varpi^c : \{x \in \mathbb{R}^I_+ : e\cdot x \in \mathbb{Z}\} \to \mathbb{Z}^I_+$ be a map defined as

$$\varpi^c(x) \doteq \Big\{\lfloor x_1\rfloor, \lfloor x_2\rfloor, \dots, e\cdot x - \sum_{i=1}^{I-1}\lfloor x_i\rfloor\Big\}$$

and $\varpi^s : \{x \in \mathbb{R}^J_+ : e\cdot x \in \mathbb{Z}\} \to \mathbb{Z}^J_+$ be a map defined as $\varpi^s(x) \doteq \big\{\lfloor x_1\rfloor, \lfloor x_2\rfloor, \dots, e\cdot x - \sum_{i=1}^{J-1}\lfloor x_i\rfloor\big\}$.

For a precise stationary Markov control $v = (v^c, v^s) \in \mathfrak{U}_{\mathrm{SM}}$, define maps $q^n[v] : \mathbb{R}^I \to \mathbb{Z}^I_+$ and $y^n[v] : \mathbb{R}^I \to \mathbb{Z}^J_+$ by

$$\begin{aligned} q^n[v](\hat{x}) &\doteq \varpi^c\Big(\big(e\cdot(\sqrt{n}\hat{x} + nx^*) - (e\cdot N^n)\big)^+ v^c(\hat{x})\Big)\,, \\ y^n[v](\hat{x}) &\doteq \varpi^s\Big(\big(e\cdot(\sqrt{n}\hat{x} + nx^*) - (e\cdot N^n)\big)^- v^s(\hat{x})\Big)\,. \end{aligned}$$

Define

$$z^n[v](\hat{x}) \doteq \Upsilon(x - q^n[v](\hat{x}), N^n - y^n[v](\hat{x}))\,. \tag{3.10}$$

Also, from the definition of ϖ^c and ϖ^s, we can conclude that

$$\begin{aligned} \max\Big\{&\Big\|\frac{1}{\sqrt{n}}\varpi^c\Big(\big(e\cdot(\sqrt{n}x + nx^*) - (e\cdot N^n)\big)^+ v^c(x)\Big) - (e\cdot x)^+ v^c(x)\Big\|, \\ &\Big\|\frac{1}{\sqrt{n}}\varpi^s\Big(\big(e\cdot(\sqrt{n}x + nx^*) - (e\cdot N^n)\big)^- v^s(x)\Big) - (e\cdot x)^- v^s(x)\Big\|\Big\} \le \frac{2(I\vee J)}{\sqrt{n}}\,. \end{aligned} \tag{3.11}$$

We now prove below that for an appropriate class of stationary Markov controls, a property that is an exponential analog of uniform integrability holds; see (3.13). To that end, let $\mathfrak{U}_{\mathrm{SM}}(l)$ be the collection of all $v \in \mathfrak{U}_{\mathrm{SM}}$ that are of the form $v(x) = v^*$, whenever $x \in B_l^c$.

Proposition 3.2. *Suppose Assumptions 2.1 and 2.2 hold. For $l > 0$ and $v \in \mathfrak{U}_{\mathrm{SM}}(l)$, let $z^n[v](\hat{x}) \doteq \Upsilon(x - q^n[v](\hat{x}), N^n - y^n[v](\hat{x}))$. Then, the following hold.*

(i) *There exist constants $C_1, C_2 > 0$ such that for $\eta > 0$ and $\widetilde{\mathcal{V}}_\eta$ defined in (3.7), we have*

$$\widehat{\mathcal{L}}_n^{z^n[v](\hat{x})}\widetilde{\mathcal{V}}_\eta(\hat{x}) \le \Big(C_1\eta - C_2\eta\|\hat{x}\|\Big)\widetilde{\mathcal{V}}_\eta(\hat{x})\,. \tag{3.12}$$

(ii) *Under the SCP $Z^n(t) \doteq z^n[v]\big(\hat{X}^n(t)\big)$,*

$$\limsup_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\log\mathbb{E}^{Z^n}\Big[\exp\Big((1+\rho)\int_0^T \hat{r}(\hat{Q}^n(t),\hat{Y}^n(t))\mathrm{d}t\Big)\Big] < \infty, \tag{3.13}$$

for every $\rho > 0$. In particular,

$$\limsup_{n\to\infty} J^n\big(\hat{X}^n(0), Z^n\big) < \infty \text{ and thereby, } \limsup_{n\to\infty}\hat{\Lambda}^n\big(\hat{X}^n(0)\big) < \infty\,.$$

Remark 3.3. The aforementioned exponential analog of uniform integrability is implied by the presence of $1+\rho$ in (3.13). In the proof of the upper bound (Lemma 6.2), we will see that this implies that the ERSC cost of the nth model associated with the SCP constructed as above for $v \in \mathfrak{U}_{\mathrm{SM}}(l)$, $l > 0$, can be approximated arbitrarily well using an appropriate truncated version of the ERSC cost. This will help us invoke Proposition 6.2 upon the application of Lemma 5.4 which ensures all the important tightness of the family of MEMs of the “extended” diffusion-scaled processes.

Proof. Fix $v(\cdot) = \big(v^c(\cdot), v^s(\cdot)\big) \in \mathfrak{U}_{\mathrm{SM}}(l)$. Then,

$$q^n[v](\hat{x}) \doteq \varpi^c\Big(\big(e\cdot\big(\sqrt{n}\hat{x} + nx^*\big) - (e\cdot N^n)\big)^+ v^c(\hat{x})\Big),$$
$$y^n[v](\hat{x}) \doteq \varpi^s\Big(\big(e\cdot\big(\sqrt{n}\hat{x} + nx^*\big) - (e\cdot N^n)\big)^- v^s(\hat{x})\Big).$$

For the proof of part (i), define, $\widehat{\mathcal{L}}_n^* f \doteq \widehat{\mathcal{L}}_n^{z^n[v]} f$, for any $f \in \mathcal{C}^2(\mathbb{R}^I)$. For $\eta > 0$ and $\widetilde{\mathcal{V}}_\eta$ defined in (3.7), we have

$$\begin{aligned}\widehat{\mathcal{L}}_n^* \widetilde{\mathcal{V}}_\eta(\hat{x}) = \sum_{i\in\mathcal{I}} \lambda_i^n \mathfrak{d}\widetilde{\mathcal{V}}_\eta(\hat{x}; n^{-\frac{1}{2}}e_i) + \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}(i)} \mu_{ij}^n z_{ij}^n[v](\hat{x})\mathfrak{d}\widetilde{\mathcal{V}}_\eta(\hat{x}; -n^{-\frac{1}{2}}e_i)\\ + \sum_{i\in\mathcal{I}} \gamma_i^n q_i^n[v](\hat{x})\mathfrak{d}\widetilde{\mathcal{V}}_\eta(\hat{x}; -n^{-\frac{1}{2}}e_i)\,. \end{aligned} \tag{3.14}$$

From the form of $\widetilde{\mathcal{V}}_\eta$, we know that

$$|\mathfrak{d}\widetilde{\mathcal{V}}_\eta(\hat{x}; \pm n^{-\frac{1}{2}}e_i) \mp \frac{1}{\sqrt{n}}\frac{\partial}{\partial \hat{x}_i}\widetilde{\mathcal{V}}_\eta(\hat{x})| \le \frac{C\eta^2}{n}\widetilde{\mathcal{V}}_\eta(\hat{x}),$$

for some $C > 0$. From the above display and (3.14), we obtain

$$\begin{aligned}\widehat{\mathcal{L}}_n^* \widetilde{\mathcal{V}}_\eta(\hat{x}) &\le \sum_{i\in\mathcal{I}} \lambda_i^n \frac{1}{\sqrt{n}}\frac{\partial}{\partial \hat{x}_i}\widetilde{\mathcal{V}}_\eta(\hat{x}) - \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}(i)} \mu_{ij}^n z_{ij}^n[v](\hat{x})\frac{1}{\sqrt{n}}\frac{\partial}{\partial \hat{x}_i}\widetilde{\mathcal{V}}_\eta(\hat{x}) - \sum_{i\in\mathcal{I}} \gamma_i^n q_i^n[v](\hat{x})\frac{1}{\sqrt{n}}\frac{\partial}{\partial \hat{x}_i}\widetilde{\mathcal{V}}_\eta(\hat{x})\\ &\quad + \frac{C\eta^2}{n}\Big(\sum_{i\in\mathcal{I}}\lambda_i^n + \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}(i)}\mu_{ij}^n |z_{ij}^n[v](\hat{x})| + \sum_{i\in\mathcal{I}}\gamma_i^n |q_i^n[v](\hat{x})|\Big)\widetilde{\mathcal{V}}_\eta(\hat{x})\\ &\le \sum_{i\in\mathcal{I}} \ell_i^n \frac{\partial}{\partial \hat{x}_i}\widetilde{\mathcal{V}}_\eta(\hat{x}) - \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}(i)} \mu_{ij}^n \hat{z}_{ij}^n[v](\hat{x})\frac{\partial}{\partial \hat{x}_i}\widetilde{\mathcal{V}}_\eta(\hat{x}) - \sum_{i\in\mathcal{I}} \gamma_i^n \hat{q}_i^n[v](\hat{x})\frac{\partial}{\partial \hat{x}_i}\widetilde{\mathcal{V}}_\eta(\hat{x})\\ &\quad + \frac{C\eta^2}{n}\Big(\sum_{i\in\mathcal{I}}\lambda_i^n + \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}(i)}\mu_{ij}^n |z_{ij}^n[v](\hat{x})| + \sum_{i\in\mathcal{I}}\gamma_i^n |q_i^n[v](\hat{x})|\Big)\widetilde{\mathcal{V}}_\eta(\hat{x})\,. \end{aligned} \tag{3.15}$$

Here,

$$\hat{z}_{ij}^n[v](\hat{x}) \doteq \frac{z_{ij}^n[v](\hat{x}) - nz_{ij}^*}{\sqrt{n}} \quad \text{and} \quad \hat{q}_j^n[v](\hat{x}) \doteq \frac{q_j^n[v](\hat{x})}{\sqrt{n}}\,.$$

From the definition of $q^n[v](\hat{x})$ and $z^n[v](\hat{x})$ and Assumption 2.1, it is clear that for every $\epsilon > 0$, we can take large enough n such that

$$\big|\mu_{ij}^n \hat{z}_{ij}^n[v](\hat{x}) - \mu_{ij}\Upsilon_{ij}(\hat{x} - (e\cdot\hat{x})^+ v^c(\hat{x}), (e\cdot\hat{x})^- v^s(\hat{x}))\big| + \big|\gamma_i^n \hat{q}_i^n[v](\hat{x}) - \gamma_i(e\cdot\hat{x})^+ v_i^c(\hat{x})\big| + |\ell_i^n - \ell_i| < \epsilon,$$

for $(i,j) \in \mathcal{E}$. Using the above fact in (3.15), we obtain

$$\begin{aligned}\widehat{\mathcal{L}}_n^* \widetilde{\mathcal{V}}_\eta(\hat{x}) \le \sum_{i\in\mathcal{I}} (\ell_i + \epsilon)\frac{\partial}{\partial \hat{x}_i}\widetilde{\mathcal{V}}_\eta(\hat{x}) - \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}(i)} \big(\mu_{ij}\Upsilon_{ij}(\hat{x} - (e\cdot\hat{x})^+ v^c(\hat{x}), (e\cdot\hat{x})^- v^s(\hat{x})) - \epsilon\big)\frac{\partial}{\partial \hat{x}_i}\widetilde{\mathcal{V}}_\eta(\hat{x})\\ - \sum_{i\in\mathcal{I}} \big(\gamma_i(e\cdot\hat{x})^+ v^c(\hat{x}) - \epsilon\big)\frac{\partial}{\partial \hat{x}_i}\widetilde{\mathcal{V}}_\eta(\hat{x})\end{aligned}$$

$$
\begin{aligned}
&+\frac{C\eta^2}{n}\Big(\sum_{i\in\mathcal{I}}\lambda_i^n+\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}(i)}\mu_{ij}^n|z_{ij}^n[v](\hat{x})|+\sum_{i\in\mathcal{I}}\gamma_i^n|q_i^n[v](\hat{x})|\Big)\widetilde{\mathcal{V}}_\eta(\hat{x})\\
&\le b\big(\hat{x},v(\hat{x})\big)\cdot\nabla\widetilde{\mathcal{V}}_\eta(\hat{x})+\frac{C\eta^2}{n}\Big(\sum_{i\in\mathcal{I}}\lambda_i^n+\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}(i)}\mu_{ij}^n|z_{ij}^n[v](\hat{x})|+\sum_{i\in\mathcal{I}}\gamma_i^n|q_i^n[v](\hat{x})|\Big)\widetilde{\mathcal{V}}_\eta(\hat{x})\\
&\quad+\epsilon\Big(2\sum_{i\in\mathcal{I}}\frac{\partial}{\partial\hat{x}_i}\widetilde{\mathcal{V}}_\eta(\hat{x})+\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}(i)}\frac{\partial}{\partial\hat{x}_i}\widetilde{\mathcal{V}}_\eta(\hat{x})\Big)\,.
\end{aligned}
$$

Since $|\frac{\partial}{\partial\hat{x}_i}\widetilde{\mathcal{V}}_\eta(\hat{x})|\le C\eta\widetilde{\mathcal{V}}_\eta(\hat{x})$, for large enough $C>0$, we can further simplify the last inequality as follows:

$$
\begin{aligned}
\widehat{\mathcal{L}}_n^*\widetilde{\mathcal{V}}_\eta(\hat{x})&\le b\big(\hat{x},v(\hat{x})\big)\cdot\nabla\widetilde{\mathcal{V}}_\eta(\hat{x})+\frac{C\eta^2}{n}\Big(\sum_{i\in\mathcal{I}}\lambda_i^n+\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}(i)}\mu_{ij}^n|z_{ij}^n[v](\hat{x})|+\sum_{i\in\mathcal{I}}\gamma_i^n|q_i^n[v](\hat{x})|\Big)\widetilde{\mathcal{V}}_\eta(\hat{x})\\
&\quad+\epsilon\big(2I+|\mathcal{E}|\big)\eta C\widetilde{\mathcal{V}}_\eta(\hat{x})\,.
\end{aligned}
$$

Recall that $|\mathcal{E}|$ denotes the number of edges in $\mathcal{E}$. Using (2.16), we can infer that

$$
|z_{ij}^n[v](\hat{x})|\le n|z_{ij}^*|+\sqrt{n}C_0'\|\hat{x}\|\quad\text{and}\quad|q_i^n[v](\hat{x})|\le C_0'\sqrt{n}\|\hat{x}\|\,.
$$

This gives us

$$
\begin{aligned}
\widehat{\mathcal{L}}_n^*\widetilde{\mathcal{V}}_\eta(\hat{x})&\le b\big(\hat{x},v(\hat{x})\big)\cdot\nabla\widetilde{\mathcal{V}}_\eta(\hat{x})\\
&\quad+\Big(\frac{C\eta^2}{n}\sum_{i\in\mathcal{I}}\lambda_i^n+\frac{C\eta^2}{n}\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}(i)}\mu_{ij}^n\big(n|z_{ij}^*|+\sqrt{n}C_0'\|\hat{x}\|\big)\\
&\qquad+\frac{C\eta^2}{n}\sum_{i\in\mathcal{I}}\gamma_i^n\sqrt{n}C_0'\|\hat{x}\|+\epsilon\big(2I+|\mathcal{E}|\big)\eta C\Big)\widetilde{\mathcal{V}}_\eta(\hat{x})\,.
\end{aligned}
\tag{3.16}
$$

We now compute $b\big(\hat{x},v(\hat{x})\big)\cdot\nabla\widetilde{\mathcal{V}}_\eta(\hat{x})$. To do this, we first observe the following:

$$
\max_{x\in\mathbb{R}^I}\|b(x,v(x))-b(x,v^*)\|=\max_{x\in B_l}\|b(x,v(x))-b(x,v^*)\|\doteq M(l)<\infty\,.
$$

From Remark 3.2 and the proof of (3.8) in the Appendix, we know that for large enough $C>0$, we have

$$
b(\hat{x},v^*)\cdot\nabla\widetilde{\mathcal{V}}_\eta(\hat{x})\le C\Big(\eta-\eta\|\hat{x}\|\Big)\widetilde{\mathcal{V}}_\eta(\hat{x})\,.
$$

Combining the above two displays and using the fact that $\|\nabla\widetilde{\mathcal{V}}_\eta(\hat{x})\|\le C\eta\widetilde{\mathcal{V}}_\eta(\hat{x})$, for large enough $C>0$, we get

$$
\begin{aligned}
b\big(\hat{x},v(\hat{x})\big)\cdot\nabla\widetilde{\mathcal{V}}_\eta(\hat{x})&\le\Big(C\eta-C\eta\|\hat{x}\|\Big)\widetilde{\mathcal{V}}_\eta(\hat{x})+M(l)C\eta\widetilde{\mathcal{V}}_\eta(\hat{x})\\
&\le C\Big(\eta+M(l)\eta-\eta\|\hat{x}\|\Big)\widetilde{\mathcal{V}}_\eta(\hat{x})\,.
\end{aligned}
\tag{3.17}
$$

Noting that $\limsup_{n\to\infty}\Big(\frac{1}{n}\sum_{i\in\mathcal{I}}\lambda_i^n+\max_{(i,j)\in\mathcal{E}}\mu_{ij}^n+\max_{i\in\mathcal{I}}\gamma_i^n\Big)\le C$ is uniformly bounded in n, substituting (3.17) into (3.16), and choosing large enough $C>0$, we further have

$$
\begin{aligned}
\widehat{\mathcal{L}}_n^*\widetilde{\mathcal{V}}_\eta(\hat{x})&\le C\Big(\eta+M(l)\eta-\eta\|\hat{x}\|\Big)\widetilde{\mathcal{V}}_\eta(\hat{x})\\
&\quad+\Big(C\eta^2+C\eta^2\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}(i)}|z_{ij}^*|+\frac{|\mathcal{E}|CC_0'\eta^2}{\sqrt{n}}\|\hat{x}\|\\
&\qquad+\frac{CC_0'I\eta^2}{\sqrt{n}}\|\hat{x}\|+\epsilon\big(2I+|\mathcal{E}|\big)\eta C\Big)\widetilde{\mathcal{V}}_\eta(\hat{x})
\end{aligned}
$$

$$\leq \Big(C_1\eta - C_2\eta\|\hat{x}\|\Big)\widetilde{\mathcal{V}}_\eta(\hat{x}), \tag{3.18}$$

where

$$C_1 = C_1(\eta) \doteq C + M(l)C + C\eta + C\eta\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}(i)}|z^*_{ij}| + \epsilon C\big(2I + |\mathcal{E}|\big)\,,$$

$$C_2 = C_2(\eta) \doteq C - \frac{(|\mathcal{E}|+I)CC_0'\eta}{\sqrt{n}}\,.$$

This completes the proof of part (i).

We now move on to part (ii). Note that $Z^n(t) = \Upsilon\big(X^n(t) - q^n[v]\big(\hat{X}^n(t)\big), N^n - y^n[v]\big(\hat{X}^n(t)\big)\big)$. We first observe that for η large enough, the function

$$-C_1\eta + C_2\eta\|\hat{x}\| - (1+\rho)\hat{r}(\hat{q},\hat{y}) \tag{3.19}$$

is inf-compact, for every $\rho > 0$. Therefore, using (3.18) and applying Itô's formula to $e^{\eta\int_0^t\big(C_2\|\hat{X}^n(s)\|-C_0\big)\mathrm{d}s}\widetilde{\mathcal{V}}_\eta(\hat{X}^n(t))$ up to the stopping time $T\wedge\tau^n_R$ gives us

$$\begin{aligned}
&\mathbb{E}^{Z^n}\Big[\exp\Big(\eta\int_0^{T\wedge\tau^n_R}\big(C_2\|\hat{X}^n(t)\| - C_1\big)\mathrm{d}t\Big)\widetilde{\mathcal{V}}_\eta(\hat{X}^n(T\wedge\tau^n_R))\Big]\\
&= \widetilde{\mathcal{V}}_\eta(\hat{X}^n(0)) + \mathbb{E}^{Z^n}\Big[\int_0^{T\wedge\tau^n_R} e^{\eta\int_0^t\big(C_2\|\hat{X}^n(s)\|-C_1\big)\mathrm{d}s}\Big(\widehat{\mathcal{L}}^*\widetilde{\mathcal{V}}_\eta(\hat{X}^n(t)) + \eta\big(C_2\|\hat{X}^n(t)\| - C_1\big)\widetilde{\mathcal{V}}_\eta(\hat{X}^n(t))\Big)\mathrm{d}t\Big]\\
&\leq \widetilde{\mathcal{V}}_\eta(\hat{X}^n(0))\,.
\end{aligned}$$

The last inequality is obtained from (3.18). From here, using the fact that $\widetilde{\mathcal{V}}_\eta \geq 1$ and then applying Fatou's Lemma, we get

$$\mathbb{E}^{Z^n}\Big[\exp\Big(\eta\int_0^T\big(C_2\|\hat{X}^n(t)\| - C_1\big)\mathrm{d}t\Big)\Big] \leq \widetilde{\mathcal{V}}_\eta(\hat{X}^n(0))\,.$$

This implies

$$\limsup_{T\to\infty}\frac{1}{T}\log\mathbb{E}^{Z^n}\Big[\exp\Big(\eta\int_0^T\big(C_2\|\hat{X}^n(t)\| - C_1\big)\mathrm{d}t\Big)\Big] \leq 0\,.$$

Using the above display and the inf-compactness of the function in (3.19), we have

$$\int_0^T\hat{r}(\hat{Q}^n(t),\hat{Y}^n(t))\mathrm{d}t \leq (1+\rho)\int_0^T\hat{r}(\hat{Q}^n(t),\hat{Y}^n(t))\mathrm{d}t \leq \eta\int_0^T\big(C_2\|\hat{X}^n(t)\| - C_1\big)\mathrm{d}t + CT,$$

for some large enough $C > 0$. This consequently proves (3.13) and subsequently, also proves that $J^n\big(X^n(0), Z^n\big) \leq C$. From here, we can immediately see that $\hat{\Lambda}\big(\hat{X}^n(0)\big) < \infty$. This completes the proof of part (ii) and the result. □

4. Optimality characterization of the ERSC problem for the limiting diffusion

In Proposition 3.1, we have shown that the ERSC problem for the limiting diffusion model is well-posed. In this section, we show that there exist optimal stationary Markov controls and provide their characterization *via.* the Hamilton-Jacobi-Bellman (HJB) equation. We make use of the results on the ERSC problem for diffusions under a general structural hypothesis obtained in [1].

