On the ergodic properties and invariant measure of a two-dimensional reflected Ornstein-Uhlenbeck process

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Abstract. We consider a two-dimensional reflected Ornstein-Uhlenbeck (ROU) process, that arises as the diffusion approximation for a parallel server network with a randomly split Hawkes arrival process (or a multivariate Hawkes arrival process) in heavy traffic. We study the ergodic properties of the process, including positive recurrence, and rates of convergence in both the total variation distance and the Wasserstein distance. We also provide a numerical scheme based on a Monte-Carlo method to approximate the invariant measure of the process.

1. Introduction

Reflected stochastic processes have attracted significant attention in recent years, because of their close connections to stochastic networks and queueing systems that arise in computer networks, telecommunications, mathematical biology, and transportation problems. Among the various areas of research, investigations into their ergodic properties have made significant progress over the years. For instance, the recurrence of reflected Brownian motions (RBMs) and reflected diffusions has been extensively studied in the literature, including but not limited to the references [41, 13, 8, 7, 2, 21]. In particular, the stationary distributions of two-dimensional RBMs have been thoroughly investigated, with explicit analytical expressions available. Relevant references include [41, 10, 11, 12, 16, 18]. However, in higher dimensions, results on the stationary distribution are limited. Harrison and Williams [20] showed that, under a certain skew symmetry condition, the stationary densities of RBMs take an exponential product form. Moreover, Kang and Ramanan [24] provided a characterization of stationary distributions of reflected diffusions.

However, when focusing specifically on the reflected Ornstein-Uhlenbeck (ROU) process, the existing literature concentrates on the one-dimensional case. Ward and Glynn [39] conducted a comprehensive study of the one-dimensional ROU (1-d ROU) process, including its explicit stationary density, a perturbation expansion for the transition density, and approximations for level crossing times. Furthermore, stationary distributions have been explored for certain generalized 1-d ROU processes. Specifically, Xing et al. [43] investigated the 1-d ROU process with jumps and the Markov-modulated 1-d ROU process, while Zhang and

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Jiang [44] studied the 1-d ROU process with two-sided barriers. The main purpose of this paper is to expand the scope of research by studying a special case of two-dimensional ROU (2-d ROU) processes with normal reflections, as introduced below.

The 2-d ROU process (X, Y) in $\mathbb{R}_+^2$ studied in this paper is defined as follows:

$$\begin{aligned} dX_t &= \theta_1(\mu_1 - X_t)dt + \sigma_1 dW_t + dL_t^X, \\ dY_t &= \theta_2(\mu_2 - Y_t)dt + \sigma_2 dB_t + dL_t^Y, \end{aligned} \tag{1.1}$$

with initial condition $(X_0, Y_0) = (x_0, y_0) \in \mathbb{R}_+^2$, where $\mu_1, \mu_2 \in \mathbb{R}$, $\theta_1, \theta_2, \sigma_1, \sigma_2 > 0$, (W, B) is a two-dimensional Brownian motion with covariance matrix $\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$ with $-1 < \rho < 1$, and (L_t^X, L_t^Y) denotes the local times at the boundaries. Specifically, $(L_0^X, L_0^Y) = (0, 0)$, L_t^X and L_t^Y are nondecreasing and $\int_0^\infty \mathbf{1}_{(0,\infty)}(X_s)dL_s^X = 0$ and $\int_0^\infty \mathbf{1}_{(0,\infty)}(Y_s)dL_s^Y = 0$. The existence and uniqueness of a strong solution to the 2-d ROU process are guaranteed by a careful extension of the results of [32].

Before studying our 2-d ROU process, we first discuss how ROU processes arise from various applications. The 1-d ROU process arises as the diffusion-scaling limit of a sequence of single-server queues with balking or reneging under heavy traffic conditions (see [38, 40]). Such queueing models have broad applications across various domains, including telecommunications, supply chain management, and computing and cloud services. Borovkov [3] and Srikant and Whitt [36] showed that the 1-d ROU process can approximate the number-in-system process in a G/M/s/0 queueing model when both the number of servers and the arrival rate are large. Additionally, Boxma et al. [4] established that the 1-d ROU process arises as the scaling limit for a sequence of reflected AR(1) processes. Recently, Li and Pang [31] showed that the 1-d ROU process can be one possible scaling limit for the single-server queues with state-dependent Hawkes arrivals. These results focus primarily on the one-dimensional case. In contrast, our 2-d ROU process arises as the diffusion scaling limit of parallel single-server queues with abandonment in the critically loaded regime which have a randomly split Hawkes process or a bivariate Hawkes process as the arrival processes (see [30]). Notably, in this case, only positive correlation $\rho \in (0, 1)$ is considered. See Figure 1 for an illustration of these queueing models with split-Hawkes arrivals. The correlation between these parallel queues emerges from dependencies in the arrival processes, which in turn translate into the correlated Brownian motions that drive the dynamics of the ROU process. Our 2-d ROU process is, of course, not the most general form of ROU processes; particularly, the covariance coefficient is constant and the reflection is normal. Furthermore, since there are no other interactions among the queues other than the arrivals, the reflection terms will be exactly like the reflection for each separate single-server queue with abandonment, as studied in [40].

The analysis of our 2-d ROU process contributes to the understanding of stationary distributions and transient performance measures of these parallel server queueing models, where explicit expressions are often intractable. Furthermore, we believe that our results also provide valuable insights into the stationary distribution of coupled M/G/1 queues with (possibly positively or negatively) correlated arrivals and parallel services and with abandonment in heavy traffic, as long as the arrival processes lead to a scaling limit as a 2-d Brownian motion (see the open problems posed in [34]).

For the 1-d ROU process, the ergodic properties are well understood, as discussed in [39], where the invariant measure is explicitly shown to have a truncated Gaussian distribution.

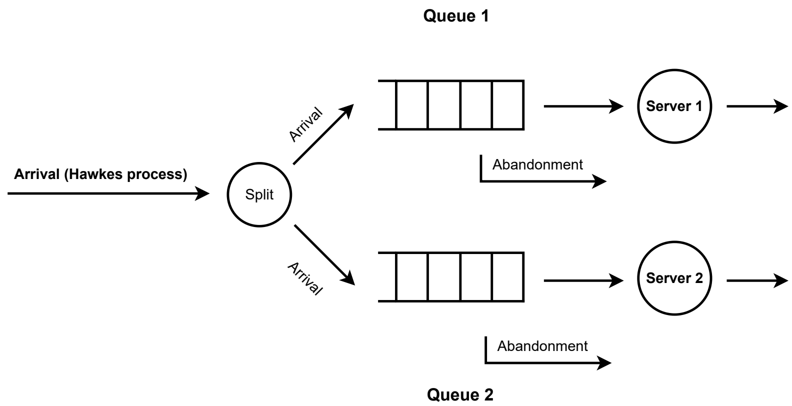


FIGURE 1. Parallel Single-Server Queues with abandonment

Additionally, the exponential rate of convergence in the total variation distance is studied in [33], and the rate in Wasserstein distance is investigated in [35]. In this paper, our goal is to understand such ergodic properties for the 2-d ROU process with normal reflections arising from the parallel server queueing models mentioned above.