Recall that the running cost is $r(x,u) = \hat{r}\big((e\cdot x)^+u^c, (e\cdot x)^-u^s\big)$ with $\hat{r}$ given by (2.20) and the generator of the diffusion X is $\mathcal{L}^u$ given by (2.28).

Definition 4.1. We say the ERSC problem for the limiting diffusion model satisfies the general structural hypothesis if the following hold: for some open set $\mathcal{K} \subset \mathbb{R}^I$,

(i) the set $\{(x,u) \in \mathbb{R}^I \times \mathbb{U} : r(x,u) \leq l\} \cap (\mathcal{K}\times\mathbb{U})$ is compact, for every $l \geq 0$;

(ii) there exist inf-compact functions $\mathcal{V} \in \mathcal{C}^2(\mathbb{R}^I)$, $h \in \mathcal{C}(\mathbb{R}^I \times \mathbb{U})$ and positive constants l_1, l_2, l_3 with $l_3 < 1$ such that

$$\begin{aligned} \mathcal{L}^u \mathcal{V}(x) &\le \big(l_1 - h(x,u)\big)\mathcal{V}(x), \qquad \text{for} \quad (x,u) \in \mathcal{K}^c \times \mathbb{U}, \\ \mathcal{L}^u \mathcal{V}(x) &\le \big(l_2 + l_3 r(x,u)\big)\mathcal{V}(x), \qquad \text{for} \quad (x,u) \in \mathcal{K} \times \mathbb{U}\,. \end{aligned} \tag{4.1}$$

Remark 4.1. Letting $\mathcal{H} \doteq (\mathcal{K} \times \mathbb{U}) \times \{(x,u) \in \mathbb{R}^I \times \mathbb{U} : r(x,u) > h(x,u)\}$, from (4.3) we can easily infer that with $l_4 = l_1 \wedge l_2$,

$$\mathcal{L}^u \mathcal{V}(x) \le \big(l_4 - h(x,u)\mathbb{1}_{\mathcal{H}^c}(x,u) + l_3 r(x,u)\mathbb{1}_{\mathcal{H}}(x,u)\big)\mathcal{V}(x)\,. \tag{4.2}$$

On the other hand, if (4.2) holds for inf-compact functions $\mathcal{V} \in \mathcal{C}^2(\mathbb{R}^I)$, $h \in \mathcal{C}(\mathbb{R}^I \times \mathbb{U})$ and positive constants l_3 and l_4 with $l_3 < 1$ for a set $\mathcal{H} \subset \mathbb{R}^I \times \mathbb{U}$, then, it is easy to see that (4.3) holds with $\mathcal{K} = \mathcal{H}$, $\mathcal{V}$, h, $l_1 = l_2 = l_4$ and l_3.

To understand Definition 4.1 better, observe that when $\mathcal{K} = \mathbb{R}^I$, Definition 4.1(i) implies that the running cost $r(\cdot,u)$ is near-monotone (in particular, inf-compact), and when $\mathcal{K} = \emptyset$, (4.3) reduces to

$$\mathcal{L}^u \mathcal{V}(x) \le \big(l_1 - h(x,u)\big)\mathcal{V}(x), \qquad \text{for} \quad (x,u) \in \mathbb{R}^I \times \mathbb{U}\,.$$

This is the notion of uniform stability in the context of ERSC problem; see [7, Pg. 210] for an equivalent definition. For this reason, general structural hypothesis can be regarded as a mixed condition of uniform stability and near-monotonicity (in particular, inf-compactness).

In contrast to the above display, in the context of CEC (see [6, 10, 13] for definitions of general hypothesis in the context of CEC for various models), when $\mathcal{K} = \emptyset$, the inequality analogous to (4.3) reduces to

$$\mathcal{L}^u \mathcal{V}(x) \le l_1 - h(x,u), \qquad \text{for} \quad (x,u) \in \mathbb{R}^I \times \mathbb{U}\,.$$

Comparing the above two displays, one can see that on the complement of a large ball, $\big(l_1 - h(x,u)\big)\mathcal{V}(x) \le \big(l_1 - h(x,u)\big)$ - meaning that the general hypothesis given by Definition 4.1 is more restrictive than the analogous condition in the context of CEC. This is unsurprising because of the exponential nature of the ERSC problem at hand.

The following result shows that our ERSC problem for the limiting diffusion model indeed satisfies the general structural hypothesis.

Proposition 4.1. *Suppose Assumptions 2.1 and 2.2 hold. Then, for an inf-compact $\mathcal{C}^2(\mathbb{R}^I)$ function $\mathcal{V} : \mathbb{R}^I \to [1,\infty)$ such that it is equal to $\exp\big(\frac{\eta}{2}(x^{\mathsf{T}}Qx)^{\frac{1}{2}}\big)$ on $\{x \in \mathbb{R}^I : \|x\| \ge 1\}$ and $\mathcal{K} = \mathcal{K}_\delta \doteq \{x \in \mathbb{R}^I : |e\cdot x| > \delta\|x\|\}$, we have*

$$\mathcal{L}^u \mathcal{V}(x) \le \Big(C_0\eta - C_1\eta\|x\|\mathbb{1}_{\mathcal{K}_\delta^c}(x) + C_2\eta|e\cdot x|\mathbb{1}_{\mathcal{K}_\delta}(x)\Big)\mathcal{V}(x) \tag{4.3}$$

for some positive constants C_i, $i = 0,1,2$ such that $C_2 < 1$. In particular, for small enough η, the ERSC problem for the limiting diffusion model satisfies the general structural hypothesis with $\mathcal{V}$, $h(x,u) = C_1\|x\|$, $\mathcal{K}$ and $C_0\eta, C_1, C_2$. Here, Q is a $I \times I$ positive diagonal matrix from Remark 3.1.

Proof. To begin with, it is clear that on $\{x \in \mathbb{R}^I : \|x\| < 1\}$, $b(x,u)\cdot\nabla\mathcal{V}(x) \le C$, for some $C > 0$. On $\{x \in \mathbb{R}^I : \|x\| \ge 1\}$, we have

$$\begin{aligned} \big(\eta\mathcal{V}(x)\big)^{-1} b(x,u)\cdot\nabla\mathcal{V}(x) &= \ell\cdot\nabla\mathcal{V}(x) - \frac{1}{2}(x^{\mathsf{T}}Qx)^{-\frac{1}{2}}x^{\mathsf{T}}(QB + B^{\mathsf{T}}Q)x \\ &\quad + (x^{\mathsf{T}}Qx)^{-\frac{1}{2}}Qx^{\mathsf{T}}\big((B-\Gamma)(e\cdot x)^+u^c + \widehat{B}(e\cdot x)^-u^s\big) \\ &\le \big(\ell^{\mathsf{T}}Qx + C\|x\||e\cdot x| - 8\|x\|^2\big)(x^{\mathsf{T}}Qx)^{-\frac{1}{2}}, \end{aligned}$$

for large enough $C > 0$. Choosing $\delta = C^{-1}$, we have

$$\big(\eta\mathcal{V}(x)\big)^{-1} b(x,u)\cdot\nabla\mathcal{V}(x) \le C - 7(x^{\mathsf{T}}Qx)^{-\frac{1}{2}}\|x\|^2, \qquad \text{for every } x \in \mathcal{K}_\delta^c \cap \{x \in \mathbb{R}^I : \|x\| \ge 1\}\,,$$

for large enough $C > 0$. On the other hand, we have

$$\big(\eta \mathcal{V}(x)\big)^{-1} b(x,u)\cdot\nabla\mathcal{V}(x) \le C(1+|e\cdot x|), \qquad \text{for every } x\in\mathcal{K}_\delta \cap \{x\in\mathbb{R}^I : \|x\|\ge 1\}\,,$$

for large enough $C > 0$. Therefore, combining the last two displays, we have

$$b(x,u)\cdot\nabla\mathcal{V}(x) \le \Big(C - \underline{C}\|x\|\mathbb{1}_{\mathcal{K}_\delta^c}(x) + C|e\cdot x|\mathbb{1}_{\mathcal{K}_\delta}(x)\Big)\eta\mathcal{V}(x),$$

on $\{x\in\mathbb{R}^I : \|x\|\ge 1\}$, for large enough $C > 0$ and a constant $\underline{C} > 0$.

Hence, for every $x\in\mathbb{R}^I$, we obtain that

$$b(x,u)\cdot\nabla\mathcal{V}(x) \le \Big(C + \frac{C}{\eta\mathcal{V}(x)} - \underline{C}\|x\|\mathbb{1}_{\mathcal{K}_\delta^c}(x) + C|e\cdot x|\mathbb{1}_{\mathcal{K}_\delta}(x)\Big)\eta\mathcal{V}(x)\,. \tag{4.4}$$

On the other hand, from the form of $\mathcal{V}$, we also get that

$$\sum_{i=1}^{I}\lambda_i\frac{\partial^2}{\partial x_i^2}\mathcal{V}(x) \le (C + C\eta^2)\mathcal{V}(x), \qquad \text{for} \quad x\in\mathbb{R}^I,$$

for large enough $C > 0$. Combining the above two displays and the fact that $\mathcal{V}(x)\ge 1$, we have

$$\mathcal{L}^u\mathcal{V}(x) \le \Big(\frac{2C}{\eta} + C + C\eta - \underline{C}\|x\|\mathbb{1}_{\mathcal{K}_\delta^c}(x) + C|e\cdot x|\mathbb{1}_{\mathcal{K}_\delta}(x)\Big)\eta\mathcal{V}(x)\,.$$

This proves (4.3) with $C_0 = 2C\eta^{-1} + C + C\eta$, $C_1 = \underline{C}$ and $C_2 = C$.

From (2.27), we have constants $r_1, r_2 > 0$, $r_1|e\cdot x| \le r(x,u) \le r_2|e\cdot x|$. Therefore, for η small enough, we can ensure that $\eta C < r_1$ and in conjunction with Remark 4.1 (in particular, (4.2)), we thereby have shown the second part of the proposition. This completes the proof. □

Remark 4.2. From the proof, the reader may notice that the linear growth property of r in (2.27) plays a key role. The Lyapunov inequality in (4.3) also illustrates why such a linear growth property of running cost is considered in the paper. With minor modifications to the proof, the above proposition can also be shown to hold if $r(x,u) = \hat{r}\big((e\cdot x)^+u^c, (e\cdot x)^-u^s\big)$ is of the form

$$\sum_{i\in\mathcal{I}}\xi_i(e\cdot x)^+u_i^c \qquad \text{or} \qquad \sum_{j\in\mathcal{J}}\zeta_j(e\cdot x)^-u_j^s\,.$$

We now provide the result which includes the well-posedness of the associated HJB equation and the characterization for the optimal stationary Markov controls to the ERSC problem for the limiting diffusion model.

Theorem 4.1. *Under Assumptions 2.1 and 2.2, there exists a pair* $(V,\widetilde{\Lambda})\in\mathcal{C}^2(\mathbb{R}^I)\times\mathbb{R}_+$ *such that*

$$\min_{u\in\mathbb{U}}\big[\mathcal{L}^uV(x) + r(x,u)\,V(x)\big] \;=\; \widetilde{\Lambda}\,V(x) \qquad \forall\, x\in\mathbb{R}^I\,.$$

Moreover,

(i) $\widetilde{\Lambda} = \Lambda_{SM} = \Lambda$ *and the function* V *is unique up to a multiplicative constant.*

(ii) $v\in\mathfrak{U}_{\mathrm{SM}}$ *is optimal if and only if* v *satisfies*

$$\mathcal{L}^vV(x) + r^v(x)\,V(x) \;=\; \min_{u\in\mathbb{U}}\,\big[\mathcal{L}^uV(x) + r(x,u)\,V(x)\big] \quad a.e.\ x\in\mathbb{R}^I\,. \tag{4.5}$$

Proof. From Propositions 3.1 and 4.1, we can see that the hypothesis of [1, Theorem 2.1] is verified. This immediately gives us the result. □

We next present a result on nearly optimal controls for the ERSC problem defined in (2.30), which will be used in the proof of the upper bound for asymptotic optimality.

Recall that $\mathfrak{U}_{\mathrm{SM}}(l)$ is the collection of all $v\in\mathfrak{U}_{\mathrm{SM}}$ that are of the form $v(x) = v^*$, whenever $x\in B_l^c$ with v^* defined in (3.1).

Proposition 4.2. *For every $\delta > 0$, there exist $l_\delta > 0$ and $v^\delta \in \mathfrak{U}_{\mathrm{SM}}(l_\delta)$ such that $J(x, v^\delta) \leq \Lambda + \delta$ and $v^\delta(\cdot) : \mathbb{R}^I \to \mathbb{U}$ is continuous.*

Remark 4.3. The above proposition holds for a much more general ERSC problem for diffusions than the one considered in (2.30). More precisely, it holds, whenever there exists a stationary Markov control such that a condition analogous to (4.7) holds and the ERSC problem is well posed, *i.e.,* the HJB equation is well posed and the optimal stationary Markov controls are characterized as minimizers of the HJB equation. For instance, this condition is satisfied by: (a) the ERSC problem for a diffusion under uniform stability and an appropriate growth of the running cost. See [7, Pg. 37–38] for the definition of uniform stability and the relevant growth condition on the running cost. We refer to [2, Lemma A.1] for the analogous result, which is discussed the context of the 'V' model (that is shown to be uniformly stable in [9, Theorem 2.2]) and a running cost with linear growth; and (b) the ERSC problem for a diffusion with either a near-monotonicity (as defined in [3, Pg. 1487]) of bounded running cost (see [3, Theorem 1.2]) or inf-compactness of the running cost (that is, all levels sets of the running cost are compact; a special case of near-monotonicity), with a stationary Markov control satisfying (4.7) and finite ERSC cost (see [1, Theorem 3.1]). We remark that the case of near-monotone running cost that is neither bounded nor inf-compact is much harder to analyze. In particular, for the above discussion to hold, it is required to verify a certain monotonicity condition (investigated in detail in [7] - this condition is a priori very difficult to verify) of principle general values of $\mathcal{L}^v$ with v being a stationary optimal Markov control; see [3, Proposition 1.3].

Proof. The proof involves the construction of a family of ERSC problems which is then shown to approximate our ERSC problem for limiting diffusion model, in other words, the associated family of optimal ERSC values approach the optimal ERSC value for our limiting diffusion model, in the limit. To that end, we define the aforementioned family of ERSC problems. For $l > 0$, let

$$b_l(x,u) \doteq \begin{cases} b(x,u), & \text{if } x \in B_l \\ b(x,v^*), & \text{otherwise}. \end{cases} \quad \text{and} \quad r_l(x,u) \doteq \begin{cases} r(x,u), & \text{if } x \in B_l \\ r(x,v^*), & \text{otherwise}. \end{cases}$$

Also, let

$$\Lambda_l \doteq \inf_{x\in\mathbb{R}^I} \inf_{v\in\mathfrak{U}_{\mathrm{SM}}} \limsup_{T\to\infty} \frac{1}{T} \log \mathbb{E}_x^v \Big[\exp\Big(\int_0^T r_l\big(X_l(t), v(X_l(t))\big)\mathrm{d}t \Big) \Big] \tag{4.6}$$

with the process X_l given as the unique strong solution to

$$X_l(t) = x + \int_0^t b_l\big(X_l(s), v(X_l(s))\big)\mathrm{d}s + \Sigma W(t)\,.$$

It is clear that $\sup_{l>0} \Lambda_l \leq \inf_{x\in\mathbb{R}^I} J(x, v^*) < \infty$, from Proposition 3.1, *i.e.,* $\{\Lambda_l\}_{l>0}$ is uniformly bounded in l. For every $l > 0$ and $v \in \mathfrak{U}_{\mathrm{SM}}(l)$, it is easy to see that $b_l(x, v(x)) = b(x, v(x))$ and $r_l(x, v(x)) = r(x, v(x))$, for every $x \in \mathbb{R}^I$. This in turn means that

$$\Lambda_l = \inf_{x\in\mathbb{R}^I} \inf_{v\in\mathfrak{U}_{\mathrm{SM}}(l)} \limsup_{T\to\infty} \frac{1}{T} \log \mathbb{E}_x^v \Big[\exp\Big(\int_0^T r\big(X(t), v(X(t))\big)\mathrm{d}t \Big) \Big]$$

and consequently, we have $\Lambda \leq \sup_{l>0} \Lambda_l$, for every $l > 0$, as $\mathfrak{U}_{\mathrm{SM}}(l) \subset \mathfrak{U}_{\mathrm{SM}}$. To summarize, we have shown that $\Lambda \leq \sup_{l>0} \Lambda_l < \infty$ and any optimal stationary Markov control for Λ_l (if it exists) lies in $\mathfrak{U}_{\mathrm{SM}}(l)$.

Since $\mathcal{L}_l^v f(x) = \mathcal{L}^v f(x)$ on B_l^c, for $f \in \mathcal{C}^2(\mathbb{R}^I)$, using Lemma 3.1, we obtain that for any $v \in \mathfrak{U}_{\mathrm{SM}}(l)$,

$$\mathcal{L}_l^v \mathcal{V}_\eta(x) \leq \Big(C_0 + C_1\|x\| - C_2\|x\|^2 \Big) \mathcal{V}_\eta(x), \qquad \text{for} \quad x \in B_l^c, \tag{4.7}$$

where $\mathcal{V}_\eta$, C_0, C_1 and C_2 are as in Lemma 3.1 and $\mathcal{L}^v_l$ is the generator associated with X_l, *i.e.*,

$$\mathcal{L}^v_l f(x) \doteq \sum_{i=1}^{I} b_{i,l}(x, v(x)) \frac{\partial}{\partial x_i} f(x) + \sum_{j=1}^{I} \lambda_i \frac{\partial^2}{\partial x_i^2} f(x) \,.$$

Using (4.7) and [7, Theorem 4.1], we obtain that there exists a pair $(V_l, \widetilde{\Lambda}_l) \in \mathcal{C}^2(\mathbb{R}^I) \times \mathbb{R}_+$ such that

$$\min_{u\in\mathbb{U}} \bigl[\mathcal{L}^u_l V_l(x) + r_l(x,u)\, V_l(x)\bigr] \;=\; \widetilde{\Lambda}_l\, V_l(x) \qquad \forall\, x \in \mathbb{R}^I \,. \tag{4.8}$$

Moreover, $\widetilde{\Lambda}_l = \Lambda_l$ and there exists $v \in \mathfrak{U}_{\mathrm{SM}}(l)$ that is optimal for Λ_l. Since $\{\Lambda_l\}_{l>0}$ is uniformly bounded in l, following the arguments from the proof of [1, Proposition 6.3], we have the existence of a pair $(\check{V}, \check{\Lambda})$ such that $\lim_{l\to\infty} \Lambda_l = \check{\Lambda}$ and

$$\min_{u\in\mathbb{U}} \bigl[\mathcal{L}^u \check{V}(x) + r(x,u)\, \check{V}(x)\bigr] \;=\; \check{\Lambda}\, \check{V}(x) \qquad \forall\, x \in \mathbb{R}^I \,.$$

It is clear that $\Lambda \le \check{\Lambda}$ and from Theorem 4.1, we know that $\Lambda = \check{\Lambda}$ and $\check{V}$ is identical to V (from Theorem 4.1) up to a multiplicative constant.

Therefore, for any δ, we can choose l large enough such that optimal stationary Markov control $v \in \mathfrak{U}_{\mathrm{SM}}(l)$ for Λ_l, that is δ–optimal for Λ. However, the obtained $v \in \mathfrak{U}_{\mathrm{SM}}(l)$ can a priori only Borel measurable. Following the arguments from the proof of [2, Lemma A.1], we can further obtain $v \in \mathfrak{U}_{\mathrm{SM}}(l)$, for larger l, if necessary, that is continuous and is δ–optimal for Λ. This completes the proof. □

5. Variational formulations and "extended" processes

5.1. **Variational formulation for the limiting diffusion.** In this section, we formulate the ERSC cost for the limiting diffusion model under any admissible control as a variational problem using the well-known variational representation for the exponential functionals of the Brownian motion ([35, Theorem 5.1]; see also [37] for its extensive application in the context of the theory of large deviations).

Recall the controlled diffusion X in (2.24) with I-dimensional Brownian motion W. Let $\{\mathcal{G}_t : t \ge 0\}$ be the filtration generated by $\{W(s) : 0 \le s \le t\}$ such that $\mathcal{G}_0$ includes all the $\mathbb{P}$–null sets.

Lemma 5.1. *[35, Theorem 5.1] For $T > 0$, suppose that $G : \mathfrak{C}^I_T \to \mathbb{R}$ is a non-negative Borel measurable functional. Then the following holds:*

$$\frac{1}{T} \log \mathbb{E}\bigl[\exp\bigl(TG(W)\bigr)\bigr] = \sup_{w\in\mathcal{A}} \mathbb{E}\Bigl[G\Bigl(W + \int_0^{\cdot} w(s)\mathrm{d}s\Bigr) - \frac{1}{2T}\int_0^T \|w(t)\|^2 \mathrm{d}t\Bigr] \,,$$

where $\mathcal{A}$ is the set of all $\mathcal{G}_t$–progressively measurable functions $w : \mathbb{R}_+ \to \mathbb{R}^I$ such that $\mathbb{E}\Bigl[\int_0^T \|w(t)\|^2 \mathrm{d}t\Bigr] < \infty$, for every $T > 0$.

Our interest in the above theorem lies in applying it to $\frac{1}{T}\log \mathbb{E}^U_x \Bigl[\exp\Bigl(\int_0^T r(X(t),U(t))\mathrm{d}t\Bigr)\Bigr]$, from (2.29). Since X is a strong solution, for every $T > 0$, there exists a Borel measurable function $\mathcal{X} : \mathbb{R}_+ \times \mathbb{R}^I \times \mathfrak{C}^I_T \to \mathfrak{C}^I_T$ such that $X([0,t]) = \mathcal{X}\bigl(t, x, W([0,t])\bigr)$, for $t \in [0,T]$. Recall that for a path f in $\mathfrak{C}^I_T$, we write $f([0,t])$ to denote the entire path over the interval $[0,t]$. On the other hand, $U \in \mathfrak{U}$ (see Definition 2.2) which implies that $U(t) = U\bigl(t, W([0,t])\bigr)$ is a Borel measurable functional of W. To summarize, this means that

$$G(W) \doteq \frac{1}{T}\int_0^T r(X(t),U(t))\mathrm{d}t = \frac{1}{T}\int_0^T r\Bigl(\mathcal{X}\bigl(t,x,W([0,t])\bigr), U\bigl(t, W([0,t])\bigr)\Bigr)\mathrm{d}t$$

is a non-negative Borel measurable function on $\mathfrak{C}_T^I$. To understand the right hand side of the representation (in Lemma 5.1) in this case, we consider $G\big(W+\int_0^\cdot w(t)\mathrm{d}t\big)$, for the functional G as chosen above and obtain

$$G\Big(W+\int_0^\cdot w(s)\mathrm{d}s\Big)=\frac{1}{T}\int_0^T r\Big(\mathfrak{X}\big(t,x,W([0,t])+\overline{w}([0,t])\big),U\big(t,W([0,t])+\overline{w}([0,t])\big)\Big)\mathrm{d}t, \quad (5.1)$$

where $\overline{w}(t)\doteq\int_0^t w(s)\mathrm{d}s$. Now observe that the process X^* defined *via.* $X^*([0,t])\doteq\mathfrak{X}\big(t,x,W([0,t])+\overline{w}([0,t])\big)$ is the unique strong solution to the following SDE:

$$dX^*(t)=b\big(X^*(t),U(t,W([0,t])+\overline{w}([0,t]))\big)\mathrm{d}t+\Sigma w(t)\mathrm{d}t+\Sigma\mathrm{d}W(t),\ X^*(0)=x\,. \quad (5.2)$$

In terms of X^*, (5.1) reduces to

$$G\Big(W+\int_0^\cdot w(s)\mathrm{d}s\Big)=\frac{1}{T}\int_0^T r\Big(X^*(t),U\big(t,W([0,t])+\overline{w}([0,t])\big)\Big)\mathrm{d}t\,. \quad (5.3)$$

From the above discussion, Lemma 5.1 can be applied (to $G(\cdot)$ chosen as above) to give us the following lemma.

Lemma 5.2. *For $U\in\mathfrak{U}$ and $x\in\mathbb{R}^I$, the ERSC cost $J(x,U)$ in (2.29) can be equivalently written as*

$$J(x,U)=\limsup_{T\to\infty}\sup_{w\in\mathcal{A}}\mathbb{E}_x^{U,w}\left[\frac{1}{T}\int_0^T\Big(r\big(X^*(t),U\big(t,W([0,t])+\overline{w}([0,t])\big)\big)-\frac{1}{2}\|w(t)\|^2\Big)\mathrm{d}t\right] \quad (5.4)$$

with $\overline{w}(t)\doteq\int_0^t w(s)\mathrm{d}s$ and the process X^* being the unique strong solution to (5.2).

We refer to $w\in\mathcal{A}$ as auxiliary control and the process X^* defined above as the "extended" diffusion under controls U and w. In the above, the dependence on $U\in\mathfrak{U}$ and $w\in\mathcal{A}$ is expressed only through $\mathbb{E}_x^{U,w}$.

Remark 5.1. Observe that we stated the above corollary for $U\in\mathfrak{U}$ which possibly includes U for which $J(x,U)=\infty$, for some $x\in\mathbb{R}^I$. This is because the representation in Lemma 5.1 holds even when the left hand side is not finite. In particular, for $U\in\mathfrak{U}^*$ (which from Proposition 3.1 is non-empty), the representation in the above corollary holds with both the left and the right hand sides being finite.

Remark 5.2. For $v\in\mathfrak{U}_{\mathrm{SM}}$ and $w\in\mathcal{A}$, the process X^* is the unique strong solution to

$$dX^*(t)=b\big(X^*(t),v(X^*(t))\big)\mathrm{d}t+\Sigma w(t)\mathrm{d}t+\Sigma\mathrm{d}W(t),\quad X^*(0)=x\,. \quad (5.5)$$

In this case, the process X^* resembles a well-studied process (denoted by $\widetilde{X}^*$) referred to as 'the ground diffusion' in the literature (see [3, 7]). The process $\widetilde{X}^*$, if it exists, is given as the unique strong solution to

$$d\widetilde{X}^*(t)=b\big(\widetilde{X}^*(t),v(\widetilde{X}^*(t))\big)\mathrm{d}t+\Sigma\Sigma^{\mathsf{T}}\nabla\Phi^v\big(\widetilde{X}^*(t)\big)\mathrm{d}t+\Sigma\mathrm{d}W(t),\quad \widetilde{X}^*(0)=x\,. \quad (5.6)$$

Here, $\Phi^v\in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^I)$ (the Sobolov space that contains functions with first two weak derivatives that lie in $L^p_{\mathrm{loc}}(\mathbb{R}^I)$) is the principal eigenfunction (or the ground state) associated with the operator given by $\mathcal{L}^v f(x)+r^v(x)f(x)$, for any $f\in\mathcal{C}^2(\mathbb{R}^I)$; see [7, Pg. 9] for the definition of this notion. There is one fundamental difference between X^* and $\widetilde{X}^*$: the process X^* is defined for $v\in\mathfrak{U}_{\mathrm{SM}}$ and $w\in\mathcal{A}$, where for every $t\geq 0$, $w(t)$ is a Borel measurable function of $W([0,t])$ as it is progressively measurable with respect to $\mathcal{G}_t$ and the process $\widetilde{X}^*$ is defined for $v\in\mathfrak{U}_{\mathrm{SM}}$, but in place of $w\in\mathcal{A}$, we have $\Sigma^{\mathsf{T}}\nabla\Phi^v\big(\widetilde{X}^*(t)\big)$. In other words, in terms of the function $\mathfrak{X}$, we can express $\widetilde{X}^*$ as

$$\widetilde{X}^*([0,t])\doteq\mathfrak{X}\big(t,x,W([0,t])+\widetilde{w}([0,t])\big)\text{ with }\widetilde{w}(t)=\Sigma^{\mathsf{T}}\nabla\Phi^v\big(\widetilde{X}^*(t)\big)\,.$$

Due to the implicit nature of the expression above, it is not a priori clear if the process $\widetilde{X}^*$ can be defined at all times $t \geq 0$ as the unique strong solution to (5.6). We refer the reader to [7, Lemma 2.5] and [3, Theorem 1.5] for conditions under which one can ensure the unique existence of the process $\widetilde{X}^*$, for all $t \geq 0$. This 'restrictive existence' of the ground diffusion process $\widetilde{X}^*$ makes it less desirable to work with for our purposes than the "extended" diffusion process X^*.

Since $w \in \mathcal{A}$ in Lemma 5.2 resembles a control, we define the notion of admissibility, stationary Markov control in this context, just as we did in the case of U.

Definition 5.1. A $\mathbb{R}^I$–valued process w is said to be admissible if it satisfies the following: if $w(t) = w(t,\omega)$ is jointly measurable in $(t,\omega) \in \mathbb{R}^+ \times \Omega$ and for every $0 \leq s < t$, $W(t) - W(s)$ is independent of the completed filtration (with respect to $(\mathcal{F}, \mathbb{P})$) generated by $\{X(0), w(r), W(r) : r \leq s\}$. The set of all such controls is denoted by $\mathfrak{W}$.

We denote $\mathfrak{W}_{\mathrm{SM}} \subset \mathfrak{W}$ as the set of stationary Markov controls (including the relaxed controls). For $l > 0$, let $\mathfrak{W}_{\mathrm{SM}}(l) \subset \mathfrak{W}_{\mathrm{SM}}$ be the set of all stationary Markov controls which are such that $\sup_{x\in\mathbb{R}^I} \|w(x)\| \leq l$ and of the form $w = w(\cdot)$ on B_l and $w \equiv 0$ on B_l^c.

Suppose that for $v \in \mathfrak{U}_{\mathrm{SM}}$, the operations '$\limsup_{T\to\infty}$' and '$\sup_{w\in\mathcal{A}}$' in Lemma 5.2 are interchangeable, *i.e.,* suppose we have

$$\text{“}J(x,v) = \sup_{w\in\mathcal{A}} \limsup_{T\to\infty} \mathbb{E}_x^{v,w}\Big[\frac{1}{T}\int_0^T \Big(r\big(X^*(t), v\big(X^*(t)\big)\big) - \frac{1}{2}\|w(t)\|^2\Big)\mathrm{d}t\Big]\,.\text{”}$$

Then, we can regard $J(x,v)$ as the maximum value of a CEC problem in terms of controls $w \in \mathcal{A}$. Proposition 5.1 below shows that the aforementioned interchangeability indeed holds, in an appropriate form. For $v \in \mathfrak{U}_{\mathrm{SM}}$ and $w \in \mathcal{A}$, define

$$J_{v,w}(x) \doteq \limsup_{T\to\infty} \mathbb{E}_x^{v,w}\Big[\frac{1}{T}\int_0^T \Big(r^v\big(X^*(t)\big) - \frac{1}{2}\|w(t)\|^2\Big)\mathrm{d}t\Big]\,.$$

Proposition 5.1. *Under Assumptions 2.1 and 2.2, the following holds: for every $v \in \mathfrak{U}^*_{\mathrm{SM}}$ (recall the definition of $\mathfrak{U}^*_{\mathrm{SM}}$ from (2.31)) and $x \in \mathbb{R}^I$,*

$$J(x,v) = \sup_{w\in\mathfrak{W}_{\mathrm{SM}}} J_{v,w}(x)\,.$$

Here, $J(x,v)$ is defined in (2.29).

Proof. From Propositions 3.1 and 4.1, and [1, Proposition 6.1], we immediately get the result. ☐

Remark 5.3. From the above proposition, we see that not only the operations '$\limsup_{T\to\infty}$' and '$\sup_{w\in\mathcal{A}}$' in Lemma 5.2 commute, we can replace '$\sup_{w\in\mathcal{A}}$' with '$\sup_{w\in\mathfrak{W}_{\mathrm{SM}}}$' which is clearly over a smaller set. We briefly discuss the proof of [1, Proposition 6.1] and the key arguments involved. There are two main components of the proof: (i) tightness of the family of MEMs of the "extended" diffusion X^* associated with (5.4) applied to $U = v$ and the nearly optimal auxiliary controls, and (ii) construction of a perturbed ERSC problem with running cost r^ε (with ERSC cost denoted by $J^\varepsilon(x,v)$ and analogously defined $J^\varepsilon_{v,w}(x)$) for which it is easier to show a result analogous to Proposition 5.1, *i.e.,* $J^\varepsilon(x,v) = \sup_{w\in\mathfrak{W}_{\mathrm{SM}}} J^\varepsilon_{v,w}(x)$. We focus mainly on the second component below. To begin with, Proposition 4.1 ensures the existence of an inf-compact function $h : \mathbb{R}^I \to \mathbb{R}$ such that

$$\limsup_{T\to\infty} \frac{1}{T}\log \mathbb{E}_x^v\Big[\exp\Big(\int_0^T h\big(X(t)\big)\mathrm{d}t\Big)\Big]$$

is finite, whenever $J(x,v)$ is finite. Using the function h, the perturbation r^ε is chosen as $r + \varepsilon h$, for sufficiently small $\varepsilon > 0$. The perturbed ERSC problem is then shown to satisfy an 'exponential uniform integrability' property. This property, in conjunction with the application of Lemma 5.1 (which gives a result analogous to Lemma 5.2 for r^ε) implies that the family of MEMs of the nearly optimal auxiliary controls (associated with r^{ε}) is tight and more importantly, it is uniformly integrable with respect to r^{ε}. This helps us in showing that $J^{\varepsilon}(x,v) = \sup_{w\in\mathfrak{W}_{\mathrm{SM}}} J^{\varepsilon}_{v,w}(x)$ by constructing further a family of truncated CEC problems (with a running cost that is a truncated version of r^{ε} and allows controls that have only compact support) and then taking the truncation to the limit, owing to the aforementioned uniform integrability of the family of MEMs with respect to r^{ε}. Finally, once $J^{\varepsilon}(x,v) = \sup_{w\in\mathfrak{W}_{\mathrm{SM}}} J^{\varepsilon}_{v,w}(x)$ is shown, upon observing that $J^{\varepsilon}_{v,w}(x)$ is linear in εh, using the tightness of the family of MEMs of X^* and taking $\varepsilon \to 0$ concludes [1, Proposition 6.1].

Remark 5.4. Observe that for every $v \in \mathfrak{U}^*_{\mathrm{SM}}$ and $w \in \mathfrak{W}_{\mathrm{SM}}$, $J_{v,w}(x)$ is the CEC cost with the associated running cost being

$$r^v(x) - \frac{1}{2}\|w(x)\|^2 .$$

Therefore, another consequence of Proposition 5.1 is that $J(x,v)$, for every $v \in \mathfrak{U}^*_{\mathrm{SM}}$, is exactly the optimal value (in particular, the maximum value) of the CEC problem given by $\sup_{w\in\mathfrak{W}_{\mathrm{SM}}} J_{v,w}(x)$. However, we remark that this CEC problem is fundamentally very difficult to analyze than the CEC problem extensively studied in [8, Chapter 3] - in that, the running cost (see the above display) is neither bounded from above nor from below. Also, the control w takes values in a non-compact space, *viz.,* in $\mathbb{R}^I$.

Lemma 5.2 and the fact that $\Lambda = \Lambda_{\mathrm{SM}}$ (from Theorem 4.1) clearly imply that

$$\Lambda = \inf_{x\in\mathbb{R}^I}\ \inf_{v\in\mathfrak{U}_{\mathrm{SM}}}\ \limsup_{T\to\infty}\ \sup_{w\in\mathcal{A}} \mathbb{E}^{v,w}_x\Big[\frac{1}{T}\int_0^T \Big(r^v\big(X^*(t)\big) - \frac{1}{2}\|w(t)\|^2\Big)\mathrm{d}t\Big] .$$

Using Proposition 5.1 above, we can express Λ as

$$\Lambda = \inf_{x\in\mathbb{R}^I}\ \inf_{v\in\mathfrak{U}_{\mathrm{SM}}}\ \sup_{w\in\mathfrak{W}_{\mathrm{SM}}} J_{v,w}(x) .$$

This expression clearly indicates that Λ is closely related to a min-max problem involving $J_{v,w}(x)$ as the optimizing criterion. Proposition 5.2 below in fact, shows that Λ is equal to the value of a two-person zero-sum stochastic game with ergodic pay-off criterion $J_{v,w}(x)$ involving the "extended" diffusion X^*, controls v and w being the minimizing and maximizing strategies, respectively. This can easily be regarded as the most important consequence of the variational formulation given above.

Proposition 5.2. *Under Assumptions 2.1 and 2.2, the following hold:*

(i) *For every $l > 0$ and $\delta > 0$, there exists $w^*_l \in \mathfrak{W}_{\mathrm{SM}}(l)$ such that*

$$\inf_{x\in\mathbb{R}^I}\ \inf_{v\in\mathfrak{U}_{\mathrm{SM}}} J_{v,w^*_l}(x) \ge \inf_{x\in\mathbb{R}^I}\ \sup_{w\in\mathfrak{W}_{\mathrm{SM}}(l)}\ \inf_{v\in\mathfrak{U}_{\mathrm{SM}}} J_{v,w}(x) - \delta ,$$

and $w^*_l(\cdot)$ is continuous.

(ii)

$$\Lambda = \inf_{x\in\mathbb{R}^I}\ \sup_{w\in\mathfrak{W}_{\mathrm{SM}}}\ \inf_{v\in\mathfrak{U}_{\mathrm{SM}}} J_{v,w}(x) \tag{5.7}$$

$$= \inf_{x\in\mathbb{R}^I}\ \lim_{l\to\infty}\ \sup_{w\in\mathfrak{W}_{\mathrm{SM}}(l)}\ \inf_{v\in\mathfrak{U}_{\mathrm{SM}}} J_{v,w}(x) . \tag{5.8}$$

Proof. The proof of (5.7) and (5.8) follows from the combination Lemma 6.5 and Theorem 2.1(iv) of [1]. The proof of part (i) follows from the combination of Theorems 5.2, 5.3 and 6.1 of [1]. ☐

Remark 5.5. At the first glance, the above result may seem to follow directly from well-known existing results in the literature on two-person zero-sum stochastic game with an ergodic pay-off criterion (for example, see [34, Section 4.2]). But this is not true for reasons similar to those stated in Remark 5.4. Even though the above proof invokes several results from [1], the basic proof methodology of these results is very similar to the one discussed in Remark 5.4. We begin by working with the same r^ε and construct a truncated version of the two-person zero-sum stochastic game in which the maximizing strategies have compact support and the truncated running cost is bounded and continuous. Then, we use results from [34, Section 4.2], to conclude the result analogous to Proposition 5.2 for this truncated version. Finally, using the tightness of the family of MEMs of the "extended" diffusion X^* (mentioned in Remark 5.4), we take the truncation parameter to infinity and then the perturbation parameter ε to zero.

5.2. **Variational formulation for the diffusion-scaled queueing processes.** In this section, we formulate the ERSC problem for the nth model using the variational representation of exponential functionals of the Poisson process. This formulation was first studied in [2].

For a one-dimensional Poisson process $\widetilde{N}$ with rate $\lambda > 0$, and let $\{\mathcal{F}_t : t \geq 0\}$ be the filtration generated by the process $\widetilde{N}$ such that $\mathcal{F}_0$ contains all $\mathbb{P}$–null sets. Let $\varkappa : \mathbb{R}_+ \to \mathbb{R}_+$ be defined as $\varkappa(z) \doteq z \ln z - z + 1$. Define $\widetilde{\mathcal{A}}$ as the set of $\mathbb{R}_+$–valued functions ϕ which are progressively measurable (with respect to $\{\mathcal{F}_t\}$) such that for every $T > 0$, $\mathbb{E}\big[\int_0^T \varkappa\big(\phi(t)\big)\mathrm{d}t\big] < \infty$. Also, define $\widetilde{\mathcal{A}}_M$ as the set of all $\phi \in \widetilde{\mathcal{A}}$ such that for every $T > 0$, $\int_0^T \varkappa\big(\phi(t)\big)\mathrm{d}t \leq M$. Finally, let $\widetilde{N}^\phi$ denote a "controlled" Poisson process which is the unique solution to the martingale problem below: for $f \in \mathcal{C}^2(\mathbb{R})$,

$$f(\widetilde{N}^\phi(t)) - f(0) - \lambda \int_0^t \phi(s)\big[f(\widetilde{N}^\phi(s)+1) - f(\widetilde{N}^\phi(s))\big]\mathrm{d}s$$

is a martingale with respect to $\{\mathcal{F}_t\}$.

Lemma 5.3. *For $T > 0$, suppose that $G : \mathfrak{D}_T \to \mathbb{R}$ is a Borel measurable function, where G is either bounded or non-negative valued. Then the following holds:*

$$\frac{1}{T}\log \mathbb{E}\Big[\exp\Big(TG(\widetilde{N})\Big)\Big] = \sup_{\phi\in\widetilde{\mathcal{A}}} \mathbb{E}\Big[G(\widetilde{N}^\phi) - \frac{\lambda}{T}\int_0^T \varkappa(\phi(t))\mathrm{d}t\Big]. \tag{5.9}$$

Moreover, for every $\delta > 0$, if G is bounded,

$$\frac{1}{T}\log \mathbb{E}\Big[\exp\Big(TG(\widetilde{N})\Big)\Big] \leq \sup_{\phi\in\widetilde{\mathcal{A}}_M} \mathbb{E}\Big[G(\widetilde{N}^\phi) - \frac{\lambda}{T}\int_0^T \varkappa(\phi(t))\mathrm{d}t\Big] + \delta, \tag{5.10}$$

for some $M > 0$ depending only on δ and $\sup_{x\in\mathfrak{D}_T} |G(x)|$.