In pursuit of this goal, one potential method involves extending the methodologies used for 1-d ROU processes. However, this strategy appears to be intractable. Indeed, in both [39] and [43], the study of the stationary distributions and the positive recurrence relies crucially on the explicit solutions to the corresponding ODEs. When transitioning from the 1-d case to the 2-d case, these ODEs transform into PDEs, whose explicit solutions seem to be impossible to find. Moreover, their proofs use the result that the 1-d ROU processes are regenerative. However, in the two-dimensional case, this property appears no longer to hold, as 2-d processes are unlikely to revisit the same point infinitely often.

It might be tempting to consider the 2-d ROU process in (1.1) as a special case in the general class of constrained diffusion processes which has been extensively studied in the literature (see, e.g., [2, 7, 24, 6]). However, these results cannot be directly applied to our 2-d ROU process with parameters $\mu_1, \mu_2 \in \mathbb{R}$. Indeed, the positive recurrence result of constrained diffusions in [2] can only be applied to our 2-d ROU process when both μ_1 and μ_2 are ≤ 0 (see Condition 2.3 in [2]). On the other hand, a 2-d ROU process with positive μ_1, μ_2 does arise in practical applications. Specifically, if a sequence of parallel single-server queues with abandonment has traffic intensities $(\rho_1^{(n)}, \rho_2^{(n)})$ such that the limits of $\sqrt{n}(\rho_1^{(n)} - 1)$ and $\sqrt{n}(\rho_2^{(n)} - 1)$ exist and are positive (which is allowed in the case with abandonment), then such a 2-d ROU process with $\mu_1, \mu_2 \ge 0$ emerges as the diffusion scaling limit. Although this result was not explicitly stated in [30], the derivation of the 2-d ROU process does not impose any constraints on the limits of the traffic intensities (similarly as in [40]). Thus, it is important to explore the positive recurrence of the 2-d ROU process when $\mu_1, \mu_2 \ge 0$. Indeed, when $\mu_1, \mu_2 \ge 0$, the 2-d ROU process (X_t^1, Y_t^1) "dominates" the 2-d ROU process (X_t^2, Y_t^2) when $\mu_1, \mu_2 \le 0$, meaning that $X_t^1 \ge X_t^2$ and $Y_t^1 \ge Y_t^2$ for all $t \ge 0$. Therefore, the positive recurrence of (X_t^1, Y_t^1) immediately implies that of (X_t^2, Y_t^2). Hence, it is sufficient to prove the positive recurrence of (X_t^1, Y_t^1) with $\mu_1, \mu_2 \ge 0$. A detailed explanation will be provided in Section 2. In this paper, we will establish the positive recurrence of the 2-d

ROU process with the parameters $\mu_1, \mu_2 \in \mathbb{R}$, thereby slightly extending the corresponding positive recurrence results in [2] when applied to this process.

A common tool for studying the positive recurrence of reflected diffusion processes involves the use of the so-called "fluid path" or "fluid model", which is a solution to the ODEs obtained by eliminating the randomness from the SDEs which drive the reflected diffusion processes. It has been shown that the "fluid path" tending to the origin generally implies the recurrence of the reflected diffusion process. For more details, we refer to [2, 5, 13]. However, this tool is not applicable to our 2-d ROU process when μ_1 and μ_2 are both positive. In fact, solving the following ODEs:

$$\begin{aligned} dX_t &= \theta_1(\mu_1 - X_t)dt + dL_t^X, \\ dY_t &= \theta_2(\mu_2 - Y_t)dt + dL_t^Y, \end{aligned}$$

subject to the initial conditions $X_0 = x_0 \geq 0$ and $Y_0 = y_0 \geq 0$, yields the "fluid path":

$$\begin{aligned} X_t &= x_0\, e^{-\theta_1 t} + \mu_1 \left(1 - e^{-\theta_1 t}\right), \\ Y_t &= y_0\, e^{-\theta_2 t} + \mu_2 \left(1 - e^{-\theta_2 t}\right), \end{aligned}$$

which tends to $(\mu_1, \mu_2) \neq (0, 0)$ as $t \to \infty$. As a result, we need a new approach to study the stability of our process. Using a method which shares the same flavor as those in [21] and [15], we have succeeded in establishing the positive recurrence of the 2-d ROU process, regardless of the values of μ_1 and μ_2. Indeed, we will prove that the process reaches an arbitrary neighborhood of the origin within a finite mean time. The strategy is summarized as follows:

- We first construct a discrete-time jump process (X_{S_n}, Y_{S_n}) from the original 2-d ROU process by restricting it to a sequence of elaborately selected stopping times, such that one of X_{S_n} and Y_{S_n} is 0 for $n \geq 1$.
- We then demonstrate that the above jump process reaches a given neighborhood of the origin within a finite mean time when starting from a position outside that neighborhood. Consequently, the original process exhibits the same property.
- Finally, by focusing on the distance between (X_t, Y_t) and the origin and using the aforementioned property, we derive a similar regenerative process, enabling us to establish the positive recurrence.

It is worth mentioning that we believe our method, in conjunction with the Skorokhod maps for 1-d ROU processes, may be extended to address the case of oblique reflections. We conjecture that 2-d ROU processes are always positive recurrent, irrespective of the reflection directions. We leave this question for future investigation.

The second contribution of this paper lies in studying the exponential rates of convergence to the stationary distribution. The existence and uniqueness of the stationary distribution, which are the prerequisites for our study, have been previously explored in [26]. Our results shall provide an alternative proof of these properties. Following a similar argument as in Section 4.4 of [25], the existence can be established through the positive recurrence, while our subsequent results on convergence rates ensure the uniqueness.

Inspired by [33] and [35], we establish the convergence rates of the transition probability measures to the invariant measure in both the total variation distance and Wasserstein distance. The derivation of the convergence rate in the total variation distance relies on the tail estimate of the first hitting time for the Ornstein–Uhlenbeck process (see Lemma 13), which affects the exponent involved in the convergence rate. On the other hand, for the Wasserstein distance, we establish a concise bound for the Wasserstein distance between two transition probability measures, resulting in an explicit exponent being $\min\{\theta_1, \theta_2\}$, governing the rate of convergence to the invariant measure.

We next investigate the stationary distribution of the 2-d ROU process by a numerical scheme based on a Monte-Carlo method. For diffusions, two common approaches for investigating the stationary distributions are the PDE method (see, e.g., [9, 27, 42]) and the Monte-Carlo method (see, e.g., [28, 37, 17]). However, explicitly solving the PDEs derived from our process appears to be impossible. Therefore, we only focus on the Monte-Carlo method. For reflected diffusions in polyhedral domains, Budhiraja et al. [6] studied the numerical computation of their stationary distributions using an Euler scheme. However, their results cannot be directly applied to our case, as Conditions 1.3 and 1.4 in [6] are not satisfied. Therefore, exploiting the unique characteristics of our process, we develop our own numerical scheme for approximating the stationary distribution and also analyze the convergence of our scheme to the stationary distribution. This study is also in a similar spirit to the recent work in [22, 23] on numerical methods using Euler-Maruyama schemes to compute the stationary distributions of limiting diffusions (without reflections) arising in approximations of queueing networks in heavy traffic. Similar to [6] and [22], our numerical scheme takes a decreasing step size. We also conduct simulation experiments to illustrate the implementation of the numerical scheme (see Section 4.5).