The above result in the case of bounded G can be found in [37, Theorem 3.23] and the case of non-negative G, follows from a straightforward adaptation of the proof of [35, Theorem 5.1], in conjunction with the result in the bounded case.

Remark 5.6. The content of (5.10) in the above theorem is that for every $\delta > 0$, there are 2δ–optimal controls $\phi \in \widetilde{\mathcal{A}}_M$. This in particular asserts that there exists nearly optimal ϕ for which $\frac{\lambda}{T}\int_0^T \varkappa\big(\phi(t)\big)\mathrm{d}t$ is not only bounded in expectation, but also bounded pointwise.

Definition 5.2. For a Poisson process $\widetilde{N}^n$ with rate $n\lambda$ ($\lambda > 0$) (adapted with respect to a filtration $\{\mathcal{F}_t : t \geq 0\}$) and a $\mathbb{R}_+$–valued process ϕ (that is adapted with respect to $\{\mathcal{F}_t\}$), the process $\widetilde{N}^{n,\phi}$ is defined as the unique solution to the martingale problem below: for $f \in \mathcal{C}^2(\mathbb{R})$,

$$f\Big(\frac{\widetilde{N}^{n,\phi}(t) - n\lambda t}{\sqrt{n}}\Big) - f(0) - \int_0^t n\lambda\phi(s)\Big[f\Big(\frac{\widetilde{N}^{n,\phi}(s) + 1 - n\lambda s}{\sqrt{n}}\Big) - f\Big(\frac{\widetilde{N}^{n,\phi}(s) - n\lambda s}{\sqrt{n}}\Big)\Big]\mathrm{d}s$$

is an $\{\mathcal{F}_t\}$-martingale.

We now apply the above theorem in the context of our nth model. To do this, we first remark that a multi-dimensional version of Lemma 5.3 also holds. Below, we only state it in the context that is relevant for us, *viz.,* for the $2I + |\mathcal{E}|$–dimensional vector of independent Poisson processes given by

$$N^n = \Big(\breve{A}_i^n, \breve{S}_{ij}^n, \breve{R}_i^n : 1 \le i \le I, j \in \mathcal{J}(i)\Big) \tag{5.11}$$

with rates $\big(\lambda_i^n, n\mu_{ij}^n, n\gamma_i^n : 1 \le i \le I, j \in \mathcal{J}(i)\big)$.

We denote the filtration of the process N^n by $\bar{\mathcal{G}}_t^n$, for $t \ge 0$ and we include all $\mathbb{P}$–null sets in $\bar{\mathcal{G}}_0^n$. In the following, using the processes $\breve{A}_i^n$, $\breve{S}_{ij}^n$ and $\breve{R}_i^n$, for $1 \le i \le I$ and $j \in \mathcal{J}(i)$, we re-define the processes $\hat{X}^n$, $\hat{Q}^n$ and $\hat{Y}^n$, for a given admissible SCP Z^n. Since the re-defined processes have the same laws, we reserve the original notation to denote them. In the rest of the paper, we always consider the re-defined versions of these processes. In terms of N^n, under an admissible SCP Z^n, $\hat{X}^n(t)$ is the diffusion-scaled queueing process given by

$$\begin{aligned}
\hat{X}_i^n(t) = \hat{X}_i^n(0) + \ell_i^n t - \sum_{j\in\mathcal{J}(i)} \mu_{ij}^n \int_0^t \hat{Z}_{ij}^n(s)\mathrm{d}s - \gamma_i^n \int_0^t \hat{Q}_i^n(s)\mathrm{d}s,& \\
+ \frac{1}{\sqrt{n}}\Big(\breve{A}_i^n(t) - \lambda_i^n t\Big) - \frac{1}{\sqrt{n}} \sum_{j\in\mathcal{J}(i)} \bigg(\breve{S}_{ij}^n\bigg(\int_0^t \frac{Z_{ij}^n(s)}{n}\mathrm{d}s\bigg) - \mu_{ij}^n \int_0^t Z_{ij}^n(s)\mathrm{d}s\bigg)& \\
- \frac{1}{\sqrt{n}}\bigg(\breve{R}_i^n\bigg(\int_0^t \frac{Q_i^n(s)}{n}\mathrm{d}s\bigg) - \gamma_i^n \int_0^t Q_i^n(s)\mathrm{d}s\bigg),&
\end{aligned}$$

Recall that any given admissible SCP Z^n can be written as $Z^n(t) = Z^n\big(t, N^n([0,t])\big)$.

Let $\psi \doteq \big(\phi_i, \psi_{ij}, \varphi_i : 1 \le i \le I, j \in \mathcal{J}(i)\big)$ be such that $\phi_i, \psi_{ij}, \varphi_i$ are $\mathbb{R}_+$–valued functions for $1 \le i \le I$ and $j \in \mathcal{J}(i)$. Also, let $\hat{X}^{n,\psi}$ be the solution to the following equation:

$$\begin{aligned}
\hat{X}_i^{n,\psi}(t) = \hat{X}_i^{n,\psi}(0) + \ell_i^n t - \sum_{j\in\mathcal{J}(i)} \mu_{ij}^n \int_0^t \hat{Z}_i^{n,\psi}(s)\mathrm{d}s - \gamma_i^n \int_0^t \hat{Q}_i^{n,\psi}(s)\mathrm{d}s + \frac{1}{\sqrt{n}}\Big(\breve{A}_i^{n,\phi}(t) - \lambda_i^n t\Big)& \\
- \frac{1}{\sqrt{n}} \sum_{j\in\mathcal{J}(i)} \bigg(\breve{S}_{ij}^{n,\psi}\bigg(\int_0^t \frac{Z_{ij}^{n,\psi}(s)}{n}\mathrm{d}s\bigg) - \mu_{ij}^n \int_0^t Z_{ij}^{n,\psi}(s)\mathrm{d}s\bigg)& \\
- \frac{1}{\sqrt{n}}\bigg(\breve{R}_i^{n,\varphi}\bigg(\int_0^t \frac{Q_i^{n,\psi}(s)}{n}\mathrm{d}s\bigg) - \gamma_i^n \int_0^t Q_i^{n,\psi}(s)\mathrm{d}s\bigg)&
\end{aligned} \tag{5.12}$$

with

$$\hat{X}_i^{n,\psi}(0) = \hat{X}_i^n(0)\,, \quad \hat{X}_i^{n,\psi}(t) = \hat{Q}_i^{n,\psi}(t) + \sum_{j\in\mathcal{J}(i)} \hat{Z}_{ij}^{n,\psi}(t)\,, \tag{5.13}$$

for $i \in \mathcal{I}$ and $Z^{n,\psi}(t) = Z^n\big(t, N^{n,\psi}([0,t])\big)$. Here, $N^{n,\psi} = \big(\breve{A}_i^{n,\phi}, \breve{S}_{ij}^{n,\psi}, \breve{R}_i^{n,\varphi} : 1 \le i \le I, j \in \mathcal{J}(i)\big)$ with $\breve{A}_i^{n,\phi}$, $\breve{S}_{ij}^{n,\psi}$, $\breve{R}_i^{n,\varphi}$ being defined according to Definition 5.2. From here, we also have

$$Y_j^{n,\psi}(t) = N_j^n - \sum_{i\in\mathcal{I}(j)} Z_{ij}^{n,\psi}(t), \text{ for } j \in \mathcal{J}.$$

Define

$$\mathfrak{K}^n(\psi, T) \doteq \frac{1}{T}\int_0^T \mathfrak{k}^n(\psi(t))\mathrm{d}t\,, \tag{5.14}$$

where

$$\mathfrak{k}^n(\psi(t)) \doteq \sum_{i\in\mathcal{I}} \big(\lambda_i^n \varkappa(\phi_i(t)) + n\gamma_i^n \varkappa(\varphi_i(t))\big) + \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}(i)} n\mu_{ij}^n \varkappa(\psi_{ij}(t))\,. \tag{5.15}$$

Let $\mathcal{A}^n$ be the set of all functions ψ that are $\bar{\mathcal{G}}_t^n$–progressively measurable such that for every $T > 0$, $\mathbb{E}\big[\mathfrak{K}^n(\psi, T)\big] < \infty$. Also for every $M > 0$, define $\mathcal{A}_M^n$ as the set of all $\psi \in \mathcal{A}^n$ such that $\mathfrak{K}^n(\psi, T) \le M$, for every $T > 0$. We are now in a position to state crucial variational representations. We again remark that these results are stated in the form that is relevant for us to work with.

Lemma 5.4. *For every n and every SCP $Z^n \in \mathfrak{Z}^n$, the following relations hold: for every $T, \rho > 0$,*

$$\frac{1}{T}\log \mathbb{E}^{Z^n}\Big[\exp\Big(\int_0^T \hat{r}(\hat{Q}^n(t), \hat{Y}^n(t))\mathrm{d}t\Big)\Big] = \sup_{\psi\in\mathcal{A}^n} \mathbb{E}^{Z^n,\psi}\Big[\frac{1}{T}\int_0^T \hat{r}(\hat{Q}^{n,\psi}(t), \hat{Y}^{n,\psi}(t))\mathrm{d}t - \mathfrak{K}^n(\psi, T)\Big],$$

$$\frac{1}{T}\log \mathbb{E}^{Z^n}\Big[\exp\Big(\int_0^T \rho\|\hat{X}^n(t)\|\mathrm{d}t\Big)\Big] = \sup_{\psi\in\mathcal{A}^n} \mathbb{E}^{Z^n,\psi}\Big[\frac{\rho}{T}\int_0^T \|\hat{X}^{n,\psi}(t)\|\mathrm{d}t - \mathfrak{K}^n(\psi, T)\Big]. \tag{5.16}$$

In the above and in what follows, we use $\mathbb{E}^{Z^n,\psi}$ to emphasize the underlying SCP Z^n and the underlying auxiliary control ψ.

At this point, the reader may wonder if the results analogous to Propositions 5.1 and 5.2 can be proved in the case of diffusion-scaled processes. To clarify this, we remark that the proofs of Propositions 5.1 and 5.2 use the tools that are exclusive to diffusion models and it is not clear if one can adapt these tools to the case of diffusion-scaled queueing processes driven by Poisson processes.

5.3. **Relating the "extended" processes.** In this section, we discuss conditions under which the family of "extended" processes discussed in the previous section relates to the "extended" diffusion processes discussed in Section 5.1. This is done by defining a suitable topology for which the compactness criterion is easy to verify. Much of the content of this section follows very closely the content of [2, Section 3.3], and we also refer the reader to this work for the motivation behind the topology defined below and an in-depth relevant discussion.

Consider a collection $\mathcal{R} \doteq \{\psi^n : \psi^n \in \mathcal{A}^n,\ \text{for } n \in \mathbb{N}\}$. We are interested in obtaining compactness of $\mathcal{R}$ in an appropriate sense, under the assumption that, for some M_1,

$$\limsup_{n\to\infty}\,\limsup_{T\to\infty} \mathfrak{K}^n(\psi^n, T) \le M_1\,. \tag{5.17}$$

To that end, observe that $\sqrt{\varkappa(\phi(\cdot))} \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R})$ (the set of locally square integrable Borel measurable functions on $\mathbb{R}_+$ under the usual strong topology), whenever we have $T^{-1}\int_0^T \varkappa(\phi(s))\mathrm{d}s < \infty$, for $T > 0$. This means that $\mathcal{R}$ can be regarded as a subset of $\mathcal{Z} \doteq L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^I \times \mathbb{R}^{\mathcal{G}} \times \mathbb{R}^I)$; recall that $\mathbb{R}^{\mathcal{G}} \doteq \{\xi_{ij} \in \mathbb{R}^{I\times J} : \xi_{ij} = 0 \text{ for } i \nsim j\}$. The aforementioned topology is defined on $\mathcal{Z}$ as follows: define, $\mathcal{Z}_T \doteq L^2([0,T], \mathbb{R}^I \times \mathbb{R}^{\mathcal{G}} \times \mathbb{R}^I)$ and equip it with the corresponding weak* topology. Now, $\mathcal{Z}$ is equipped with the coarsest topology under which the map $\mathcal{Z} \ni u \mapsto u_{|[0,T]} \in \mathcal{Z}_T$ is continuous, for every $T > 0$. $\mathcal{Z}$ is denoted by $L^{2,*}_\infty$ when equipped with the above topology.

The following result which is a direct adaptation of Corollary 3.3 of [2] illustrates that (5.17) is indeed, a sufficient condition for tightness under the topology defined above. For $t \ge 0$ and $\psi^n = (\phi^n, \psi^n, \varphi^n) \in \mathcal{A}^n$, define

$$h^n[\psi^n](t) \doteq \Big(\sqrt{n}\big(1 - \phi_i^n(t)\big), \sqrt{n}\big(1 - \psi_{ij}^n(t)\big), \sqrt{n}\big(1 - \varphi_i^n(t)\big) : 1 \le i \le I, j \in \mathcal{J}(i)\Big)\,. \tag{5.18}$$

Lemma 5.5. *Suppose for every n, $\psi^n \in \mathcal{A}_M^n$, for some $M > 0$ independent of n. Then, $\{h^n[\psi^n] : n \in \mathbb{N}\}$ is a tight family of $L^{2,*}_\infty$–valued random variables.*

Below, we state a sample-path weak convergence result of the family of processes

$$\big\{\big(\hat{X}^{n,\psi^n}, U^{n,\psi^n}, h^n[\psi^n]\big) : n \in \mathbb{N}\big\}$$

under appropriate conditions, *viz.,* uniform boundedness of $\mathfrak{K}^n(\uppsi^n, T)$ in T and n. We also provide the characterization of the limit which turns out to be the "extended" diffusion. This is necessary to identify the weak limit of the family of MEMs (considered in both lower and upper bounds) as ergodic occupation measures associated with the aforementioned "extended" diffusion. Let $\mathfrak{D}_T^{\mathbb{U}}$ be the set of $\mathbb{U}$–valued càdlàg functions on $[0,T]$ equipped with the Skorohod topology. We provide the sketch proof in the Appendix (see also an analogous result for the multiclass 'V' network model in [2, Theorem 3.6]).

Theorem 5.1. *Suppose* $\uppsi^n = (\phi^n, \psi^n, \varphi^n) \in \mathcal{A}^n$, *for every* n, *is such that*

$$\limsup_{n\to\infty}\limsup_{T\to\infty} \mathfrak{K}^n(\uppsi^n, T) < \infty.$$

Then the family of processes $\{(\hat{X}^{n,\uppsi^n}, U^{n,\uppsi^n}, h^n[\uppsi^n]) : n \in \mathbb{N}\}$ *(with* $\hat{X}^{n,\uppsi^n}$ *defined by* (5.12) *with* $\uppsi = \uppsi^n$*) is tight in* $\mathfrak{D}_T^I \times \mathfrak{D}_T^{\mathbb{U}} \times L_\infty^{2,*}$, *for every* $T > 0$. *Moreover, every limit* $(X^*, U, w) \doteq (X^*_{U,w}, U, w)$ *with* $w = \big(w_i^1, w_{ij}^2, w_i^3 : 1 \le i \le I, j \in \mathcal{J}(i)\big)$ *satisfies*

$$\mathrm{d}X^*(t) = b(X^*(t), U(t))\mathrm{d}t + \Sigma\widetilde{w}(t)\mathrm{d}t + \Sigma\mathrm{d}W(t)\,. \tag{5.19}$$

Here, W *is an* I*–dimensional Brownian motion and* $\widetilde{w}$ *is defined as*

$$\widetilde{w}_i(t) \doteq \lambda_i w_i^1(t) + \sum_{j\in\mathcal{J}(i)} \mu_{ij} z_{ij}^* w_{ij}^2(t)\,.$$

6. Proof of Asymptotic Optimality (Theorem 2.1)

Before we proceed with the proof of Theorem 2.1, we establish two key results below: Proposition 6.1 is used in the proof of the lower bound, where we begin by choosing a sequence of SCPs for every n such that their ERSC cost is uniformly bounded. It will also imply that an ERSC cost associated with an inf-compact function is also bounded. In Proposition 6.2, for a SCP Z^n, we provide sufficient condition on the auxiliary control $\uppsi^n$ which in conjuction with the uniform boundedness (in n) of ERSC cost under Z^n gives us the uniform boundedness (in n) of the CEC of the functional $\|\hat{X}^{n,\uppsi^n}(t)\|$ associated with the "extended" process $\hat{X}^{n,\uppsi^n}$. This is then used later in the proof of upper and lower bounds to obtain the tightness of MEMs of an appropriately chosen family of "extended" diffusion-scaled processes which is fundamental to the proofs of both the lower and upper bounds.

Proposition 6.1. *Suppose Assumptions 2.1 and 2.2 hold, and for every* n, *let* $Z^n \in \mathfrak{Z}^n$. *Then, there exist positive constants* ρ_0 *and* C_0 *and* C_1 *such that for* $\rho < \rho_0$, *we have*

$$\limsup_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\mathbb{E}^{Z^n}\Big[\exp\Big(\rho\int_0^T \|\hat{X}^n(t)\|\mathrm{d}t\Big)\Big] \le C_0 + \max\Big\{C_1, \frac{1}{2}\limsup_{n\to\infty} J^n(\hat{X}^n(0), Z^n)\Big\}. \tag{6.1}$$

Proof. For every n, let $Z^n \in \mathfrak{Z}^n$. Suppose for now that there exist an inf-compact $\mathcal{C}^2(\mathbb{R}^I)$ function $\mathcal{W} : \mathbb{R}^I \to [1,\infty)$ and positive constants C, $\underline{C}$, and $\overline{C}$ such that $\overline{C} < 1$ and

$$\widehat{\mathcal{L}}_n^z\mathcal{W}(\hat{x}) \le \big(C - \underline{C}\mathbb{1}_{\widehat{\mathcal{K}}^c}(\hat{x})\|\hat{x}\| + \overline{C}\mathbb{1}_{\widehat{\mathcal{K}}}(\hat{x})\hat{r}\big(\hat{q}(\hat{x},\hat{z}), \hat{y}(\hat{z})\big)\big)\mathcal{W}(\hat{x}), \tag{6.2}$$

for every $\hat{x} \in \mathbb{R}^I$ and $z \in \mathcal{Z}^n(x)$. Here, $\widehat{\mathcal{K}} = \{\hat{x} \in \mathbb{R}^I : |e\cdot\hat{x}| > \delta\|\hat{x}\|\}$, for $\delta > 0$. Recall that $\widehat{\mathcal{L}}_n^z$ is the infinitesimal generator of $\hat{X}^n$; see (2.17) for the definition. We next show that (6.2) implies (6.1). To that end, we begin by applying Itô's formula to

$$\exp\Big(\int_0^{T\wedge\tau_R^n}\Big(\underline{C}\|\hat{X}^n(t)\|\mathbb{1}_{\widehat{\mathcal{K}}^c}\big(\hat{X}^n(t)\big) - \overline{C}\hat{r}\big(\hat{Q}^n(t), \hat{Y}^n(t)\big)\mathbb{1}_{\widehat{\mathcal{K}}}\big(\hat{X}^n(t)\big) - C\Big)\mathrm{d}t\Big)\mathcal{W}(X_{T\wedge\tau_R^n})$$

and using (6.2), to get

$$\mathbb{E}^{Z^n}\Big[\exp\Big(\int_0^{T\wedge\tau_R^n}\Big(\underline{C}\|\hat{X}^n(t)\|\mathbb{1}_{\widehat{\mathcal{K}}^c}\big(\hat{X}^n(t)\big)-\overline{C}\hat{r}\big(\hat{Q}^n(t),\hat{Y}^n(t)\big)\mathbb{1}_{\widehat{\mathcal{K}}}\big(\hat{X}^n(t)\big)-C\Big)\mathrm{d}t\Big)\mathcal{W}(X_{T\wedge\tau_R^n})\Big]$$
$$\leq \mathcal{W}\big(\hat{X}^n(0)\big)\,.$$

Then, taking $R\to\infty$ and applying Fatou's lemma gives us

$$\limsup_{T\to\infty}\frac{1}{T}\log\mathbb{E}^{Z^n}\Big[\exp\Big(\int_0^T\Big(\underline{C}\|\hat{X}^n(t)\|\mathbb{1}_{\widehat{\mathcal{K}}^c}\big(\hat{X}^n(t)\big)-\overline{C}\hat{r}\big(\hat{Q}^n(t),\hat{Y}^n(t)\big)\mathbb{1}_{\widehat{\mathcal{K}}}\big(\hat{X}^n(t)\big)\Big)\mathrm{d}t\Big)\Big]\leq C\,. \tag{6.3}$$

We now note that for small enough $\eta>0$, we have $\eta\underline{C}\|\hat{x}\|\leq\underline{C}\mathbb{1}_{\widehat{\mathcal{K}}^c}(\hat{x})\|\hat{x}\|+\mathbb{1}_{\widehat{\mathcal{K}}}(\hat{x})|e\cdot\hat{x}|$. Using (2.21), we consequently have

$$\eta\underline{C}\|\hat{x}\|\leq\underline{C}\mathbb{1}_{\widehat{\mathcal{K}}^c}(\hat{x})\|\hat{x}\|+\frac{1}{r_1}\mathbb{1}_{\widehat{\mathcal{K}}}(\hat{x})\hat{r}(\hat{q},\hat{y})+\frac{\epsilon_n}{r_1}\,. \tag{6.4}$$