Finally, it is worth mentioning that unlike the extensively studied ergodicity properties and stationary distributions of 2-d RBM (see [14], [11] and [16]), such studies on 2-d ROU processes are lacking. While the 2-d ROU processes investigated in this paper are not the most general, we contribute to this area by examining those properties for a specific case. In future work, we aim to extend our results to more general ROU processes.

In summary, our contributions in this paper are as follows:

- Prove the positive recurrence of the 2-d ROU process defined in (1.1); see Theorem 12.
- Establish the exponential rates of convergence to its stationary distribution; see Theorems 15 and 17.
- Provide a numerical scheme to approximate its stationary distribution; see Theorem 18.

1.1. **Organization of the paper.** We first summarize the notation used in this paper at the end of this section. Section 2 is devoted to investigating the positive recurrence of the 2-d ROU process defined in (1.1), employing the aforementioned strategy. In Section 3, we examine the exponential rates of convergence to the stationary distribution in both the total variation distance and Wasserstein distance. Finally, Section 4 focuses on analyzing a numerical scheme based on the Monte-Carlo method to approximate the stationary distribution of the 2-d ROU process.

1.2. **Notation.** Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be the natural filtration space generated by the two-dimensional Brownian motion (W_t, B_t). Since (X_t, Y_t) are the solutions to the SDE (1.1), they are also processes on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$. Throughout the paper, $\mathbb{R}_+^2 := \{(x, y) : x, y \geq 0\}$ denotes the two-dimensional positive orthant, and $\partial\mathbb{R}_+^2$ represents its boundary. Additionally, we use $\mathcal{O}_B$ to denote the set $\{(x, y) \in \mathbb{R}_+^2 : x + y \leq B\}$. For $(x, y) \in \mathbb{R}_+^2$, we use $P^{(x,y)}$ and $E^{(x,y)}$ to denote the probability and the expectation, respectively, when the 2-d ROU process starts from (x, y). Similarly, for a 1-d process starting from $z \in \mathbb{R}$, P^z and

E^z are the corresponding probability and expectation. Moreover, for $x, y \in \mathbb{R}$, we define $x \wedge y := \min\{x, y\}$, $x \vee y := \max\{x, y\}$, and $x^+ := \max\{x, 0\}$. Additionally, $\delta_{(x,y)}$ is the Dirac measure on $\mathbb{R}^2$ concentrated at (x, y). For a set A, we use $\mathbb{1}_A$ to represent the indicator function of A.

When studying the convergence rates to the stationary distribution, we will need the following notation. We denote the stationary distribution as π. Furthermore, for $(x, y) \in \mathbb{R}_+^2$, $P_t\left((x, y), \cdot\right)$ represents the transition probability measure for the 2-d ROU process. Finally, we introduce the total variation distance and Wasserstein distance. Given two measures m_1 and m_2 on $\mathbb{R}_+^2$, the total variation distance is defined by

$$d_{TV}\left(m_1, m_2\right) := \sup_{A \in \mathcal{B}(\mathbb{R}_+^2)} \left|m_1(A) - m_2(A)\right|,$$

where $\mathcal{B}(\mathbb{R}_+^2)$ is the family of all Borel sets on $\mathbb{R}_+^2$. Further, if both $\int \|x\|\, m_1(dx)$ and $\int \|x\|\, m_2(dx)$ are finite, we define the Wasserstein distance between m_1 and m_2 by

$$d_W\left(m_1, m_2\right) := \sup_{f \in \mathrm{Lip}(1)} \left| \int f(x)\, m_1(dx) - \int f(x)\, m_2(dx) \right|,$$

where Lip(1) is the set of all Lipschitz functions on $\mathbb{R}_+^2$ with Lipschitz constant ≤ 1.

Additional notation will be introduced in the paper as needed.

2. Positive recurrence

The aim of this section is to study the positive recurrence of the 2-d ROU process defined in (1.1), following the strategy outlined in Section 1.

2.1. **Preliminary.**

We start by citing a lemma from [39], which demonstrates that a 1-d ROU process is "controlled" or "dominated" by a 1-d reflected Brownian motion (RBM) with the same parameters. This lemma allows us to exploit the properties of the 1-d RBM to study the ROU process. Subsequently, we extend this lemma to compare between two 1-d ROU processes, and the result is presented in Lemma 2.

Lemma 1. *Suppose $\theta > 0, \mu \in \mathbb{R}, \sigma > 0$ and W_t is a standard 1-d Brownian motion. Let X be a 1-d ROU process satisfying*

$$dX_t = \theta(\mu - X_t)dt + \sigma dW_t + dL_t^X$$

starting from $x_0 \geq 0$, where L^X represents the local time of X at 0. Also let $\widetilde{X}$ be a 1-d RBM associated with X, satisfying

$$d\widetilde{X}_t = \theta\mu\, dt + \sigma dW_t + dL_t^{\widetilde{X}}$$

starting from $x_0 \geq 0$, where $L^{\widetilde{X}}$ represents the local time of $\widetilde{X}$ at 0. Then, with probability 1, for all $t \geq 0$,

$$X_t \leq \widetilde{X}_t.$$

Proof. See the proof of Proposition 2 in [39]. □

Lemma 2. *Suppose that $\theta, \tilde{\theta}, \sigma > 0$, $\mu, \tilde{\mu} \in \mathbb{R}$, and W_t is a standard 1-d Brownian motion. Let X be a 1-d ROU process satisfying*

$$dX_t = \theta(\mu - X_t)dt + \sigma dW_t + dL_t^X$$

starting from $x_0 \geq 0$*, where* L^X *represents the local time of* X *at* 0*. Furthermore, let* $\widetilde{X}$ *be another 1-d ROU process driven by the same Brownian motion* W_t*, satisfying*

$$d\widetilde{X}_t = \tilde{\theta}(\tilde{\mu} - \widetilde{X}_t)dt + \sigma dW_t + dL_t^{\widetilde{X}}$$

starting from $\tilde{x}_0 \geq 0$*, where* $L^{\widetilde{X}}$ *represents the local time of* $\widetilde{X}$ *at* 0*. Suppose that* $\tilde{x}_0 \geq x_0$ *and* $\tilde{\theta}(\tilde{\mu} - x) \geq \theta(\mu - x)$ *for any* $x \geq 0$*. Then, with probability* 1*, for all* $t \geq 0$*,*

$$X_t \leq \widetilde{X}_t.$$

Proof. Suppose that there exists $t > 0$ for which $X_t > \widetilde{X}_t$. By assumptions, $X_0 \leq \widetilde{X}_0$. Furthermore, noting that $X - \widetilde{X}$ has continuous sample paths, there exists $s \in [0, t)$ such that $X_s = \widetilde{X}_s$ but $X_u > \widetilde{X}_u$ for $s < u \leq t$. Together with the given assumptions and the fact that $X_u \geq 0$, we obtain for $s < u \leq t$,

$$\tilde{\theta}\left(\tilde{\mu} - \widetilde{X}_u\right) > \tilde{\theta}\left(\tilde{\mu} - X_u\right) \geq \theta(\mu - X_u).$$

Therefore,

$$\begin{aligned}
X_t - \widetilde{X}_t &= \left(X_s + \int_s^t \theta(\mu - X_u)\,du + \int_s^t \sigma\,dW_u + \int_s^t dL_u^X\right) \\
&\quad - \left(\widetilde{X}_s + \int_s^t \tilde{\theta}\left(\tilde{\mu} - \widetilde{X}_u\right)\,du + \int_s^t \sigma\,dW_u + \int_s^t dL_u^{\widetilde{X}}\right) \\
&= X_s - \widetilde{X}_s + \int_s^t \left[\theta(\mu - X_u) - \tilde{\theta}\left(\tilde{\mu} - \widetilde{X}_u\right)\right]\,du + \left(L_t^X - L_s^X\right) - \left(L_t^{\widetilde{X}} - L_s^{\widetilde{X}}\right) \\
&\leq \left(L_t^X - L_s^X\right) - \left(L_t^{\widetilde{X}} - L_s^{\widetilde{X}}\right).
\end{aligned}$$

Since $L^{\widetilde{X}}$ is nondecreasing, it follows

$$X_t - \widetilde{X}_t \leq L_t^X - L_s^X.$$

Noting that $X_u > \widetilde{X}_u \geq 0$ for $u \in (s, t]$ and L^X is a continuous process that increases only when X equals 0, we conclude $L_t^X = L_s^X$. Therefore, $X_t \leq \widetilde{X}_t$, which is a contradiction. □

Remark 3. *By using a similar argument, the result in Lemma 2 still holds if the 1-d ROU processes are replaces by two 1-d OU processes. Furthermore, we can also prove that a 1-d OU process is bounded above by a 1-d ROU process with the same parameters.*

With the above lemmas in hand, we are ready to show that it is sufficient to prove the positive recurrence when both μ_1 and μ_2 are non-negative, as promised in Section 1. Indeed, if at least one of μ_1 and μ_2 is negative, we can construct a new 2-d ROU process $(\bar{X}, \bar{Y})$ by following SDEs:

$$\begin{aligned}
d\bar{X}_t &= \theta_1(|\mu_1| - \bar{X}_t)dt + \sigma_1 dW_t + dL_t^{\bar{X}} \\
d\bar{Y}_t &= \theta_2(|\mu_2| - \bar{Y}_t)dt + \sigma_2 dB_t + dL_t^{\bar{Y}}
\end{aligned}$$

subject to $\bar{X}_0 = x_0$ and $\bar{Y}_0 = y_0$. Suppose (X, Y) is the 2-d ROU derived from (1.1). Applying Lemma 2, we obtain that (X, Y) is "controlled" by $(\bar{X}, \bar{Y})$, that is, with probability 1, for any $t \geq 0$, $X_t \leq \bar{X}_t$ and $Y_t \leq \bar{Y}_t$. Then, for any neighborhood of the origin, the hitting time of (X, Y) to that neighborhood is always bounded by the hitting time of $(\bar{X}, \bar{Y})$. Hence, the positive recurrence of $(\bar{X}, \bar{Y})$ implies the positive recurrence of (X, Y). As a result, in the rest of the section, we will always assume that both μ_1 and μ_2 are non-negative.

With the above preparation, we now turn to the proof of the positive recurrence. For any neighborhood $\mathcal{N}$ of the origin, we define

$$T := \inf\{t \geq 0 : (X_t, Y_t) \in \mathcal{N}\}.$$

In what follows, we will prove that $E^{(x,y)}[T] < \infty$ for any $(x, y) \in \mathbb{R}_+^2$. Consequently, the positive recurrence follows. Here the symbol $E^{(x,y)}$ represents the expectation when the 2-d ROU process starts from (x, y). The proof will be divided into three subsections, aligning with the three steps described in Section 1.

2.2. **Stopping times.** In this subsection, we will first introduce certain stopping times and estimate their expectations. These will serve as the basis for defining a sequence of stopping times $\{S_n\}_{n=1}^{\infty}$, enabling us to construct the jump process. Moreover, we will provide estimates for the expectation of S_1, as well as the position of (X, Y) at S_1.

Define

$$\tau^X := \inf\{t : X(t) = 0\},$$
$$\tau^Y := \inf\{t : Y(t) = 0\}.$$

Indeed, τ^X and τ^Y are the first hitting times of X and Y to 0, respectively. If X starts at 0, we define

$$\zeta^X := \inf\{t : X(t) = 1\}$$

and

$$\delta^X := \inf\{t > \zeta^X : X(t) = 0\}.$$

Thus, δ^X is the first time that X returns to 0 after it hits 1. Additionally, δ^X can also be written as

$$\delta^X = \zeta^X + \tau^X \circ \theta(\zeta^X).$$

Here $\theta(S)$ is a shift operator, the effect of which on a path ω is to cut off the part of the path before $S(\omega)$ and to shift the remaining part in time. Similarly, if Y starts at 0, we define

$$\zeta^Y := \inf\{t : Y(t) = 1\}$$

and

$$\delta^Y := \inf\{t > \zeta^Y : Y(t) = 0\}.$$

Again, δ^Y is the first time that Y returns to 0 after it hits 1, and δ^Y can be written as

$$\delta^Y = \zeta^Y + \tau^Y \circ \theta(\zeta^Y).$$

Once these stopping times are defined, we can proceed to provide estimates for their expectations, which are summarized in Lemmas 4, 5, and 6.

Lemma 4. *For any $(x, y) \in \mathbb{R}_+^2$, we have*

$$E^{(x,y)}\left[\tau^X\right] = \frac{2}{\sigma_1^2} \int_0^x e^{\frac{\theta_1(v-\mu_1)^2}{\sigma_1^2}} \int_v^{\infty} e^{-\frac{\theta_1(u-\mu_1)^2}{\sigma_1^2}} du dv, \tag{2.1}$$

$$E^{(x,y)}\left[\tau^Y\right] = \frac{2}{\sigma_2^2} \int_0^y e^{\frac{\theta_2(v-\mu_2)^2}{\sigma_2^2}} \int_v^{\infty} e^{-\frac{\theta_2(u-\mu_2)^2}{\sigma_2^2}} du dv. \tag{2.2}$$

Furthermore,

$$E^{(x,y)}\left[\tau^X\right] \le C_1 + \frac{1}{\theta_1}\log(x+1), \tag{2.3}$$

$$E^{(x,y)}\left[\tau^Y\right] \le C_2 + \frac{1}{\theta_2}\log(y+1), \tag{2.4}$$

where

$$C_1 := \frac{2}{\sigma_1}\sqrt{\frac{\pi}{\theta_1}}\int_0^{\mu_1+1} e^{\frac{\theta_1(v-\mu_1)^2}{\sigma_1^2}}\,dv,$$

$$C_2 := \frac{2}{\sigma_2}\sqrt{\frac{\pi}{\theta_2}}\int_0^{\mu_2+1} e^{\frac{\theta_2(v-\mu_2)^2}{\sigma_2^2}}\,dv.$$

Proof. The first statement is an immediate consequence of Proposition 4 in [39]. We then turn to the second statement. By the symmetry, we only need to derive (2.3) from (2.1).