From here, choosing $\theta>0$, we get

$$\begin{aligned}
&\mathbb{E}^{Z^n}\Big[\exp\Big(\theta\eta\underline{C}\int_0^T\|\hat{X}^n(t)\|\mathrm{d}t\Big)\Big]\\
&\leq\mathbb{E}^{Z^n}\Big[\exp\Big(\int_0^T\Big(\theta\underline{C}\|\hat{X}^n(t)\|\mathbb{1}_{\widehat{\mathcal{K}}^c}\big(\hat{X}^n(t)\big)+\frac{\theta}{r_1}\hat{r}\big(\hat{Q}^n(t),\hat{Y}^n(t)\big)\mathbb{1}_{\widehat{\mathcal{K}}}\big(\hat{X}^n(t)\big)+\frac{\theta\epsilon_n}{r_1}\Big)\mathrm{d}t\Big)\Big]\\
&\leq\frac{1}{2}\mathbb{E}^{Z^n}\Big[\exp\Big(\int_0^T\Big(2\Big(\theta\underline{C}\|\hat{X}^n(t)\|\mathbb{1}_{\widehat{\mathcal{K}}^c}\big(\hat{X}^n(t)\big)-\frac{\overline{C}}{2}\hat{r}\big(\hat{Q}^n(t),\hat{Y}^n(t)\big)\mathbb{1}_{\widehat{\mathcal{K}}}\big(\hat{X}^n(t)\big)\Big)+\frac{2\theta\epsilon_n}{r_1}\Big)\mathrm{d}t\Big)\Big]\\
&\quad+\frac{1}{2}\mathbb{E}^{Z^n}\Big[\exp\Big(\int_0^T\big(\frac{2\theta}{r_1}+\overline{C}\big)\Big(\hat{r}\big(\hat{Q}^n(t),\hat{Y}^n(t)\big)\mathbb{1}_{\widehat{\mathcal{K}}}\big(\hat{X}^n(t)\big)\Big)\mathrm{d}t\Big)\Big]\,.
\end{aligned} \tag{6.5}$$

In the above, the first inequality is obtained from (6.4) and the second inequality is obtained from the application of Young's inequality for the product of two non-negative real numbers (recall, for $a,b\geq 0$ and $p,q>1$ such that $p^{-1}+q^{-1}=1$, we have $ab\leq\frac{a^p}{p}+\frac{b^q}{q}$ and equality holds if and only if $a^p=b^q$) with $p=q=2$,

$$a=\exp\Big(\int_0^T\Big(\theta\underline{C}\|\hat{X}^n(t)\|\mathbb{1}_{\widehat{\mathcal{K}}^c}\big(\hat{X}^n_t\big)-\frac{\overline{C}}{2}\hat{r}\big(\hat{Q}^n(t),\hat{Y}^n(t)\big)\mathbb{1}_{\widehat{\mathcal{K}}}\big(\hat{X}^n(t)\big)\Big)\mathrm{d}t\Big)\,,$$

$$b=\exp\Big(\int_0^T\Big(\big(\frac{\theta}{r_1}+\frac{\overline{C}}{2}\big)\hat{r}\big(\hat{Q}^n(t),\hat{Y}^n(t)\big)\mathbb{1}_{\widehat{\mathcal{K}}}\big(\hat{X}^n(t)\big)\Big)\mathrm{d}t\Big)\,.$$

Choosing $\theta<\min\{\frac{r_1(1-\overline{C})}{2},\frac{1}{2}\}$, (6.5) becomes

$$\begin{aligned}
&\limsup_{T\to\infty}\frac{1}{T}\log\mathbb{E}^{Z^n}\Big[\exp\Big(\theta\eta\underline{C}\int_0^T\|\hat{X}^n(t)\|\mathrm{d}t\Big)\Big]\\
&\leq\max\Big\{\limsup_{T\to\infty}\frac{1}{2T}\log\mathbb{E}^{Z^n}\Big[\exp\Big(\int_0^T\Big(2\theta\underline{C}\|\hat{X}^n(t)\|\mathbb{1}_{\widehat{\mathcal{K}}^c}\big(\hat{X}^n(t)\big)\\
&\qquad-\overline{C}\hat{r}\big(\hat{Q}^n(t),\hat{Y}^n(t)\big)\mathbb{1}_{\widehat{\mathcal{K}}}\big(\hat{X}^n(t)\big)+\frac{2\epsilon_n}{r_1}\Big)\Big)\mathrm{d}t\Big],\\
&\qquad\quad\limsup_{T\to\infty}\frac{1}{2T}\log\mathbb{E}^{Z^n}\Big[\exp\Big(\int_0^T\big(\frac{2\theta}{r_1}+\overline{C}\big)\Big(\hat{r}\big(\hat{Q}^n(t),\hat{Y}^n(t)\big)\mathbb{1}_{\widehat{\mathcal{K}}}\big(\hat{X}^n(t)\big)\Big)\mathrm{d}t\Big)\Big]\Big\}\\
&\leq\max\Big\{\frac{C}{2}+\frac{\epsilon_n}{r_1},\frac{1}{2}J^n\big(\hat{X}^n(0),Z^n\big)\Big\}\,.
\end{aligned}$$

In the above, the first term in the maximum of second inequality is obtained from (6.3) and the fact that $2\theta < 1$ and the second term in the maximum is obtained from the fact that $2\theta r_1^{-1} + \overline{C} < 1$. Now taking $n \to \infty$ and using the fact that $\epsilon_n \to 0$ (see (2.21)), we get (6.1) with $\rho_0 = \theta\eta\underline{C}$.

Therefore, to prove part (i) of the lemma, it suffices to prove that (6.2) holds and this is what we do below. Define $\mathcal{W}(\hat{x}) \doteq \mathcal{V}(\hat{x})$ with $\mathcal{V}$ as constructed in the proof of Proposition 4.1. Recall

$$\widehat{\mathcal{L}}_n^z \mathcal{W}(\hat{x}) = \sum_{i\in\mathcal{I}} \lambda_i^n \mathfrak{d}\mathcal{W}(\hat{x}; n^{-\frac{1}{2}}e_i) + \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}(i)} \mu_{ij}^n z_{ij} \mathfrak{d}\mathcal{W}(\hat{x}; -n^{-\frac{1}{2}}e_i) + \sum_{i\in\mathcal{I}} \gamma_i^n q_i(x,z) \mathfrak{d}\mathcal{W}(\hat{x}; -n^{-\frac{1}{2}}e_i)\,. \tag{6.6}$$

To simplify the above, we use the following identity: if $f \in \mathcal{C}^2(\mathbb{R}^I)$, then

$$\mathfrak{d}f(\hat{x}; \pm n^{-\frac{1}{2}}e_i) \mp \frac{1}{\sqrt{n}}\frac{\partial}{\partial \hat{x}_i} f(\hat{x}) = \int_0^1 (1-s)\frac{\partial^2}{\partial \hat{x}_i^2} f(\hat{x} \pm s n^{-\frac{1}{2}} e_i)\mathrm{d}s\,.$$

From the above display and the form of $\mathcal{W}(\cdot)$, we get

$$\Big|\mathfrak{d}\mathcal{W}(\hat{x}; \pm n^{-\frac{1}{2}}e_i) \mp \frac{1}{\sqrt{n}}\frac{\partial}{\partial \hat{x}_i}\mathcal{W}(\hat{x})\Big| \le \frac{C\eta^2}{n}\mathcal{W}(\hat{x}),$$

for large enough $C > 0$. From the above display, (6.6) becomes

$$\begin{aligned}
\widehat{\mathcal{L}}_n^z \mathcal{W}(\hat{x}) &\le \sum_{i\in\mathcal{I}} \frac{\lambda_i^n}{\sqrt{n}}\frac{\partial}{\partial \hat{x}_i}\mathcal{W}(\hat{x}) - \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}(i)} \frac{\mu_{ij}^n z_{ij}}{\sqrt{n}}\frac{\partial}{\partial \hat{x}_i}\mathcal{W}(\hat{x}) - \sum_{i\in\mathcal{I}} \frac{\gamma_i^n q_i(x,z)}{\sqrt{n}}\frac{\partial}{\partial \hat{x}_i}\mathcal{W}(\hat{x}) \\
&\quad + \frac{C\eta^2}{n}\Big(\sum_{i\in\mathcal{I}}\lambda_i^n + \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}(i)}\mu_{ij}^n z_{ij} + \sum_{i\in\mathcal{I}}\gamma_i^n q_i(x,z)\Big)\mathcal{W}(\hat{x}) \\
&\le \sum_{i\in\mathcal{I}} \ell_i^n \frac{\partial}{\partial \hat{x}_i}\mathcal{W}(\hat{x}) - \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}(i)} \mu_{ij}^n \hat{z}_{ij}\frac{\partial}{\partial \hat{x}_i}\mathcal{W}(\hat{x}) - \sum_{i\in\mathcal{I}} \gamma_i^n \hat{q}_i(x,z)\frac{\partial}{\partial \hat{x}_i}\mathcal{W}(\hat{x}) \\
&\quad + \frac{C\eta^2}{n}\Big(\sum_{i\in\mathcal{I}}\lambda_i^n + \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}(i)}\mu_{ij}^n z_{ij} - \sum_{i\in\mathcal{I}}\gamma_i^n q_i(x,z)\Big)\mathcal{W}(\hat{x}) \qquad (6.7)\\
&\le \sum_{i\in\mathcal{I}} \ell_i^n \frac{\partial}{\partial \hat{x}_i}\mathcal{W}(\hat{x}) - \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}(i)} \mu_{ij}^n \hat{z}_{ij}\frac{\partial}{\partial \hat{x}_i}\mathcal{W}(\hat{x}) - \sum_{i\in\mathcal{I}} \gamma_i^n \hat{q}_i(x,z)\frac{\partial}{\partial \hat{x}_i}\mathcal{W}(\hat{x}) \\
&\quad + \frac{C\eta^2}{n}\big(\sum_{i\in\mathcal{I}}\lambda_i^n + C\|\hat{x}\|\big)\mathcal{W}(\hat{x})\,. \qquad (6.8)
\end{aligned}$$

To get (6.7), we use the definitions of ℓ^n, $\hat{q}^n$ and $\hat{z}^n$ and to get (6.8), we use Assumption 2.1 and (2.16), and the fact that, for large enough $C > 0$,

$$\Big|\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}(i)}\mu_{ij}^n z_{ij} - \sum_{i\in\mathcal{I}}\gamma_i^n q_i(x,z)\Big| \le C\|\hat{x}\|\,.$$

From Assumption 2.1 and the definition of ℓ^n, we know that for every $\epsilon > 0$ and large enough η (depending only on ϵ), we have

$$\max_{(i,j)\in\mathcal{E}}\Big(|\ell_i^n - \ell_i| + |\mu_{ij}^n - \mu_{ij}^n| + |\gamma_i^n - \gamma_i|\Big) < \epsilon\,. \tag{6.9}$$

Also from the form of $\mathcal{W}$, we observe that for every $\hat{x} \in \mathbb{R}^I$,

$$\big|\frac{\partial}{\partial \hat{x}_i}\mathcal{W}(\hat{x})\big| \le \eta\widehat{C}\mathcal{W}(\hat{x}), \tag{6.10}$$

for large enough $C > 0$. Using (6.9) in (6.8) and choosing n large enough gives us

$$\begin{aligned}
\widehat{\mathcal{L}}_n^z \mathcal{W}(\hat{x}) \le& \sum_{i\in\mathcal{I}} (\ell_i + \epsilon) \frac{\partial}{\partial \hat{x}_i} \mathcal{W}(\hat{x}) - \sum_{i\in\mathcal{I}} \sum_{j\in\mathcal{J}(i)} (\mu_{ij} - \epsilon) \hat{z}_{ij} \frac{\partial}{\partial \hat{x}_i} \mathcal{W}(\hat{x}) - \sum_{i\in\mathcal{I}} (\gamma_i - \epsilon) \hat{q}_i(x,z) \frac{\partial}{\partial \hat{x}_i} \mathcal{W}(\hat{x}) \\
&+ \frac{C\eta^2}{n} \big(\sum_{i\in\mathcal{I}} \lambda_i^n + C\|\hat{x}\| \big) \mathcal{W}(\hat{x}) \\
\le& \sum_{i\in\mathcal{I}} \ell_i \frac{\partial}{\partial \hat{x}_i} \mathcal{W}(\hat{x}) - \sum_{i\in\mathcal{I}} \sum_{j\in\mathcal{J}(i)} \mu_{ij} \hat{z}_{ij} \frac{\partial}{\partial \hat{x}_i} \mathcal{W}(\hat{x}) - \sum_{i\in\mathcal{I}} \gamma_i \hat{q}_i(x,z) \frac{\partial}{\partial \hat{x}_i} \mathcal{W}(\hat{x}) \\
&+ \frac{C\eta^2}{n} \big(\sum_{i\in\mathcal{I}} \lambda_i^n + C\|\hat{x}\| \big) \mathcal{W}(\hat{x}) + \epsilon\eta C \big(I + C\|\hat{x}\| \big) \mathcal{W}(\hat{x}),
\end{aligned} \tag{6.11}$$

for large enough $C > 0$. In the second line above, we use (6.10) and the fact that

$$\sum_{i\in\mathcal{I}} \sum_{j\in\mathcal{J}(i)} |\hat{z}_{ij}| + \sum_{i\in\mathcal{I}} |\hat{q}_i(x,z)| \le C\|\hat{x}\|$$

which follows from (2.16). From (4.4), we know that

$$\begin{aligned}
\sum_{i\in\mathcal{I}} \ell_i \frac{\partial}{\partial \hat{x}_i} \mathcal{W}(\hat{x}) -& \sum_{i\in\mathcal{I}} \sum_{j\in\mathcal{J}(i)} \mu_{ij} \hat{z}_{ij} \frac{\partial}{\partial \hat{x}_i} \mathcal{W}(\hat{x}) - \sum_{i\in\mathcal{I}} \gamma_i \hat{q}_i(x,z) \frac{\partial}{\partial \hat{x}_i} \mathcal{W}(\hat{x}) \\
&\le \big(C\eta - \underline{C}\eta \|\hat{x}\| \mathbb{1}_{\widehat{\mathcal{K}}^c}(\hat{x}) + \frac{C\eta}{r_1} \hat{r}(\hat{q},\hat{y}) \mathbb{1}_{\widehat{\mathcal{K}}}(\hat{x}) \big) \mathcal{W}(\hat{x})
\end{aligned}$$

with C and $\underline{C}$ as in the proof of Proposition 4.1. From here, (6.11) becomes

$$\begin{aligned}
\widehat{\mathcal{L}}_n^z \mathcal{W}(\hat{x}) \le& \big(C\eta - \underline{C}\eta \|\hat{x}\| \mathbb{1}_{\widehat{\mathcal{K}}^c}(\hat{x}) + \frac{C\eta}{r_1} \hat{r}(\hat{q},\hat{y}) \mathbb{1}_{\widehat{\mathcal{K}}}(\hat{x}) \big) \mathcal{W}(\hat{x}) \\
&+ \Big(\frac{C\eta^2}{n} \sum_{i\in\mathcal{I}} \lambda_i^n + \frac{C^2\eta^2}{n} \|\hat{x}\| + \epsilon\eta C I + \epsilon\eta C^2 \|\hat{x}\| \Big) \mathcal{W}(\hat{x}) \\
=& \Big(C\eta + \frac{C\eta^2}{n} \sum_{i\in\mathcal{I}} \lambda_i^n + C\epsilon\eta I - \underline{C}\eta \|\hat{x}\| \mathbb{1}_{\widehat{\mathcal{K}}^c}(\hat{x}) + \frac{C\eta}{r_1} \hat{r}(\hat{q},\hat{y}) \mathbb{1}_{\widehat{\mathcal{K}}}(\hat{x}) \Big) \mathcal{W}(\hat{x}) \\
&+ \Big(\frac{C^2\eta^2}{n} \|\hat{x}\| + \epsilon\eta C^2 \|\hat{x}\| \Big) \mathbb{1}_{\widehat{\mathcal{K}}^c}(\hat{x}) \mathcal{W}(\hat{x}) + \Big(\frac{C^2\eta^2}{n} \|\hat{x}\| + \epsilon\eta C^2 \|\hat{x}\| \Big) \mathbb{1}_{\widehat{\mathcal{K}}}(\hat{x}) \mathcal{W}(\hat{x}) \\
=& \big(C\eta + \frac{C}{n} \sum_{i\in I} \lambda_i^n + C\epsilon I \big) \eta \mathcal{W}(\hat{x}) - \Big(\underline{C}\eta - \frac{C^2\eta^2}{n} - \epsilon\eta C^2 \Big) \|\hat{x}\| \mathbb{1}_{\widehat{\mathcal{K}}^c}(\hat{x}) \mathcal{W}(\hat{x}) \\
&+ \Big(\frac{C\eta}{r_1} \hat{r}(\hat{q},\hat{y}) + \big(\frac{C^2\eta^2}{n} + \epsilon\eta C^2 \big) \|\hat{x}\| \Big) \mathbb{1}_{\widehat{\mathcal{K}}}(\hat{x}) \mathcal{W}(\hat{x}) \,.
\end{aligned}$$

Therefore, for large enough $C > 0$ we have

$$\begin{aligned}
\widehat{\mathcal{L}}_n^z \mathcal{W}(\hat{x}) \le& C\eta \mathcal{W}(\hat{x}) - \Big(\underline{C}\eta - \frac{C^2\eta^2}{n} - \epsilon\eta C^2 \Big) \|\hat{x}\| \mathbb{1}_{\widehat{\mathcal{K}}^c}(\hat{x}) \mathcal{W}(\hat{x}) \\
&+ \Big(\frac{C\eta}{r_1} \hat{r}(\hat{q},\hat{y}) + \big(\frac{C^2\eta^2}{n} + \epsilon\eta C^2 \big) \frac{|e\cdot\hat{x}|}{\delta} \Big) \mathbb{1}_{\widehat{\mathcal{K}}}(\hat{x}) \mathcal{W}(\hat{x}) \\
\le& C\eta \mathcal{W}(\hat{x}) - \Big(\underline{C}\eta - \frac{C^2\eta^2}{n} - \epsilon\eta C^2 \Big) \|\hat{x}\| \mathbb{1}_{\widehat{\mathcal{K}}^c}(\hat{x}) \mathcal{W}(\hat{x}) \\
&+ \Big(\frac{C\eta}{r_1} + \big(\frac{C^2\eta^2}{n} + \epsilon\eta C^2 \big) \frac{1}{r_1\delta} \Big) \hat{r}(\hat{q},\hat{y}) \mathbb{1}_{\widehat{\mathcal{K}}}(\hat{x}) \mathcal{W}(\hat{x}) \,.
\end{aligned}$$

To get the first inequality, we use the definition of $\widehat{\mathcal{K}}$ in the third term on the right hand side and to get the second inequality, we use (2.21) in the third term. Taking $\epsilon = \eta$ small enough and then n large enough, we can ensure that

$$\underline{C}\eta - \frac{C^2\eta^2}{n} - \eta^2 C^2 > 0, \quad \text{and} \quad \frac{C\eta}{r_1} + \big(\frac{C^2\eta^2}{n} + \eta^2 C^2\big)\frac{1}{r_1\delta} < 1.$$

This proves (6.2) and the proposition. ☐

Proposition 6.2. *Suppose Assumptions 2.1 and 2.2 hold, and for every n, let $Z^n \in \mathfrak{Z}^n$. Also, suppose*

$$\limsup_{n\to\infty} J^n(\hat{X}^n(0), Z^n) \le M_0, \quad \textit{and} \quad \limsup_{n\to\infty}\limsup_{T\to\infty} \mathfrak{K}^n(\psi^n, T) \le M_1,$$

for some $M_i > 0$, $i = 0, 1$. Then, for some $M_2 > 0$, we have

$$\limsup_{n\to\infty}\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}^{Z^n,\psi^n}\Big[\int_0^T \|\hat{X}^{n,\psi^n}(t)\|\mathrm{d}t\Big] \le M_2\,.$$

Proof. We note that from Proposition 6.1,

$$\limsup_{n\to\infty}\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}^{Z^n}\Big[\exp\Big(\rho\int_0^T \|\hat{X}^n(t)\|\mathrm{d}t\Big)\Big] \le C_0 + \max\{C_1, \frac{M_0}{2}\}\,.$$

Here, $\rho < \rho_0$ with ρ_0 obtained from Proposition 6.1. Now using (5.16), we get

$$\begin{aligned}\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}^{Z^n}\Big[\exp\Big(\rho\int_0^T \|\hat{X}^n(t)\|\mathrm{d}t\Big)\Big] &= \limsup_{T\to\infty}\sup_{\psi\in\mathcal{E}^n} \mathbb{E}^{Z^n,\psi}\Big[\frac{\rho}{T}\int_0^T \|\hat{X}^{n,\psi}(t)\|\mathrm{d}t - \mathfrak{K}^n(\psi, T)\Big]\\ &\ge \limsup_{T\to\infty} \mathbb{E}^{Z^n,\psi^n}\Big[\frac{\rho}{T}\int_0^T \|\hat{X}^{n,\psi^n}(t)\|\mathrm{d}t\Big] - M_1.\end{aligned}$$