By substituting the variable $z = \sqrt{2\theta_1/\sigma_1^2}(u-\mu_1)$, we have

$$\int_v^\infty e^{-\frac{\theta_1(u-\mu_1)^2}{\sigma_1^2}}\,du = \sqrt{\frac{\sigma_1^2}{2\theta_1}}\int_{\sqrt{\frac{2\theta_1}{\sigma_1^2}}(v-\mu_1)}^\infty e^{-\frac{z^2}{2}}\,dz.$$

If $v \le \mu_1+1$,

$$\int_v^\infty e^{-\frac{\theta_1(u-\mu_1)^2}{\sigma_1^2}}\,du = \sqrt{\frac{\sigma_1^2}{2\theta_1}}\int_{\sqrt{\frac{2\theta_1}{\sigma_1^2}}(v-\mu_1)}^\infty e^{-\frac{z^2}{2}}\,dz \le \sqrt{\frac{\sigma_1^2}{2\theta_1}}\int_{-\infty}^\infty e^{-\frac{z^2}{2}}\,dz = \sqrt{\frac{\pi\sigma_1^2}{\theta_1}}.$$

If $v > \mu_1+1$, using the upper bound for the tail of the standard normal distribution,

$$\begin{aligned}\int_v^\infty e^{-\frac{\theta_1(u-\mu_1)^2}{\sigma_1^2}}\,du &= \sqrt{\frac{\sigma_1^2}{2\theta_1}}\int_{\sqrt{\frac{2\theta_1}{\sigma_1^2}}(v-\mu_1)}^\infty e^{-\frac{z^2}{2}}\,dz \\ &\le \sqrt{\frac{\sigma_1^2}{2\theta_1}}\cdot\frac{1}{\sqrt{\frac{2\theta_1}{\sigma_1^2}}(v-\mu_1)}e^{-\frac{\theta_1(v-\mu_1)^2}{\sigma_1^2}} = \frac{\sigma_1^2}{2\theta_1}\frac{1}{v-\mu_1}e^{-\frac{\theta_1(v-\mu_1)^2}{\sigma_1^2}}.\end{aligned}$$

When $x \ge \mu_1+1$, combining the above two displays, together with (2.1), we have

$$\begin{aligned}&E^{(x,y)}\left[\tau^X\right] \\ =\ & \frac{2}{\sigma_1^2}\int_0^x e^{\frac{\theta_1(v-\mu_1)^2}{\sigma_1^2}}\int_v^\infty e^{-\frac{\theta_1(u-\mu_1)^2}{\sigma_1^2}}\,dudv \\ =\ & \frac{2}{\sigma_1^2}\int_0^{\mu_1+1} e^{\frac{\theta_1(v-\mu_1)^2}{\sigma_1^2}}\int_v^\infty e^{-\frac{\theta_1(u-\mu_1)^2}{\sigma_1^2}}\,dudv + \frac{2}{\sigma_1^2}\int_{\mu_1+1}^x e^{\frac{\theta_1(v-\mu_1)^2}{\sigma_1^2}}\int_v^\infty e^{-\frac{\theta_1(u-\mu_1)^2}{\sigma_1^2}}\,dudv \\ \le\ & \frac{2}{\sigma_1^2}\int_0^{\mu_1+1} e^{\frac{\theta_1(v-\mu_1)^2}{\sigma_1^2}}\sqrt{\frac{\pi\sigma_1^2}{\theta_1}}\,dv + \frac{2}{\sigma_1^2}\int_{\mu_1+1}^x e^{\frac{\theta_1(v-\mu_1)^2}{\sigma_1^2}}\frac{\sigma_1^2}{2\theta_1}\frac{1}{v-\mu_1}e^{-\frac{\theta_1(v-\mu_1)^2}{\sigma_1^2}}\,dv \\ =\ & C_1 + \frac{1}{\theta_1}\int_{\mu_1+1}^x \frac{1}{v-\mu_1}\,dv = C_1 + \frac{1}{\theta_1}\log(x-\mu_1) \\ \le\ & C_1 + \frac{1}{\theta_1}\log(x+1).\end{aligned}$$

When $x < \mu_1 + 1$, it is immediate that $E^{(x,y)}\left[\tau^X\right] \le C_1 \le C_1 + \frac{1}{\theta_1}\log(x+1)$. Combining these two cases, (2.3) follows. $\square$

Lemma 5. *For any $x, y \ge 0$, we have*

$$E^{(0,y)}\left[\zeta^X\right] = \frac{2}{\sigma_1^2}\int_0^1 e^{\frac{\theta_1(v-\mu_1)^2}{\sigma_1^2}}\int_0^v e^{-\frac{\theta_1(u-\mu_1)^2}{\sigma_1^2}}dudv < \infty,$$

$$E^{(x,0)}\left[\zeta^Y\right] = \frac{2}{\sigma_2^2}\int_0^1 e^{\frac{\theta_2(v-\mu_2)^2}{\sigma_2^2}}\int_0^v e^{-\frac{\theta_2(u-\mu_2)^2}{\sigma_2^2}}dudv < \infty.$$

Thus, we use constants C_3 and C_4 to denote $E^{(0,y)}\left[\zeta^X\right]$ and $E^{(x,0)}\left[\zeta^Y\right]$ respectively.

Proof. See Theorem 2 in [39]. $\square$

With the above two lemmas and the fact that $\delta^X = \zeta^X + \tau^X \circ \theta(\zeta^X)$ and $\delta^Y = \zeta^Y + \tau^Y \circ \theta(\zeta^Y)$, we are ready to establish Lemma 6.

Lemma 6. *For any $x, y \ge 0$, we have*

$$E^{(0,y)}\left[\delta^X\right] \le C_5, \tag{2.5}$$

$$E^{(x,0)}\left[\delta^Y\right] \le C_6, \tag{2.6}$$

where

$$C_5 := C_1 + C_3 + \frac{1}{\theta_1}\log(2),$$

$$C_6 := C_2 + C_4 + \frac{1}{\theta_2}\log(2).$$

Proof. We only prove (2.5), and (2.6) follows similarly.

By the strong Markov property,

$$\begin{aligned} E^{(0,y)}\left[\delta^X\right] &= E^{(0,y)}\left[\zeta^X\right] + E^{(0,y)}\left[\tau^X \circ \theta(\zeta^X)\right] \\ &= E^{(0,y)}\left[\zeta^X\right] + E^{(0,y)}\left[E^{\left(1, Y_{\zeta^X}\right)}\left[\tau^X\right]\right]. \end{aligned}$$

By Lemma 5, we have

$$E^{(0,y)}\left[\zeta^X\right] \le C_3.$$

By Lemma 4, we have

$$E^{\left(1, Y_{\zeta^X}\right)}\left[\tau^X\right] \le C_1 + \frac{1}{\theta_1}\log(2).$$

The desired result follows by combining the above three displays. $\square$

The three lemmas above provide bounds for the expectations of the stopping times. To define and analyze the jump process mentioned in Section 1, we also need to evaluate the position of (X, Y) at a given stopping time. The following lemma, which provides a bound for the expectation of a 1-d OU process at any stopping time, will be a valuable tool for this purpose.

Lemma 7. *Let Z be a 1-d OU process satisfying*

$$dZ_t = \theta(\mu - Z_t)dt + \sigma dB_t$$

and starting at z, where $\theta > 0$, $\mu \geq 0$, $\sigma > 0$ and B is a standard Brownian motion. If R is a stopping time with a finite mean, then

$$E^z\left[|Z_R|\right] \leq |z| E^z\left[e^{-\theta R}\right] + \mu + \sigma \cdot \left\{E^z\left[R\right]\right\}^{\frac{1}{2}}.$$

Proof. It is immediate that Z has the expression

$$Z_t = z e^{-\theta t} + \mu\left(1 - e^{-\theta t}\right) + \sigma e^{-\theta t} \int_0^t e^{\theta s}\, dB_s.$$

By substituting R for t and applying the triangle inequality, it easily follows that

$$|Z_R| \leq |z| e^{-\theta R} + \mu + \sigma e^{-\theta R} \left| \int_0^R e^{\theta s}\, dB_s \right|. \tag{2.7}$$

We then evaluate the expectation of $e^{-\theta R}\left|\int_0^R e^{\theta s}\, dB_s\right|$. By Itô's lemma,

$$d\left(e^{-2\theta t}\left(\int_0^t e^{\theta s}\, dB_s\right)^2\right) = -2\theta e^{-2\theta t}\left(\int_0^t e^{\theta s}\, dB_s\right)^2 dt + 2e^{-\theta t}\left(\int_0^t e^{\theta s}\, dB_s\right) dB_t + dt.$$

Using the optional sampling theorem yields

$$\begin{aligned} & E^z\left[e^{-2\theta(t\wedge R)}\left(\int_0^{t\wedge R} e^{\theta s}\, dB_s\right)^2\right] \\ = \;& -2\theta E^z\left[\int_0^{t\wedge R} e^{-2\theta s}\left(\int_0^s e^{\theta u}\, dB_u\right)^2 ds\right] + E^z\left[t\wedge R\right] \\ \leq \;& E^z\left[t \wedge R\right]. \end{aligned}$$

Letting $t \to \infty$, together with Fatou's lemma and the monotone convergence theorem, we have

$$E^z\left[e^{-2\theta R}\left(\int_0^R e^{\theta s}\, dB_s\right)^2\right] \leq E^z\left[R\right].$$

It follows by Jensen's inequality that

$$E^z\left[e^{-\theta R}\left|\int_0^R e^{\theta s}\, dB_s\right|\right] \leq \left\{E^z\left[e^{-2\theta R}\left(\int_0^R e^{\theta s}\, dB_s\right)^2\right]\right\}^{\frac{1}{2}} \leq \left\{E^z\left[R\right]\right\}^{\frac{1}{2}}.$$

By taking expectation on both sides of (2.7) and combining it with the above display, the desired result follows. □

Next, we introduce a new stopping time, S, related to τ^X, τ^Y, δ^X and δ^Y. This stopping time is a key ingredient in defining the jump process and is defined as follows:

- If (X, Y) starts at $(x, 0)$, then $S = \tau^X \wedge \delta^Y$.
- If (X, Y) starts at $(0, y)$, then $S = \tau^Y \wedge \delta^X$.

In other words, if the process (X, Y) starts from a point on $\partial\mathbb{R}^2_+ \setminus \{(0,0)\} = \{(x, y) \neq (0, 0) : x = 0 \text{ or } y = 0\}$, the stopping time S means the first time when either one component of the pair (X_t, Y_t) reaches 0 or the other that starts at 0 reaches 1 and then returns to 0. In particular, if the process (X, Y) starts from $(0, 0)$, both definitions result in $S = 0$.

Before introducing the sequence of stopping times that defines the jump process, we pause to examine the properties of S. Indeed, we will estimate the expectations of S and X_S+Y_S, as summarized in Lemma 8 and Lemma 9, respectively.

We begin with Lemma 8.

Lemma 8. *For any $(x,y)\in\partial\mathbb{R}_+^2$, we have*

$$E^{(x,y)}\left[S\right]\le C_5\vee C_6,$$

where C_5 and C_6 are defined in Lemma 6.

Proof. If (X,Y) starts at $(x,0)$, it follows by Lemma 6 and the definition of S that

$$E^{(x,0)}[S]=E^{(x,0)}[\tau^X\wedge\delta^Y]\le E^{(x,0)}[\delta^Y]\le C_6.$$

Similarly, if (X,Y) starts at $(0,y)$, we have $E^{(0,y)}[S]\le C_5$. The desired result follows by combining the above two cases. $\square$

The following lemma concerns the expectation of X_S+Y_S.

Lemma 9. *For any $x,y\ge 0$, we have*

$$E^{(x,0)}\left[X_S+Y_S\right]\le C_7x+\sigma_2\theta_1^{-1/2}\sqrt{x}+C_9, \tag{2.8}$$

$$E^{(0,y)}\left[X_S+Y_S\right]\le C_8y+\sigma_1\theta_2^{-1/2}\sqrt{y}+C_{10}, \tag{2.9}$$

where

$$C_7:=E^{(x,0)}\left[e^{-\theta_1\delta^Y}\right]<1,$$

$$C_8:=E^{(0,y)}\left[e^{-\theta_2\delta^X}\right]<1,$$

and

$$C_9:=2+\mu_1+\mu_2+\sigma_1\sqrt{C_6}+\sigma_2\sqrt{C_1}+\sigma_2\theta_1^{-1/2}\sqrt{\mu_1+\sigma_1\sqrt{C_4}},$$

$$C_{10}:=2+\mu_1+\mu_2+\sigma_2\sqrt{C_5}+\sigma_1\sqrt{C_2}+\sigma_1\theta_2^{-1/2}\sqrt{\mu_2+\sigma_2\sqrt{C_3}}.$$

Proof. We only prove (2.8), and (2.9) follows similarly.