From here, the proof of the lemma is complete with $M_2 = \rho^{-1}\big(M_1 + C_0 + \max\{C_1, \frac{M_0}{2}\}\big)$. ☐

6.1. **Proof of the lower bound.**

Theorem 6.1 (Lower bound)**.** *Under Assumptions 2.1 and 2.2, the following holds:*

$$\liminf_{n\to\infty} \hat{\Lambda}^n\big(\hat{X}^n(0)\big) \ge \Lambda\,.$$

Proof. To begin with, using Proposition 3.2 we know that

$$\liminf_{n\to\infty} \hat{\Lambda}^n\big(\hat{X}^n(0)\big) \le \limsup_{n\to\infty} \hat{\Lambda}^n\big(\hat{X}^n(0)\big) < \infty\,. \tag{6.12}$$

Let n_k be a subsequence of n along which $\liminf_{k\to\infty} \hat{\Lambda}^{n_k}\big(\hat{X}^{n_k}(0)\big)$ is attained. In the rest of the proof, we confine ourselves to this subsequence and denote it by n with a slight abuse of notation. For a fixed $\delta > 0$, we choose an SCP $Z^n \in \mathfrak{Z}^n$, for every n, such that

$$J^n\big(\hat{X}^n(0), Z^n\big) - \delta \le \hat{\Lambda}^n(\hat{X}^n(0))\,. \tag{6.13}$$

Recall that

$$J^n\big(\hat{X}^n(0), Z^n\big) = \limsup_{T\to\infty} \frac{1}{T}\log\mathbb{E}^{Z^n}\Big[\exp\Big(\int_0^T \hat{r}(\hat{Q}^n(t), \hat{Y}^n(t))\mathrm{d}t\Big)\Big]\,.$$

Applying Lemma 5.4, we immediately get

$$\begin{aligned}J^n\big(\hat{X}^n(0), Z^n\big) &= \limsup_{T\to\infty}\sup_{\psi\in\mathcal{A}^n} \mathbb{E}^{Z^n,\psi}\Big[\frac{1}{T}\int_0^T \hat{r}(\hat{Q}^{n,\psi}(t), \hat{Y}^{n,\psi}(t))\mathrm{d}t - \mathfrak{K}^n(\psi, T)\Big]\\ &\ge \limsup_{T\to\infty} \mathbb{E}^{Z^n,\psi}\Big[\frac{1}{T}\int_0^T \hat{r}(\hat{Q}^{n,\psi}(t), \hat{Y}^{n,\psi}(t))\mathrm{d}t - \mathfrak{K}^n(\psi, T)\Big],\end{aligned}$$

where the above inequality holds for every $\psi \in \mathcal{A}^n$. We now make a particular choice of $\psi = \psi^n = \big(\phi_i^n, \psi_{ij}^n, \varphi_i^n : 1 \le i \le I, j \in \mathcal{J}(i)\big)$. Define

$$\widetilde{\phi}_i^n(x) \doteq 1 - \frac{w_i^*(x)}{\sqrt{2\lambda_i n}}, \quad \text{and} \quad \widetilde{\psi}_{ij}^n(x) \doteq 1 - \frac{w_i^*(x)\sqrt{z_{ij}^*}}{\sqrt{2\lambda_i n}},$$

for $1 \le i \le I$ and $j \in \mathcal{J}(i)$. Here, z^* is as defined in (2.2) and $w^* = w_l^*$ is obtained from Proposition 5.2, with large enough $l > 0$ such that

$$\Lambda \le \sup_{w \in \mathfrak{W}_{\mathrm{SM}}(l)} \inf_{v \in \mathfrak{U}_{\mathrm{SM}}} J_{v,w} + \delta = \inf_{v \in \mathfrak{U}_{\mathrm{SM}}} J_{v,w_l^*} + \delta\,. \tag{6.14}$$

Then, set $\phi_i^n(t) \doteq \widetilde{\phi}_i^n\big(\hat{X}^{n,\psi^n}(t)\big)$, $\psi_{ij}^n(t) \doteq \widetilde{\psi}_{ij}^n\big(\hat{X}^{n,\psi^n}(t)\big)$ and $\varphi_i^n(t) \doteq 1$. This immediately gives us

$$\begin{aligned} J^n\big(\hat{X}^n(0), Z^n\big) &\ge \limsup_{T\to\infty} \mathbb{E}^{Z^n,\psi^n}\Big[\frac{1}{T}\int_0^T \hat{r}(\hat{Q}^{n,\psi^n}(t), \hat{Y}^{n,\psi^n}(t))\mathrm{d}t - \mathfrak{K}^n(\psi^n, T)\Big] \\ &\ge \limsup_{T\to\infty} \mathbb{E}^{Z^n,\psi^n}\Big[\frac{1}{T}\int_0^T \hat{r}(\hat{Q}^{n,\psi^n}(t), \hat{Y}^{n,\psi^n}(t))\mathrm{d}t \\ &\qquad - \frac{1}{T}\int_0^T \Big(\sum_{i=1}^I \lambda_i^n \varkappa\big(\phi_i^n(t)\big) + \sum_{i=1}^I \sum_{j\in\mathcal{J}(i)} n\mu_{ij}^n \varkappa\big(\psi_{ij}^n(t)\big)\Big)\mathrm{d}t\Big]\,. \end{aligned}$$

To get the second line above, we use (5.14) and (5.15). From the definition of ψ^n, Assumption 2.1 and the fact that $\|w^*(x)\| \le l$ (because $w^* \in \mathfrak{W}_{\mathrm{SM}}(l)$), we can infer that

$$\limsup_{n\to\infty} \limsup_{T\to\infty} \mathfrak{K}^n(\psi^n, T) \le M_1, \tag{6.15}$$

for some $M_1 > 0$. Then, using (6.12), (6.13), (6.15), compactness of $\mathbb{U}$ and Proposition 6.2, the family of MEMs of $(\hat{X}^{n,\psi^n}, U^{n,\psi^n})$ given by

$$\pi_{X,U}^{n,T}(A \times B) \doteq \frac{1}{T}\int_0^T \mathbb{1}_{A\times B}\big(\hat{X}^{n,\psi^n}(t), U^{n,\psi^n}(t)\big)\mathrm{d}t, \text{ for Borel sets } A \subset \mathbb{R}^I \text{ and } B \subset \mathbb{U}$$

is tight in both T and n. Then, along a subsequence T_k, $\{\pi_{X,U}^{n,T_k} : k \in \mathbb{N}\}$ converges weakly to a measure $\pi_{X,U}^{n,*} \in \mathcal{P}(\mathbb{R}^I \times \mathbb{U})$. Hence, we get

$$\begin{aligned} &J^n\big(\hat{X}^n(0), Z^n\big) \\ &\ge \limsup_{T\to\infty} \mathbb{E}^{Z^n,\psi^n}\Big[\frac{1}{T}\int_0^T \hat{r}(\hat{Q}^{n,\psi}(t), \hat{Y}^{n,\psi}(t))\mathrm{d}t \\ &\qquad - \frac{1}{T}\int_0^T \Big(\sum_{i=1}^I \lambda_i^n \varkappa\big(\phi_i^n(t)\big) + \sum_{i=1}^I \sum_{j\in\mathcal{J}(i)} n\mu_{ij}^n \varkappa\big(\psi_{ij}^n(t)\big)\Big)\mathrm{d}t\Big] \\ &\ge \lim_{k\to\infty} \mathbb{E}^{Z^n,\psi^n}\Big[\int_{\mathbb{R}^I\times\mathbb{U}} \Big(\hat{r}\big(f_q^n(x,u), f_y^n(x,u)\big) \\ &\qquad - \sum_{i=1}^I \lambda_i^n \varkappa\big(\widetilde{\phi}_i^n(x)\big) - \sum_{i=1}^I \sum_{j\in\mathcal{J}(i)} n\mu_{ij}^n \varkappa\big(\widetilde{\psi}_{ij}^n(x)\big)\Big)\mathrm{d}\pi_{X,U}^{n,T_k}(x,u)\Big] \\ &\ge \int_{\mathbb{R}^I\times\mathbb{U}} \Big(\hat{r}\big(f_q^n(x,u), f_y^n(x,u)\big) - \sum_{i=1}^I \lambda_i^n \varkappa\big(\widetilde{\phi}_i^n(x)\big) - \sum_{i=1}^I \sum_{j\in\mathcal{J}(i)} n\mu_{ij}^n \varkappa\big(\widetilde{\psi}_{ij}^n(x)\big)\Big)\mathrm{d}\pi_{X,U}^{n,*}(x,u)\,. \end{aligned}$$

Here, for $u = (u^c, u^s)$,

$$f_q^n(x,u) \doteq \varpi^c\Big(\big(e\cdot(x+\sqrt{n}x^*)-\frac{(e\cdot N^n)}{\sqrt{n}}\big)^+u^c\big)\Big) \text{ and } f_y^n(x,u) \doteq \varpi^s\Big(\big(e\cdot(x+\sqrt{n}x^*)-\frac{(e\cdot N^n)}{\sqrt{n}}\big)^-u^s\Big).$$

Recall that $\varkappa(z) = z\log z - z + 1$ and it is easy to see that $n\varkappa\big(1 \pm \frac{z}{\sqrt{n}}\big) \to \frac{z^2}{2}$, as $n \to \infty$. Using this, the fact that $\sup_{x\in\mathbb{R}^I}\|w^*(x)\| \le l$ (this follows from the definition of $\mathfrak{W}_{\mathrm{SM}}(l)$ which w^* belongs to) and the definitions of $\widetilde{\phi}_i(\cdot)$ and $\widetilde{\psi}_{ij}(\cdot)$, we can conclude that

$$\Big|\lambda_i^n\varkappa\big(\widetilde{\phi}_i^n(x)\big) - \frac{1}{4}|w_i^*(x)|^2\Big| + \Big|n\mu_{ij}^n\varkappa\big(\widetilde{\psi}_{ij}^n(x)\big) - \frac{\mu_{ij}z_{ij}^*}{4\lambda_i}|w_i^*(x)|^2\Big| \to 0,$$

as $n \to \infty$, uniformly in $x \in \mathbb{R}^I$, for $1 \le i \le I$ and $j \in \mathcal{J}(i)$. Therefore, using the above display and (3.11), for large enough n, we can conclude that

$$\begin{aligned}
&J^n\big(\hat{X}^n(0), Z^n\big)\\
&\ge \int_{\mathbb{R}^I\times\mathbb{U}}\Big(\hat{r}\big((e\cdot x)^+u^c, (e\cdot x)^-u^s\big) - \Big(\sum_{i=1}^I\frac{1}{4}|w_i^*(x)|^2 + \sum_{i=1}^I\sum_{j\in\mathcal{J}(i)}\frac{\mu_{ij}z_{ij}^*}{4\lambda_i}|w_i^*(x)|^2\Big)\mathrm{d}\pi_{X,U}^{n,*}(x,u) - \delta\\
&\ge \int_{\mathbb{R}^I\times\mathbb{U}}\Big(\hat{r}\big((e\cdot x)^+u^c, (e\cdot x)^-u^s\big) - \frac{1}{2}\|w^*(x)\|^2\Big)\mathrm{d}\pi_{X,U}^{n,*}(x,u) - \delta\,.
\end{aligned}$$

To get the second line, we use Assumption 2.2 and (2.2).

From the tightness of the family $\{\pi_{X,U}^{n,*} : n \in \mathbb{N}\}$ obtained from Proposition 6.2 and following the computation similar to the one in [6, Pg. 3559-3560], we can conclude that $\pi_{X,U}^{n,*}$ converges weakly along a subsequence (still denoted by n) to an ergodic occupation measure $\pi_{X,v}^{*,*}$, corresponding to some $v \in \mathfrak{U}_{\mathrm{SM}}$, *i.e.,* $\mathrm{d}\pi_{X,v}^{*,*}(x,u) = \mathrm{d}\pi_X^*(x)\mathrm{d}v(u|x)$ with $\pi_X^* = \pi_X^*[v]$ being the unique invariant measure of X^* (defined in (2.24)) under some $v \in \mathfrak{U}_{\mathrm{SM}}$ and $w^* \in \mathfrak{W}_{\mathrm{SM}}(l)$ (chosen according to (6.14)). We remark that v is a priori only known to be a relaxed Markov control. This gives us

$$\begin{aligned}
\liminf_{n\to\infty}\hat{\Lambda}^n\big(\hat{X}^n(0)\big) &\ge \liminf_{n\to\infty} J^n\big(\hat{X}^n(0), Z^n\big) - \delta\\
&\ge \int_{\mathbb{R}^I\times\mathbb{U}}\Big(\hat{r}\big((e\cdot x)^+u^c, (e\cdot x)^-u^s\big) - \frac{1}{2}\|w^*(x)\|^2\Big)\mathrm{d}\pi_{X,v}^{*,*}(x,u) - 2\delta\\
&= J_{v,w^*} - 2\delta\\
&\ge \inf_{v\in\mathfrak{U}_{\mathrm{SM}}} J_{v,w^*} - 2\delta\\
&\ge \Lambda - 3\delta\,.
\end{aligned}$$

In the above, to get the third line, we use the definition of J_{v,w^*} and to get the last line, we use (6.14). From the arbitrariness of δ, we obtain the lower bound in Theorem 2.1. □

6.2. **Proof of the upper bound.**

Theorem 6.2 (Upper bound)**.** *Under Assumptions 2.1 and 2.2, the following holds:*

$$\liminf_{n\to\infty}\hat{\Lambda}^n\big(\hat{X}^n(0)\big) \le \Lambda\,.$$

Proof. Fix $\delta > 0$. Let $v = v^\delta$ be the stationary Markov control from Proposition 4.2 and let $Z^n[v]$ be as defined by (3.10). In the rest of the proof, we restrict ourselves to this admissible SCP. We now express below all the relevant processes which are $X^n = X^n[v]$, $Q^n = Q^n[v]$ and $Y^n = Y^n[v]$:

$$\begin{aligned}
Q^n[v](t) &\doteq q^n[v](\hat{X}^n(t)) = \varpi^c\Big(\big(e\cdot(\sqrt{n}\hat{X}^n[v](t) + nx^*) - (e\cdot N^n)\big)^+v^c(\hat{X}^n[v](t))\Big),\\
Y^n[v](t) &\doteq y^n[v](\hat{X}^n(t)) = \varpi^s\Big(\big(e\cdot(\sqrt{n}\hat{X}^n[v](t) + nx^*) - (e\cdot N^n)\big)^-v^s(\hat{X}^n[v](t))\Big),\\
Z^n[v](t) &\doteq \Upsilon\big(X^n(t) - Q^n[v](t), N^n - Y^n[v](t)\big),
\end{aligned}$$

$$X_i^n[v](t) \doteq Q_i^n[v](t) + \sum_{j\in\mathcal{J}(i)} Z_{ij}^n[v](t),\ \text{for } i\in\mathcal{I},$$
$$N_j^n = Y_j^n[v](t) + \sum_{i\in\mathcal{I}(j)} Z_{ij}^n[v](t),\ \text{for } j\in\mathcal{J}\,.$$

As before, we have

$$\hat{X}^n[v](t) = \frac{X^n[v](t) - nx^*}{\sqrt{n}},\quad \hat{Z}^n[v](t) = \frac{Z^n[v](t) - nz^*}{\sqrt{n}},$$
$$\hat{Q}^n[v](t) = \frac{Q^n[v](t)}{\sqrt{n}} \quad\text{and}\quad \hat{Y}^n[v](t) = \frac{Y^n[v](t)}{\sqrt{n}}\,.$$

To keep the expressions that follow short, we omit the dependence on $[v]$ from the above quantities.

From Proposition 4.2, it is clear that $v\in\mathfrak{U}_{\mathrm{SM}}(l)$, for some large enough $l>0$. From Proposition 3.2(ii), we know that $\limsup_{n\to\infty} J^n\big(\hat{X}^n(0), Z^n\big) < \infty$ and for any $\rho>0$,

$$\limsup_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\log\mathbb{E}^{Z^n}\Big[\exp\Big((1+\rho)\int_0^T \hat{r}(\hat{Q}^n(t),\hat{Y}^n(t))\mathrm{d}t\Big)\Big] < \infty\,. \tag{6.16}$$

For $L>0$ and $T>0$, define

$$H^{n,L}(T) \doteq \exp\Big(\int_0^T \hat{r}(\hat{Q}^n(t),\hat{Y}^n(t))\mathrm{d}t - LT\Big)\,.$$

Then, we have

$$\begin{aligned}
&\exp\big(-LT\big)\mathbb{E}^{Z^n}\Big[\exp\Big(\int_0^T \hat{r}(\hat{Q}^n(t),\hat{Y}^n(t))\mathrm{d}t\Big)\mathbb{1}_{[LT,\infty)}\big(\int_0^T \hat{r}(\hat{Q}^n(t),\hat{Y}^n(t))\mathrm{d}t\big)\Big]\\
&= \mathbb{E}^{Z^n}\Big[H^{n,L}(T)\mathbb{1}_{[1,\infty)}\big(H^{n,L}(T)\big)\Big]\\
&\le \mathbb{E}^{Z^n}\Big[(H^{n,L}(T))^{1+\rho}\Big] = \exp\big(-(1+\rho)LT\big)\mathbb{E}^{Z^n}\Big[\exp\Big((1+\rho)\int_0^T \hat{r}(\hat{Q}^n(t),\hat{Y}^n(t))\mathrm{d}t\Big)\Big],
\end{aligned}$$

for $\rho>0$. This gives us

$$\begin{aligned}
&\limsup_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\log\mathbb{E}^{Z^n}\Big[\exp\Big(\int_0^T \hat{r}(\hat{Q}^n(t),\hat{Y}^n(t))\mathrm{d}t\Big)\mathbb{1}_{[LT,\infty)}\big(\int_0^T \hat{r}(\hat{Q}^n(t),\hat{Y}^n(t))\mathrm{d}t\big)\Big]\\
&\qquad\le -\rho L + \limsup_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\log\mathbb{E}^{Z^n}\Big[\exp\Big((1+\rho)\int_0^T \hat{r}(\hat{Q}^n(t),\hat{Y}^n(t))\mathrm{d}t\Big)\Big]\,.
\end{aligned}$$

Applying (6.16), we get

$$\limsup_{L\to\infty}\limsup_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\log\mathbb{E}^{Z^n}\Big[\exp\Big(\int_0^T \hat{r}(\hat{Q}^n(t),\hat{Y}^n(t))\mathrm{d}t\Big)\mathbb{1}_{[LT,\infty)}\big(\int_0^T \hat{r}(\hat{Q}^n(t),\hat{Y}^n(t))\mathrm{d}t\big)\Big] = -\infty\,. \tag{6.17}$$

From here, we see that for any $L>0$,

$$\begin{aligned}
&\limsup_{n\to\infty} J^n(\hat{X}^n(0), Z^n)\\
&\le \max\Bigg\{\limsup_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\log\mathbb{E}^{Z^n}\Big[\exp\Big(\int_0^T \hat{r}(\hat{Q}^n(t),\hat{Y}^n(t))\mathrm{d}t\Big)\mathbb{1}_{[LT,\infty)}\big(\int_0^T \hat{r}(\hat{Q}^n(t),\hat{Y}^n(t))\mathrm{d}t\big)\Big],\\
&\qquad \limsup_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\log\mathbb{E}^{Z^n}\Big[\exp\Big(\int_0^T \hat{r}(\hat{Q}^n(t),\hat{Y}^n(t))\mathrm{d}t\Big)\mathbb{1}_{[0,LT]}\big(\int_0^T \hat{r}(\hat{Q}^n(t),\hat{Y}^n(t))\mathrm{d}t\big)\Big]\Bigg\}\,.
\end{aligned}$$

On the other hand, it is clear that for $L > 0$,

$$\limsup_{n\to\infty} J^n(\hat{X}^n(0), Z^n) \geq \limsup_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\log \mathbb{E}^{Z^n}\Big[\exp\Big(\int_0^T \hat{r}(\hat{Q}^n(t),\hat{Y}^n(t))\mathrm{d}t\Big)\mathbb{1}_{[0,LT]}\Big(\int_0^T \hat{r}(\hat{Q}^n(t),\hat{Y}^n(t))\mathrm{d}t\Big)\Big].$$

Taking $L\to\infty$ in the above two displays and using (6.17) gives us the following:

$$\limsup_{n\to\infty} J^n(\hat{X}^n(0), Z^n) = \lim_{L\to\infty}\limsup_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\log \mathbb{E}^{Z^n}\Big[\exp\Big(LT\wedge\int_0^T \hat{r}(\hat{Q}^n(t),\hat{Y}^n(t))\mathrm{d}t\Big)\Big].$$

Then, choosing L large enough, we have

$$\limsup_{n\to\infty} J^n(\hat{X}^n(0), Z^n) \leq \limsup_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\log \mathbb{E}^{Z^n}\Big[\exp\Big(LT\wedge\int_0^T \hat{r}(\hat{Q}^n(t),\hat{Y}^n(t))\mathrm{d}t\Big)\Big] + \delta\,.$$