By the definition of S, when (X,Y) starts at $(x,0)$,

$$X_S=X_{\tau^X\wedge\delta^Y}=X_{\tau^X}\cdot\mathbb{1}_{\{\tau^X\le\delta^Y\}}+X_{\delta^Y}\cdot\mathbb{1}_{\{\delta^Y<\tau^X\}}=X_{\delta^Y}\cdot\mathbb{1}_{\{\delta^Y<\tau^X\}},$$

where the last equality follows from the fact that $X_{\tau^X}=0$, as τ^X is the first time X hits 0. On the event $\{t<\tau^X\}$, the 1-d ROU process X is exactly a 1-d OU process, and hence, has the representation

$$X_t=xe^{-\theta_1 t}+\mu_1\left(1-e^{-\theta_1 t}\right)+\sigma_1 e^{-\theta_1 t}\int_0^t e^{\theta s}\,dW_s.$$

Applying Lemma 7 yields

$$\begin{aligned}&E^{(x,0)}\left[X_S\right]=E^{(x,0)}\left[X_{\delta^Y}\cdot\mathbb{1}_{\{\delta^Y<\tau^X\}}\right]\\ =\;&E^{(x,0)}\left[\left(xe^{-\theta_1\delta^Y}+\mu_1\left(1-e^{-\theta_1\delta^Y}\right)+\sigma_1 e^{-\theta_1\delta^Y}\int_0^{\delta^Y}e^{\theta s}\,dW_s\right)\mathbb{1}_{\{\delta^Y<\tau^X\}}\right]\end{aligned}$$

$$
\begin{aligned}
&\le E^{(x,0)}\left[\left|xe^{-\theta_1\delta^Y}+\mu_1\left(1-e^{-\theta_1\delta^Y}\right)+\sigma_1 e^{-\theta_1\delta^Y}\int_0^{\delta^Y} e^{\theta s}\,dW_s\right|\right]\\
&\le xE^{(x,0)}\left[e^{-\theta_1\delta^Y}\right]+\mu_1+\sigma_1\left\{E^{(x,0)}\left[\delta^Y\right]\right\}^{\frac{1}{2}}\\
&\le xE^{(x,0)}\left[e^{-\theta_1\delta^Y}\right]+\mu_1+\sigma_1\sqrt{C_6}, \qquad (2.10)
\end{aligned}
$$

where in the last inequality we have invoked Lemma 6.

Similarly, when (X,Y) starts at $(x,0)$, noting that $Y_{\delta^Y}=0$, we have

$$
\begin{aligned}
Y_S &= Y_{\tau^X\wedge\delta^Y} = Y_{\tau^X}\cdot\mathbb{1}_{\{\tau^X<\delta^Y\}}+Y_{\delta^Y}\cdot\mathbb{1}_{\{\delta^Y\le\tau^X\}} = Y_{\tau^X}\cdot\mathbb{1}_{\{\tau^X<\delta^Y\}}\\
&= Y_{\tau^X}\cdot\mathbb{1}_{\{\tau^X\le\zeta^Y\}}+Y_{\tau^X}\cdot\mathbb{1}_{\{\zeta^Y<\tau^X<\delta^Y\}}.
\end{aligned}
$$

Since $Y_t\le 1$ for $t\le\zeta^Y$, we have $Y_{\tau^X}\cdot\mathbb{1}_{\{\tau^X\le\zeta^Y\}}\le 1$, and thus,

$$
Y_S\le 1+Y_{\tau^X}\cdot\mathbb{1}_{\{\zeta^Y<\tau^X<\delta^Y\}}.
$$

Taking expectation yields

$$
\begin{aligned}
E^{(x,0)}\left[Y_S\right] &\le 1+E^{(x,0)}\left[Y_{\tau^X}\cdot\mathbb{1}_{\{\zeta^Y<\tau^X<\delta^Y\}}\right]\\
&= 1+E^{(x,0)}\left[\mathbb{1}_{\{\zeta^Y<\tau^X\}}Y_{\tau^X}\cdot\mathbb{1}_{\{\tau^X<\delta^Y\}}\right]\\
&= 1+E^{(x,0)}\left[\mathbb{1}_{\{\zeta^Y<\tau^X\}}E^{(X_{\zeta^Y},1)}\left[Y_{\tau^X}\cdot\mathbb{1}_{\{\tau^X<\tau^Y\}}\right]\right], \qquad (2.11)
\end{aligned}
$$

where the last equality follows by the strong Markov property. On the event $\{t<\tau^Y\}$, the 1-d ROU process Y is exactly a 1-d OU process, and hence, has the form

$$
Y_t=e^{-\theta_2 t}+\mu_2\left(1-e^{-\theta_2 t}\right)+\sigma_2 e^{-\theta_2 t}\int_0^t e^{\theta s}\,dB_s.
$$

Applying Lemma 7 yields

$$
\begin{aligned}
&E^{(X_{\zeta^Y},1)}\left[Y_{\tau^X}\cdot\mathbb{1}_{\{\tau^X<\tau^Y\}}\right]\\
&= E^{(X_{\zeta^Y},1)}\left[\left(e^{-\theta_2\tau^X}+\mu_2\left(1-e^{-\theta_2\tau^X}\right)+\sigma_2 e^{-\theta_2\tau^X}\int_0^{\tau^X} e^{\theta s}\,dB_s\right)\mathbb{1}_{\{\tau^X<\tau^Y\}}\right]\\
&\le E^{(X_{\zeta^Y},1)}\left[\left|e^{-\theta_2\tau^X}+\mu_2\left(1-e^{-\theta_2\tau^X}\right)+\sigma_2 e^{-\theta_2\tau^X}\int_0^{\tau^X} e^{\theta s}\,dB_s\right|\right]\\
&\le E^{(X_{\zeta^Y},1)}\left[e^{-\theta_2\tau^X}\right]+\mu_2+\sigma_2\left\{E^{(X_{\zeta^Y},1)}\left[\tau^X\right]\right\}^{\frac{1}{2}}\\
&\le 1+\mu_2+\sigma_2\cdot\left(C_1+\frac{1}{\theta_1}\log\left(X_{\zeta^Y}+1\right)\right)^{\frac{1}{2}}\\
&\le 1+\mu_2+\sigma_2\sqrt{C_1}+\sigma_2\theta_1^{-1/2}\left(\log\left(X_{\zeta^Y}+1\right)\right)^{\frac{1}{2}}\\
&\le 1+\mu_2+\sigma_2\sqrt{C_1}+\sigma_2\theta_1^{-1/2}\cdot X_{\zeta^Y}^{\frac{1}{2}},
\end{aligned}
$$

where the third last inequality is due to Lemma 4, the second last inequality follows by the inequality $\sqrt{a+b}\le\sqrt{a}+\sqrt{b}$, and in the last inequality, the inequality $\log(x+1)\le x$ is applied. Combining the last display with (2.11), after a routine calculation, we have

$$\begin{aligned} E^{(x,0)}\left[Y_S\right] &\leq 2+\mu_2+\sigma_2\sqrt{C_1}+\sigma_2\theta_1^{-1/2}\cdot E^{(x,0)}\left[\mathbb{1}_{\{\zeta^Y<\tau^X\}}X_{\zeta^Y}^{\frac{1}{2}}\right] \\ &\leq 2+\mu_2+\sigma_2\sqrt{C_1}+\sigma_2\theta_1^{-1/2}\cdot\left\{E^{(x,0)}\left[X_{\zeta^Y}\mathbb{1}_{\{\zeta^Y<\tau^X\}}\right]\right\}^{\frac{1}{2}}, \end{aligned} \tag{2.12}$$

where in the last inequality we have used Jensen's inequality. Again, using the argument that X_t is a 1-d OU process when $t\leq\tau^X$ and Lemma 7, we have