Applying Lemma 5.3 (in particular, (5.10)) to the functional $LT\wedge\int_0^T \hat{r}(\hat{Q}^n(t),\hat{Y}^n(t))\mathrm{d}t$ and noting that the pair $(\hat{Q}^n(t),\hat{Y}^n(t))$ is given by a Borel measurable functional of $N^n([0,t])$ (see (5.11)), we have

$$\limsup_{n\to\infty} J^n(\hat{X}^n(0), Z^n) \leq \limsup_{n\to\infty}\limsup_{T\to\infty}\sup_{\psi\in\mathcal{A}^n_{M_1}}\mathbb{E}^{Z^n,\psi}\Big[\Big(\frac{1}{T}\int_0^T \hat{r}(\hat{Q}^{n,\psi}(t),\hat{Y}^n(t))\mathrm{d}t\Big)\wedge L - \mathfrak{K}^n(\psi,T)\Big] + 2\delta,$$

for some $M_1 > 0$ depending only on L and δ. Choose $\psi^n = \psi^n(T)$, for every T and n such that

$$\sup_{\psi\in\mathcal{A}^n_{M_1}}\mathbb{E}^{Z^n,\psi}\Big[\Big(\frac{1}{T}\int_0^T \hat{r}(\hat{Q}^{n,\psi}(t),\hat{Y}^{n,\psi}(t))\mathrm{d}t\Big)\wedge L - \mathfrak{K}^n(\psi,T)\Big] \leq \mathbb{E}^{Z^n,\psi^n}\Big[\Big(\frac{1}{T}\int_0^T \hat{r}(\hat{Q}^{n,\psi^n}(t),\hat{Y}^{n,\psi^n}(t))\mathrm{d}t\Big)\wedge L - \mathfrak{K}^n(\psi^n,T)\Big] + \delta\,.$$

It is evident that ψ^n depends on T, but we suppress the dependence. Combining the above two displays, we get

$$\limsup_{n\to\infty} J^n(\hat{X}^n(0), Z^n) \leq \limsup_{n\to\infty}\limsup_{T\to\infty}\mathbb{E}^{Z^n,\psi^n}\Big[\Big(\frac{1}{T}\int_0^T \hat{r}(\hat{Q}^{n,\psi^n}(t),\hat{Y}^{n,\psi^n}(t))\mathrm{d}t\Big)\wedge L - \mathfrak{K}^n(\psi^n,T)\Big] + 3\delta\,.$$

Since $\psi^n\in\mathcal{A}^n_{M_1}$, we have

$$\mathfrak{K}^n(\psi^n,T)\leq M_2, \tag{6.18}$$

for some $M_2 > 0$, depending only on L and δ. In particular, the family of MEMs of the $\mathbb{R}^I\times\mathbb{R}^{\mathcal{G}}\times\mathbb{R}^I$–valued process $h^n[\psi^n]$ is tight in T and n. From the finiteness of $\limsup_{n\to\infty} J^n\big(\hat{X}^n(0), Z^n\big)$, (6.18) and using Proposition 6.2, we have

$$\limsup_{n\to\infty}\limsup_{T\to\infty}\mathbb{E}^{Z^n,\psi^n}\Big[\frac{1}{T}\int_0^T \|\hat{X}^{n,\psi^n}(t)\|\mathrm{d}t\Big] \leq M_3, \tag{6.19}$$

for some $M_3 > 0$. Therefore, the family of MEMs of the $\mathbb{R}^I$–valued process $\hat{X}^{n,\psi^n}$ given by

$$\pi_X^{n,T}(A) = \pi_X^{n,T}[Z^n,\psi^n](A) \doteq \frac{1}{T}\int_0^T \mathbb{1}_A(\hat{X}^{n,\psi^n}(t))\mathrm{d}t,\ \text{for Borel set } A\subset\mathbb{R}^I$$

is also tight in both T and n. We have

$$\limsup_{T\to\infty}\mathbb{E}^{Z^n,\psi^n}\Big[\Big(\frac{1}{T}\int_0^T \hat{r}(\hat{Q}^{n,\psi^n}(t),\hat{Y}^{n,\psi^n}(t))\mathrm{d}t\Big)\wedge L - \mathfrak{K}^n(\psi^n,T)\Big]$$

$$\le \limsup_{T\to\infty}\left(\mathbb{E}^{Z^n,\psi^n}\Big[\frac{1}{T}\int_0^T \Big(\hat r(\hat Q^{n,\psi^n}(t),\hat Y^{n,\psi^n}(t))\wedge L\Big)\mathrm{d}t\Big] - \mathbb{E}^{Z^n,\psi^n}\Big[\mathfrak{K}^n(\psi^n,T)\Big]\right)$$
$$= \limsup_{T\to\infty}\left(\mathbb{E}^{Z^n,\psi^n}\Big[\int_{\mathbb{R}^I}\Big(\hat r\big(\frac{1}{\sqrt{n}}q^n[v](x),\frac{1}{\sqrt{n}}y^n[v](x)\big)\wedge L\Big)\mathrm{d}\pi_X^{n,T}(x)\Big] - \mathbb{E}^{Z^n,\psi^n}\Big[\mathfrak{K}^n(\psi^n,T)\Big]\right).$$

From the tightness of $\{\pi_X^{n,T} : T>0\}$, we know that along a subsequence T_k, $\{\pi_X^{n,T_k} : k\in\mathbb{N}\}$ converges weakly to some measure $\pi_X^{n,*}=\pi_X^{n,*}[Z^n,\psi^n]\in\mathcal{P}(\mathbb{R}^I)$. This means that

$$\limsup_{T\to\infty}\mathbb{E}^{Z^n,\psi^n}\Big[\Big(\frac{1}{T}\int_0^T \hat r(\hat Q^{n,\psi^n}(t),\hat Y^{n,\psi^n}(t))\mathrm{d}t\Big)\wedge L - \mathfrak{K}^n(\psi^n,T)\Big]$$
$$\le \int_{\mathbb{R}^I}\Big(\hat r\big(\frac{1}{\sqrt{n}}q^n[v](x),\frac{1}{\sqrt{n}}y^n[v](x)\big)\wedge L\Big)\mathrm{d}\pi_X^{n,*}(x) - \limsup_{k\to\infty}\mathbb{E}^{Z^n,\psi^n}\Big[\mathfrak{K}^n(\psi^n,T_k)\Big].$$

We now take $n\to\infty$. From Theorem 5.1, we know that for every $T>0$, $\hat X^{n,\psi^n}$ converges weakly to $X^*=X^*_{v,\widehat w}$ on $\mathfrak{D}_T^I$, for some $L^2([0,T],\mathbb{R}^I)$–valued random variable $\widehat w$. Here, X^* is given as the solution to (5.19) with $u(t)=v(X^*(t))$ and $w(t)=\widehat w(t)$.

From (6.19), we know that there exist a subsequence n_k and a measure $\pi_X^*=\pi_X^*[v,\widehat w]$ such that $\{\pi_X^{n_k,*} : k\in\mathbb{N}\}$ converges weakly to $\pi_X^*\in\mathcal{P}(\mathbb{R}^I)$. Again, following the computation similar to the one in [6, Pg. 3559-3560], we can ensure that π_X^* is an ergodic occupation measure corresponding to process X^* under $v\in\mathfrak{U}_{\mathrm{SM}}$ and $\widehat w\in\mathcal{A}$. This gives us

$$\lim_{k\to\infty}\left\{\int_{\mathbb{R}^I}\Big(\hat r\big(\frac{1}{\sqrt{n}_k}q^{n_k}[v](x),\frac{1}{\sqrt{n}_k}y^{n_k}[v](x)\big)\wedge L\Big)\mathrm{d}\pi_X^{n_k,*}(x) - \limsup_{m\to\infty}\mathbb{E}^{Z^{n_k},\psi^{n_k}}\Big[\mathfrak{K}^{n_k}(\psi^{n_k},T_m)\Big]\right\}$$
$$= \int_{\mathbb{R}^I}\Big(\hat r\big((e\cdot x)^+v^c(x),(e\cdot x)^-v^s(x)\big)\wedge L\Big)\mathrm{d}\pi_X^*(x) - \limsup_{k\to\infty}\limsup_{m\to\infty}\mathbb{E}^{Z^{n_k},\psi^{n_k}}\Big[\mathfrak{K}^{n_k}(\psi^{n_k},T_m)\Big].$$

In the above, we use (3.11). Therefore, we have shown that

$$\limsup_{n\to\infty}J^n(\hat X^n(0),Z^n)\le \int_{\mathbb{R}^I}\Big(\hat r\big((e\cdot x)^+v^c(x),(e\cdot x)^-v^s(x)\big)\wedge L\Big)\mathrm{d}\pi_X^*(x)$$
$$- \limsup_{k\to\infty}\limsup_{m\to\infty}\mathbb{E}^{Z^{n_k},\psi^{n_k}}\Big[\mathfrak{K}^{n_k}(\psi^{n_k},T_m)\Big]+3\delta\,.$$

From Lemma 3.6 of [2], the weak lower semi-continuity of the norm and Theorem 5.1, we have

$$\limsup_{k\to\infty}\limsup_{m\to\infty}\mathbb{E}^{Z^{n_k},\psi^{n_k}}\Big[\mathfrak{K}^{n_k}(\psi^{n_k},T_m)\Big]\ge \limsup_{T\to\infty}\frac{1}{2T}\mathbb{E}_x^{v,\widehat w}\Big[\int_0^T\|\widehat w(t)\|^2\mathrm{d}t\Big].$$

To summarize, we have shown that

$$\begin{aligned}
\limsup_{n\to\infty}\hat\Lambda^n\big(\hat X^n(0)\big) &\le \limsup_{n\to\infty}J^n(\hat X^n(0),Z^n)\\
&\le \int_{\mathbb{R}^I}\Big(\hat r\big((e\cdot x)^+v^c(x),(e\cdot x)^-v^c(x)\big)\wedge L\Big)\mathrm{d}\pi_X^*(x)\\
&\qquad - \limsup_{T\to\infty}\frac{1}{2T}\mathbb{E}_x^{v,\widehat w}\Big[\int_0^T\|\widehat w(t)\|^2\mathrm{d}t\Big]+3\delta\\
&\le \limsup_{T\to\infty}\frac{1}{T}\mathbb{E}_x^{v,\widehat w}\Big[\int_0^T\Big(r\big(X^*(t),v\big(X^*(t)\big)\big)\wedge L-\frac{1}{2}\|\widehat w(t)\|^2\Big)\mathrm{d}t\Big]+3\delta\\
&\le \limsup_{T\to\infty}\sup_{w\in\mathcal{A}}\frac{1}{T}\mathbb{E}_x^{v,w}\Big[\int_0^T\Big(r\big(X^*(t),v\big(X^*(t)\big)\big)-\frac{1}{2}\|w(t)\|^2\Big)\mathrm{d}t\Big]+3\delta\\
&= J(x,v)+3\delta\\
&\le \Lambda+4\delta\,.
\end{aligned}$$

In the above, to get the third inequality, we re-write the integral over π^*_X as an integral over time and expectation, and use the definition of $r(\cdot,\cdot)$; to get to the fourth inequality, we use the monotone convergence theorem and take $L \to \infty$, and take supremum over $\mathcal{A}$; to get the equality, we use Lemma 5.2 and finally, we get the last line from the choice of $v \in \mathfrak{U}_{\mathrm{SM}}$. From the arbitrariness of δ, we have the result. □

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Appendix A. Proof of Lemma 3.1

Recall that for $\eta > 0$,

$$\mathcal{V}_\eta(x) \doteq \exp\Big(\frac{\eta}{2}\widetilde{x}^{\mathsf{T}}\widetilde{Q}\widetilde{x} + \frac{\eta^2}{2}x_I^2\Big), \tag{A.1}$$

where the $\mathbb{R}^{I-1}\times\mathbb{R}^{I-1}$ matrix $\widetilde{Q}$ is positive diagonal and satisfies (3.3). We have

$$\begin{aligned}
&\big(\mathcal{V}_\eta(x)\big)^{-1} b\big(x,v^*\big)\cdot\nabla\mathcal{V}_\eta(x)\\
&= \eta\sum_{i=1}^{I-1}\Big(\ell_i - (Bx)_i\Big)\widetilde{Q}_{ii}x_i + \eta^2\Big(-F(\widetilde{x}) - \mu_{IJ}x_I - (\gamma_I-\mu_{IJ})(e\cdot x)^+ + \ell_I\Big)x_I\\
&\le \eta\sum_{i=1}^{I-1}\ell_i\widetilde{Q}_{ii}x_i - 8\eta\|\widetilde{x}\|^2 + \eta^2\Big(-F(\widetilde{x})x_I - \mu_{IJ}x_I^2 - (\gamma_I-\mu_{IJ})(e\cdot x)^+x_I + \ell_Ix_I\Big)\,.
\end{aligned} \tag{A.2}$$

In the rest of the proof, we write $e = e^{(I)}$ and $\widetilde{e} = e^{(I-1)}$. It is clear that $0 \le (e\cdot x)^+ \le (\widetilde{e}\cdot\widetilde{x})^+ + x_I^+$. We also have $0 \le (e\cdot x)^+x_I \le \big((\widetilde{e}\cdot\widetilde{x})^+ + x_I\big)x_I$, if $x_I > 0$ and $(\widetilde{e}\cdot\widetilde{x})^+x_I \le (e\cdot x)^+x_I \le 0$, if $x_I \le 0$. This can be more concisely expressed as

$$(\widetilde{e}\cdot\widetilde{x})^+x_I \le (e\cdot x)^+x_I \le \big((\widetilde{e}\cdot\widetilde{x})^+ + x_I^+\big)x_I^+\,. \tag{A.3}$$

We now consider two cases depending on the sign of $\gamma_I - \mu_{IJ}$. Suppose $\gamma_I - \mu_{IJ} \le 0$. Using the above bound, we obtain

$$\begin{aligned}
&\big(\mathcal{V}_\eta(x)\big)^{-1} b\big(x,v^*\big)\cdot\nabla\mathcal{V}_\eta(x)\\
&\le \eta\sum_{i=1}^{I-1}\ell_i\widetilde{Q}_{ii}x_i - 8\eta\|\widetilde{x}\|^2 + \eta^2\Big(-F(\widetilde{x})x_I - \mu_{IJ}x_I^2 - \big(\gamma_I-\mu_{IJ}\big)\big((\widetilde{e}\cdot\widetilde{x})^+ + x_I^+\big)x_I^+ + \ell_Ix_I\Big)\\
&= \eta\sum_{i=1}^{I-1}\ell_i\widetilde{Q}_{ii}x_i - 8\eta\|\widetilde{x}\|^2 + \eta^2\Big(-F(\widetilde{x})x_I - \mu_{IJ}\big(x_I^2 - \big(x_I^+\big)^2\big) - \big(\gamma_I-\mu_{IJ}\big)(\widetilde{e}\cdot\widetilde{x})^+x_I^+\\
&\qquad\qquad - \gamma_I\big(x_I^+\big)^2 + \ell_Ix_I\Big)\\
&\le \eta\sum_{i=1}^{I-1}\ell_i\widetilde{Q}_{ii}x_i - 8\eta\|\widetilde{x}\|^2 + \eta^2\Big(-F(\widetilde{x})x_I - \min\{\gamma_I,\mu_{IJ}\}x_I^2 - \big(\gamma_I-\mu_{IJ}\big)(\widetilde{e}\cdot\widetilde{x})^+x_I^+ + \ell_Ix_I\Big)\\
&\le \eta\sum_{i=1}^{I-1}\ell_i\widetilde{Q}_{ii}x_i - 8\eta\|\widetilde{x}\|^2 + \eta^2\Big(-\widetilde{F}(\widetilde{x},x_I) - \min\{\gamma_I,\mu_{IJ}\}x_I^2 + \ell_Ix_I\Big)\\
&\le \eta\sum_{i=1}^{I-1}\ell_i\widetilde{Q}_{ii}x_i - 8\eta\|\widetilde{x}\|^2 + \eta^2\Big(C\|\widetilde{x}\||x_I| - \min\{\gamma_I,\mu_{IJ}\}x_I^2 + \ell_Ix_I\Big) && \text{(A.4)}\\
&\le \eta\sum_{i=1}^{I-1}\ell_i\widetilde{Q}_{ii}x_i - 8\eta\|\widetilde{x}\|^2 + \eta^2\Big(\frac{C}{2\varepsilon}\|\widetilde{x}\|^2 + \frac{C\varepsilon}{2}x_I^2 - \min\{\gamma_I,\mu_{IJ}\}x_I^2 + \ell_Ix_I\Big) && \text{(A.5)}\\
&\le \eta\sum_{i=1}^{I-1}\ell_i\widetilde{Q}_{ii}x_i + \eta^2\ell_Ix_I - \big(8\eta - \frac{\eta}{2}\big)\|\widetilde{x}\|^2 + \Big(\frac{C^2\eta^3}{2} - \eta^2\min\{\gamma_I,\mu_{IJ}\}\Big)x_I^2 && \text{(A.6)}\\
&= \eta\sum_{i=1}^{I-1}\ell_i\widetilde{Q}_{ii}x_i + \eta^2\ell_Ix_I - \frac{7\eta}{2}\|\widetilde{x}\|^2 + \Big(\frac{C^2\eta^3}{2} - \eta^2\min\{\gamma_I,\mu_{IJ}\}\Big)x_I^2\,.
\end{aligned}$$

In the above, $\widetilde{F}(\widetilde{x}, x_I) \doteq F(\widetilde{x})x_I + \big(\gamma_I - \mu_{IJ}\big)(\widetilde{e}\cdot\widetilde{x})^+ x_I^+$ and since F has at most linear growth and $\widetilde{F}(0,0) = 0$, *i.e.*, $|\widetilde{F}(\widetilde{x}, x_I)| \le C\|\widetilde{x}\||x_I|$, for some $C > 0$, we obtain (A.4). To get (A.5), we use Young's inequality: for $a, b > 0$, $ab \le \frac{1}{2\varepsilon}a^2 + \frac{\varepsilon}{2}b^2$ for $\varepsilon > 0$. In (A.6), we take $\varepsilon = C\eta > 0$. From here, it is clear that for $0 < \eta \le \eta_1 \doteq \frac{\min\{\gamma_I, \mu_{IJ}\}}{C^2}$,

$$\big(\mathcal{V}_\eta(x)\big)^{-1} b\big(x, v^*\big)\cdot\nabla\mathcal{V}_\eta(x) \le \eta\sum_{i=1}^{I-1}\ell_i\widetilde{Q}_{ii}x_i + \eta^2\ell_I x_I - \frac{7\eta}{2}\|\widetilde{x}\|^2 - \frac{\eta^2}{2}\min\{\gamma_I, \mu_{IJ}\}x_I^2\,. \tag{A.7}$$

We now estimate the following:

$$\sum_{i=1}^{I}\lambda_i\frac{\partial^2}{\partial x_i^2}\mathcal{V}_\eta(x) = \mathcal{V}_\eta(x)\bigg(\sum_{i=1}^{I-1}\lambda_i\big(\eta\widetilde{Q}_{ii} + \eta^2\widetilde{Q}_{ii}^2x_i^2\big) + \big(\eta^2\lambda_I + \eta^4\lambda_I x_I^2\big)\bigg)\,. \tag{A.8}$$

Combining (A.7) and (A.8) and defining

$$\beta \doteq \max\{\lambda_1\widetilde{Q}_{11}^2, \dots, \lambda_{I-1}\widetilde{Q}_{I-1,I-1}^2\} \quad \text{and} \quad \xi \doteq \frac{\eta^2}{2}\min\{\gamma_I, \mu_{IJ}\} - \eta^4\lambda_I,$$

we get

$$\mathcal{L}^{v^*}\mathcal{V}_\eta(x) \le \bigg(\sum_{i=1}^{I-1}\eta\big(\lambda_i\widetilde{Q}_{ii} + \ell_i\widetilde{Q}_{ii}x_i\big) + \eta^2\lambda_I + \eta^2\ell_I x_I - \big(\frac{7\eta}{2} - \eta^2\beta\big)\|\widetilde{x}\|^2 - \xi x_I^2\bigg)\mathcal{V}_\eta(x)\,.$$

Therefore, choosing $\eta < \eta_2 \doteq \min\Big\{\frac{\min\{\gamma_I,\mu_{IJ}\}}{C^2}, \frac{7}{4\beta}, \sqrt{\frac{\min\{\gamma_I,\mu_{IJ}\}}{4\lambda_I}}\Big\}$, we obtain

$$\begin{aligned}\mathcal{L}^{v^*}\mathcal{V}_\eta(x) &\le \bigg(\sum_{i=1}^{I-1}\eta\big(\lambda_i\widetilde{Q}_{ii} + \ell_i\widetilde{Q}_{ii}x_i\big) + \eta^2\lambda_I + \eta^2\ell_I x_I - \frac{7\eta}{4}\|\widetilde{x}\|^2 - \eta^2\frac{\min\{\gamma_I, \mu_{IJ}\}}{4\lambda_I}x_I^2\bigg)\mathcal{V}_\eta(x)\\ &\le \bigg(\sum_{i=1}^{I-1}\eta\big(\lambda_i\widetilde{Q}_{ii} + \ell_i\widetilde{Q}_{ii}x_i\big) + \eta^2\lambda_I + \eta^2\ell_I x_I - \min\Big\{\frac{7\eta}{4}, \eta^2\frac{\min\{\gamma_I, \mu_{IJ}\}}{4\lambda_I}\Big\}\|x\|^2\bigg)\mathcal{V}_\eta(x)\,.\end{aligned} \tag{A.9}$$

Hence, we have proved that $\mathcal{V}_\eta(x)$ indeed satisfies the Lyapunov inequality for $0 < \eta < \eta_2$, whenever $\gamma_I - \mu_{IJ} \le 0$.

For the case where $\gamma_I - \mu_{IJ} > 0$, we use the lower bound of (A.3) and substitute in (B.2), to get

$$\begin{aligned}&\big(\mathcal{V}_\eta(x)\big)^{-1} b\big(x, v^*\big)\cdot\nabla\mathcal{V}_\eta(x)\\ &\le \eta\sum_{i=1}^{I-1}\ell_i\widetilde{Q}_{ii}x_i - 8\eta\|\widetilde{x}\|^2 + \eta^2\Big(-F(\widetilde{x})x_I - \mu_{IJ}x_I^2 - \big(\gamma_I - \mu_{IJ}\big)(\widetilde{e}\cdot\widetilde{x})^+x_I + \ell_I x_I\Big)\\ &= \eta\sum_{i=1}^{I-1}\ell_i\widetilde{Q}_{ii}x_i - 8\eta\|\widetilde{x}\|^2 + \eta^2\Big(-\big(F(\widetilde{x}) + (\gamma_I - \mu_{IJ})(\widetilde{e}\cdot\widetilde{x})^+\big)x_I - \mu_{IJ}x_I^2 + \ell_I x_I\Big)\,.\end{aligned}$$

From here, we follow the similar arguments as in the case of $\gamma_I - \mu_{IJ} \le 0$ with $\big(F(\widetilde{x}) + (\gamma_I - \mu_{IJ})(\widetilde{e}\cdot\widetilde{x})^+\big)x_I$ playing the role of $\widetilde{F}(\widetilde{x}, x_I)$ and μ_{IJ} in place of $\min\{\gamma_I, \mu_{IJ}\}$. This gives us the following: for

$$\eta < \eta_3 \doteq \min\Big\{\frac{\mu_{IJ}}{C^2}, \frac{7}{4\beta}, \sqrt{\frac{\mu_{IJ}}{4\lambda_I}}\Big\},$$

$$\mathcal{L}^{v^*}\mathcal{V}_\eta(x) \le \bigg(\sum_{i=1}^{I-1}\eta\big(\lambda_i\widetilde{Q}_{ii} + \ell_i\widetilde{Q}_{ii}x_i\big) + \eta^2\lambda_I + \eta^2\ell_I x_I - \min\big\{\frac{7\eta}{4}, \frac{\mu_{IJ}\eta^2}{4\lambda_I}\big\}\|x\|^2\bigg)\mathcal{V}_\eta(x)\,. \tag{A.10}$$

From (A.9) and (A.10) and noting that $\sum_{i=1}^{I-1} \eta\ell_i\widetilde{Q}_{ii}x_i + \eta^2\ell_I x_I \le C\eta\|x\|$, for large enough $C > 0$, it is clear that the desired result follows with $\eta_0 = \min\{\eta_2, \eta_3\}$,

$$\begin{aligned} C_0 &= C_0(\eta) = \sum_{i=1}^{I-1} \eta\lambda_i\widetilde{Q}_{ii} + \eta^2\lambda_I, \\ C_1 &= C_1(\eta) = C\eta, \\ C_2 &= C_2(\eta) = \min\left\{\frac{7\eta}{4}, \eta^2\frac{\min\{\gamma_I, \mu_{IJ}\}}{4\lambda_I}, \frac{\mu_{IJ}\eta^2}{4\lambda_I}\right\}. \end{aligned}$$

□

Appendix B. Proof of (3.8)

Recall that for $\eta > 0$, $\widetilde{\mathcal{V}}_\eta \in \mathcal{C}^2(\mathbb{R}^I)$ is given by

$$\widetilde{\mathcal{V}}_\eta(x) \doteq \exp\Big(\frac{\eta}{2}(\widetilde{x}^{\mathsf{T}}\widetilde{Q}\widetilde{x})(1+\widetilde{x}^{\mathsf{T}}\widetilde{Q}\widetilde{x})^{-\frac{1}{2}} + \frac{\eta\kappa}{2}x_I^2(1+x_I^2)^{-\frac{1}{2}}\Big),$$

with the positive diagonal matrix $\widetilde{Q}$ that satisfies (3.3). As in the previous section, we write $e = e^{(I)}$ and $\widetilde{e} = e^{(I-1)}$. To begin with, from the form of $\widetilde{\mathcal{V}}_\eta$, we observe that

$$\sum_{i=1}^{I} \lambda_i \frac{\partial^2}{\partial x_i^2}\widetilde{\mathcal{V}}_\eta(x) \le \eta^2 C\widetilde{\mathcal{V}}_\eta(x), \tag{B.1}$$

for some $C > 0$. Using (3.3), we have

$$\begin{aligned} &\big(\eta\widetilde{\mathcal{V}}_\eta(x)\big)^{-1} b\big(x, v^*\big)\cdot\nabla\widetilde{\mathcal{V}}_\eta(x) \\ &= \sum_{i=1}^{I-1}\big(\ell_i - (Bx)_i\big)(\widetilde{Q}_{ii}x_i)c_i(\widetilde{x}) + \kappa\big(-F(\widetilde{x}) - \mu_{IJ}x_I - (\gamma_I - \mu_{IJ})(e\cdot x)^+ + \ell_I\big)x_I c_I(x_I) \\ &\le \sum_{i=1}^{I-1}\big((\ell_i\widetilde{Q}_{ii}x_i)c_i(\widetilde{x}) - 8\|\widetilde{x}\|^2 c_i(\widetilde{x})\big) + \kappa c_I(x_I)\big(-F(\widetilde{x})x_I - \mu_{IJ}x_I^2 - (\gamma_I - \mu_{IJ})(e\cdot x)^+ x_I + \ell_I x_I\big). \end{aligned} \tag{B.2}$$

Here, $c_i(\widetilde{x}) = (\widetilde{x}^{\mathsf{T}}\widetilde{Q}\widetilde{x}+2)(1+\widetilde{x}^{\mathsf{T}}\widetilde{Q}\widetilde{x})^{-\frac{3}{2}}$, for $1 \le i \le I-1$, and $c_I(x_I) = (x_I^2+2)(1+x_I^2)^{-\frac{3}{2}}$. We note that $|x_i c_i(\widetilde{x}_i)|$, for $1 \le i \le I-1$ and $|x_I c_I(x_I)|$ are uniformly bounded. Recall (A.3):

$$(\widetilde{e}\cdot\widetilde{x})^+ x_I \le (e\cdot x)^+ x_I \le \big((\widetilde{e}\cdot\widetilde{x})^+ + x_I^+\big)x_I^+ .$$

Again, we consider two cases depending on the sign of $\gamma_I - \mu_{IJ}$. Suppose $\gamma_I - \mu_{IJ} \le 0$. Using the above bound, we obtain

$$\begin{aligned} &\big(\eta\mathcal{V}_\eta(x)\big)^{-1} b\big(x, v^*\big)\cdot\nabla\mathcal{V}_\eta(x) \\ &\le \sum_{i=1}^{I-1}\big((\ell_i\widetilde{Q}_{ii}x_i)c_i(\widetilde{x}) - 8\|\widetilde{x}\|^2 c_i(\widetilde{x})\big) \\ &\qquad + \kappa c_I(x_I)\Big(-F(\widetilde{x})x_I - \mu_{IJ}x_I^2 - \big(\gamma_I - \mu_{IJ}\big)\big((\widetilde{e}\cdot\widetilde{x})^+ + x_I^+\big)x_I^+ + \ell_I x_I\Big) \\ &= \sum_{i=1}^{I-1}\big((\ell_i\widetilde{Q}_{ii}x_i)c_i(\widetilde{x}) - 8\|\widetilde{x}\|^2 c_i(\widetilde{x})\big) \\ &\qquad + \kappa c_I(x_I)\Big(-F(\widetilde{x})x_I - \mu_{IJ}\big(x_I^2 - \big(x_I^+\big)^2\big) - \big(\gamma_I - \mu_{IJ}\big)(\widetilde{e}\cdot\widetilde{x})^+ x_I^+ - \gamma_I\big(x_I^+\big)^2 + \ell_I x_I\Big) \end{aligned}$$

$$
\begin{aligned}
&\le \sum_{i=1}^{I-1}\big((\ell_i\widetilde{Q}_{ii}x_i)c_i(\widetilde{x}) - 8\|\widetilde{x}\|^2c_i(\widetilde{x})\big)\\
&\qquad\qquad + \kappa c_I(x_I)\Big(-F(\widetilde{x})x_I - \min\{\gamma_I,\mu_{IJ}\}x_I^2 - \big(\gamma_I-\mu_{IJ}\big)(\widetilde{e}\cdot\widetilde{x})^+x_I^+ + \ell_I x_I\Big)\\
&\le \sum_{i=1}^{I-1}\big((\ell_i\widetilde{Q}_{ii}x_i)c_i(\widetilde{x}) - 8\|\widetilde{x}\|^2c_i(\widetilde{x})\big) + \kappa c_I(x_I)\Big(C\|\widetilde{x}\||x_I| - \min\{\gamma_I,\mu_{IJ}\}x_I^2 + \ell_I x_I\Big) &\text{(B.3)}\\
&\le \sum_{i=1}^{I-1} C - 8\|\widetilde{x}\|^2\underline{c}(\widetilde{x}) + \kappa\Big(C\|\widetilde{x}\| - \min\{\gamma_I,\mu_{IJ}\}\epsilon|x_I| + C\ell_I\Big) &\text{(B.4)}\\
&\le \sum_{i=1}^{I-1} C - 4\|\widetilde{x}\|^2\underline{c}(\widetilde{x}) + \kappa\Big(-\min\{\gamma_I,\mu_{IJ}\}\epsilon|x_I| + \ell_I\Big)\,. &\text{(B.5)}
\end{aligned}
$$

Here, $\underline{c}(\widetilde{x}) \doteq \min_{1\le i\le I-1} c_i(\widetilde{x})$. In the above, to get (B.3), we use the fact that $|F(\widetilde{x}) - \big(\gamma_I - \mu_{IJ}\big)(\widetilde{e}\cdot\widetilde{x})^+x_I^+| \le C\|\widetilde{x}\||x_I|$, for some larger $C>0$; to get (B.4), we use the fact that $c_I(x_I)|x_I|^2 \ge \epsilon|x_I|$, for some $\epsilon>0$, the fact that $|c_i(\widetilde{x})x_i|$, for $1\le i\le I-1$ and $|c_I(x_I)x_I|$ are uniformly bounded, and choosing $C>0$ large enough; to get (B.5), we choose κ small enough to ensure that $\kappa C \le 4\|\widetilde{x}\|\underline{c}(\widetilde{x})$. Therefore, combining (B.1) and (B.5), we obtain that $\widetilde{\mathcal{V}}_\eta(x)$ indeed satisfies (3.8) whenever $\gamma_I - \mu_{IJ}\le 0$.

For the case where $\gamma_I - \mu_{IJ} > 0$, we use the lower bound of (A.3) and substitute in (B.2), to get

$$
\begin{aligned}
&\big(\eta\mathcal{V}_\eta(x)\big)^{-1} b\big(x,v^*\big)\cdot\nabla\mathcal{V}_\eta(x)\\
&\le \sum_{i=1}^{I-1}\big((\ell_i\widetilde{Q}_{ii}x_i)c_i(\widetilde{x}) - 8\|\widetilde{x}\|^2c_i(\widetilde{x})\big) + \kappa c_I(x_I)\Big(-F(\widetilde{x})x_I - \mu_{IJ}x_I^2 - \big(\gamma_I-\mu_{IJ}\big)(\widetilde{e}\cdot\widetilde{x})^+x_I + \ell_I x_I\Big)\\
&= \sum_{i=1}^{I-1}\big((\ell_i\widetilde{Q}_{ii}x_i)c_i(\widetilde{x}) - 8\|\widetilde{x}\|^2c_i(\widetilde{x})\big) + \kappa c_I(x_I)\Big(-\big(F(\widetilde{x}) + (\gamma_I-\mu_{IJ})(\widetilde{e}\cdot\widetilde{x})^+\big)x_I - \mu_{IJ}x_I^2 + \ell_I x_I\Big)\,.
\end{aligned}
$$

From here, to get (3.8), we follow the similar arguments as in the case of $\gamma_I - \mu_{IJ}\le 0$ with $\big(F(\widetilde{x}) + (\gamma_I-\mu_{IJ})(\widetilde{e}\cdot\widetilde{x})^+\big)x_I$ playing the role of $F(\widetilde{x})x_I + \big(\gamma_I-\mu_{IJ}\big)(\widetilde{e}\cdot\widetilde{x})^+x_I^+$ and μ_{IJ} in place of $\min\{\gamma_I,\mu_{IJ}\}$. □

Appendix C. Sketch proof of Theorem 5.1

The proof follows very closely the arguments of [2, Theorem 3.6] and below we only give the key differences. Recall that $\mathbb{R}^{\mathcal{G}} \doteq \{\xi_{ij}\in\mathbb{R}^{I\times J} : \xi_{ij} = 0 \text{ for } i \nsim j\}$ and let $\mathfrak{C}_T^{\mathcal{G}}$ be the set of $\mathbb{R}^{\mathcal{G}}$–valued continuous function on $[0,T]$ equipped with the uniform topology. To begin with, fixing $T>0$, we can re-write (5.12) for $\psi = \psi^n$ as follows:

$$
\hat{X}_i^{n,\psi^n}(t) = \hat{X}_i^{n,\psi^n}(0) + \ell_i^n t - \sum_{j\in\mathcal{J}(i)}\mu_{ij}^n\int_0^t \hat{Z}_{ij}^{n,\psi^n}(s)\mathrm{d}s - \gamma_i^n\int_0^t\hat{Q}_i^{n,\psi^n}(s)\mathrm{d}s + \hat{M}_i^{n,\psi^n}(t) + \hat{\Xi}_i^n(t)\,.
$$

Here, for $1\le i\le I$ and $t>0$,

$$
\hat{M}_i^{n,\psi^n}(t) \doteq \hat{M}_i^{n,A,\psi^n}(t) + \sum_{j\in\mathcal{J}(i)}\hat{M}_{ij}^{n,S,\psi^n}(t) + \hat{M}_i^{n,R,\psi^n}(t),
$$

$$
\hat{\Xi}_i^n(t) \doteq \frac{\lambda_i^n}{\sqrt{n}}\int_0^t(1-\phi_i^n(s))\mathrm{d}s + \sum_{j\in\mathcal{J}(i)}\mu_{ij}^n\sqrt{n}\int_0^t(1-\psi_{ij}^n(s))\frac{Z_{ij}^{n,\psi^n}(s)}{n}\mathrm{d}s + \gamma_i^n\sqrt{n}\int_0^t(1-\varphi_i^n(s))\frac{Q_i^{n,\psi^n}(s)}{n}\mathrm{d}s\,.
$$

with square integrable $\bar{\mathcal{G}}^n_t$–martingales

$$\hat{M}^{n,A,\psi^n}_i(t) \doteq \frac{1}{\sqrt{n}}\Big(\breve{A}^n_i\Big(\int_0^t \phi^n_i(s)\mathrm{d}s\Big) - \lambda^n_i\int_0^t \phi^n_i(s)\mathrm{d}s\Big),$$

$$\hat{M}^{n,S,\psi^n}_{ij}(t) \doteq \frac{1}{\sqrt{n}}\left(\breve{S}^n_{ij}\left(\int_0^t \frac{\psi^n_{ij}(s)Z^{n,\psi^n}_{ij}(s)}{n}\mathrm{d}s\right) - n\mu^n_{ij}\int_0^t \frac{\psi^n_{ij}(s)Z^{n,\psi^n}_{ij}(s)}{n}\mathrm{d}s\right),$$

$$\hat{M}^{n,R,\psi^n}_i(t) \doteq \frac{1}{\sqrt{n}}\left(\breve{R}^n_i\left(\int_0^t \frac{\varphi^n_i(s)Q^{n,\psi^n}_i(s)}{n}\mathrm{d}s\right) - n\gamma^n_i\int_0^t \frac{\varphi^n_i(s)Q^{n,\psi^n}_i(s)}{n}\mathrm{d}s\right).$$

Below we state the tightness of a certain family of random variables (which are the key aspects of the proof) and briefly explain why such a tightness holds.

(i) $\{U^{n,\psi^n} : n\in\mathbb{N}\}$: Since U^{n,ψ^n} is a càdlàg $\mathbb{U}$–valued process, the family is trivially tight in $\mathfrak{D}^{\mathbb{U}}_T$ (due to the compactness of $\mathbb{U}$) with a limit point U.

(ii) $\{h^n[\psi^n] : n\in\mathbb{N}\}$: Using Lemma 5.5 and the hypothesis of the theorem, we can conclude that this family of random variables is tight in $L^{2,*}_\infty$.

(iii) $\{\psi^n : n\in\mathbb{N}\}$: From the definition of h^n in (5.18), it is clear that $\phi^n_i,\psi^n_{ij},\varphi^n_i \Rightarrow \mathfrak{e}$, for every $1\le i\le I$ and $j\in\mathcal{J}(i)$, as $n\to\infty$. Here, $\mathfrak{e}$ is the identity function on the real line.

(iv) $\big\{(n^{-1}Z^{n,\psi^n}_{ij}, n^{-1}Q^{n,\psi^n}_i, n^{-1}Y^{n,\psi^n}_i) : n\in\mathbb{N}\big\}$: Following the arguments of the proof of [17, Proposition 1(i)], we can conclude that for $1\le i\le I$ and $j\in\mathcal{J}(i)$,

$$(n^{-1}Z^{n,\psi^n}_{ij}, n^{-1}Q^{n,\psi^n}_i, n^{-1}Y^{n,\psi^n}_i) \Rightarrow (z^*_{ij},0,0) \quad\text{in}\quad \mathfrak{D}_T\times\mathfrak{D}_T\times\mathfrak{D}_T \quad\text{as}\quad n\to\infty.$$

(v) $\{\hat{M}^{n,\psi^n} : n\in\mathbb{N}\}$: From the martingale central limit theorem, the random time change lemma, the fact that $\phi^n_i,\psi^n_{ij},\varphi^n_i\Rightarrow\mathfrak{e}$, for every $1\le i\le I$ and $j\in\mathcal{J}(i)$, as $n\to\infty$ and the above display, we can conclude that

$$\hat{M}^{n,\psi^n} \Rightarrow \Sigma W \quad\text{in}\quad \mathfrak{D}^I_T \quad\text{as}\quad n\to\infty.$$

Here, W is a I–dimensional Brownian motion and $\Sigma = \sqrt{2}\,\mathrm{diag}(\sqrt{\lambda_1},\sqrt{\lambda_2},\dots,\sqrt{\lambda_I})$.

(vi) $\{\hat{\Xi}^n : n\in\mathbb{N}\}$: From the above tightness in (ii) and the following arguments in the proof of [2, Theorem 3.9] we can conclude the tightness of $\{\int_0^\cdot h^n[\psi^n]\mathrm{d}t : n\in\mathbb{N}\}$ in $\mathfrak{C}^I_T\times\mathfrak{C}^{\mathcal{G}}_T\times\mathfrak{C}^I_T$. Combining this with tightness in (iv), gives us the following: If $h^n[\psi^n]\Rightarrow w = \big(w^1_i, w^2_{ij}, w^3_i : 1\le i\le I, j\in\mathcal{J}(i)\big)\in L^{2,*}_\infty$, then as $n\to\infty$,

$$\int_0^\cdot h^n[\psi^n](t)\mathrm{d}t \Rightarrow \Big(\int_0^\cdot w^1_i(t)\mathrm{d}t, \int_0^\cdot w^2_{ij}(t)\mathrm{d}t, \int_0^\cdot w^3_i(t)\mathrm{d}t : 1\le i\le I, j\in\mathcal{J}(i)\Big) \quad\text{in}\quad \mathfrak{C}^I_T\times\mathfrak{C}^{\mathcal{G}}_T\times\mathfrak{C}^I_T$$

and

$$\hat{\Xi}^n_i \Rightarrow \lambda_i w^1_i + \sum_{j\in\mathcal{J}(i)} \mu_i z^*_{ij} w^2_{ij}\,.$$

Notice that the weak limit points of Ξ^n do not depend on w^3_i. This happens because of the following reason: for every $1\le i\le I$ and from (iv), we know that $n^{-1}Q^{n,\psi^n}_i\Rightarrow 0$ in $\mathfrak{D}_T$, as $n\to\infty$. From here, we can conclude that the third term in the definition of Ξ^n which is $\gamma^n_i\sqrt{n}\int_0^\cdot(1-\varphi^n_i(s))\big(n^{-1}Q^{n,\psi^n}_i(s)\big)\mathrm{d}s$ converges weakly to zero in $\mathfrak{C}_T$, as $n\to\infty$.

Following the arguments of the proof of [17, Proposition 1(ii)], we can show that $\{\hat{X}^{n,\psi^n} : n\in\mathbb{N}\}$ is tight in $\mathfrak{D}^I_T$. Therefore, along a subsequence (again denoted by n),

$$\big(\hat{X}^{n,\psi^n}, U^{n,\psi^n}, h^n[\psi^n]\big) \Rightarrow (X^*_{U,w}, U, w) \quad\text{in}\quad \mathfrak{D}^I_T\times\mathfrak{D}^{\mathbb{U}}_T\times L^{2,*}_\infty \quad\text{as}\quad n\to\infty.$$

Here, $X^*_{U,w}$ and w are related according to (5.19).