$$\begin{aligned} & E^{(x,0)}\left[X_{\zeta^Y}\mathbb{1}_{\{\zeta^Y<\tau^X\}}\right] \\ \leq\ & xE^{(x,0)}\left[e^{-\theta_1\zeta^Y}\right]+\mu_1+\sigma_1\{E^{(x,0)}\left[\zeta^Y\right]\}^{\frac{1}{2}} \\ \leq\ & x+\mu_1+\sigma_1\sqrt{C_4}, \end{aligned}$$

where in the last inequality Lemma 5 is applied. Plugging the last display into (2.12) yields

$$\begin{aligned} E^{(x,0)}\left[Y_S\right] &\leq 2+\mu_2+\sigma_2\sqrt{C_1}+\sigma_2\theta_1^{-1/2}\cdot\left(x+\mu_1+\sigma_1\sqrt{C_4}\right)^{\frac{1}{2}} \\ &\leq 2+\mu_2+\sigma_2\sqrt{C_1}+\sigma_2\theta_1^{-1/2}\sqrt{\mu_1+\sigma_1\sqrt{C_4}}+\sigma_2\theta_1^{-1/2}\sqrt{x}. \end{aligned}$$

Combining the last display with (2.10), the desired result follows. □

With Lemma 9 in hand, We immediately obtain Corollary 10 as follows:

Corollary 10. *There are constants $C_{11}<1$, C_{12} and C_{13} such that for any $(x,y)\in\partial\mathbb{R}_+^2$,*

$$E^{(x,y)}\left[X_S+Y_S\right]\leq C_{11}(x+y)+C_{12}\sqrt{x+y}+C_{13}.$$

Consequently, there exist sufficiently large B and $\alpha\in[0,1)$ such that if $(x,y)\in\partial\mathbb{R}_+^2$ and $x+y\geq B$, then

$$\begin{aligned} E^{(x,y)}\left[X_S+Y_S\right] &\leq C_{11}(x+y)+C_{12}\sqrt{x+y}+C_{13} \\ &\leq \alpha(x+y). \end{aligned}$$

In the rest of the section, we will use $\mathcal{O}_B$ to denote the set $\{(x,y)\in\mathbb{R}_+^2 : x+y\leq B\}$.

With the above preparation, we are ready to introduce the stopping times and the jump process as mentioned in Section 1.

Define $S_0=0$, $S_1=S$, and

$$S_n=S_{n-1}+S\circ\theta(S_{n-1})\qquad\text{for } n\geq 2.$$

By the strong Markov property and Lemma 8, we have for $(x,y)\in\partial\mathbb{R}_+^2$,

$$E^{(x,y)}\left[S_{n+1}-S_n\right]=E^{(x,y)}\left[E^{(X_{S_n},Y_{S_n})}\left[S\right]\right]\leq E^{(x,y)}\left[C_5\vee C_6\right]=C_5\vee C_6.$$

Additionally, we define the jump process as follows:

$$(\xi_n,\eta_n)=(X_{S_n},Y_{S_n})$$

for $n=0,1,2,\ldots$. By the definition of S_n, it is immediate that $(\xi_n,\eta_n)\in\partial\mathbb{R}_+^2$. Therefore, $\{(\xi_n,\eta_n):n=1,2,\ldots\}$ is a jump process on $\partial\mathbb{R}_+^2$.

2.3. **Reachability of the jump process to the neighborhood of the origin.** In this subsection, we will prove that if the process $\{(\xi_n,\eta_n):n=0,1,2,\dots\}$ starts from a point on $\partial\mathbb{R}^2_+$, then it will eventually reach $\partial\mathbb{R}^2_+\cap\mathcal{O}_B$ within a finite mean time. Consequently, the same property holds for the original 2-d ROU process.

The preceding statements are summarized in the following lemma.

Lemma 11. *Define* $N:=\inf\{n:(\xi_n,\eta_n)\in\mathcal{O}_B\}$. *We have* $N<\infty$ *almost surely. Furthermore,*

$$E^{(x,y)}\left[S_N\right]\le C_5\vee C_6\cdot\frac{x+y}{(1-\alpha)B}\,. \tag{2.13}$$

Therefore, the 2-d ROU process (X,Y)*, starting from a point* (x,y) *on* $\partial\mathbb{R}^2_+$*, will eventually reach the region* $\partial\mathbb{R}^2_+\cap\mathcal{O}_B$ *within an expected time of at most* $(C_5\vee C_6)(x+y)/((1-\alpha)B)$.

Proof. We claim $\left\{\alpha^{-(n\wedge N)}\left(\xi_{n\wedge N}+\eta_{n\wedge N}\right)\right\}$ is a non-negative supermartingale with respect to the filtration $\{\mathcal{F}_{S_n}\}_{n\in\mathbb{N}}$. Recall that $(\mathcal{F}_t)_{t\ge0}$ is the natural filtration generated by (W,B). Therefore, $\mathcal{F}_{S_n}$ represents the corresponding σ-field stopped at S_n. It is immediate that ξ_n and η_n are $\mathcal{F}_{S_n}$-measurable, since $\xi_n=X_{S_n}$ and $\eta_n=Y_{S_n}$.

We now turn to verify the above claim. We have

$$\begin{aligned}
& E^{(x,y)}\left[\alpha^{-((n+1)\wedge N)}\left(\xi_{(n+1)\wedge N}+\eta_{(n+1)\wedge N}\right)\big|\mathcal{F}_{S_n}\right]\\
= & E^{(x,y)}\left[\alpha^{-((n+1)\wedge N)}\left(\xi_{(n+1)\wedge N}+\eta_{(n+1)\wedge N}\right)\mathbb{1}_{\{N\le n\}}\big|\mathcal{F}_{S_n}\right]\\
& +E^{(x,y)}\left[\alpha^{-((n+1)\wedge N)}\left(\xi_{(n+1)\wedge N}+\eta_{(n+1)\wedge N}\right)\mathbb{1}_{\{N>n\}}\big|\mathcal{F}_{S_n}\right]\\
= & E^{(x,y)}\left[\alpha^{-(n\wedge N)}\left(\xi_{n\wedge N}+\eta_{n\wedge N}\right)\mathbb{1}_{\{N\le n\}}\big|\mathcal{F}_{S_n}\right]\\
& +E^{(x,y)}\left[\alpha^{-(n+1)}\left(\xi_{n+1}+\eta_{n+1}\right)\mathbb{1}_{\{N>n\}}\big|\mathcal{F}_{S_n}\right]\\
= & \alpha^{-(n\wedge N)}\left(\xi_{n\wedge N}+\eta_{n\wedge N}\right)\mathbb{1}_{\{N\le n\}}\\
& +\mathbb{1}_{\{N>n\}}\cdot E^{(\xi_n,\eta_n)}\left[\alpha^{-(n+1)}\left(X_S+Y_S\right)\right]\\
\le & \alpha^{-(n\wedge N)}\left(\xi_{n\wedge N}+\eta_{n\wedge N}\right)\mathbb{1}_{\{N\le n\}}+\mathbb{1}_{\{N>n\}}\cdot\alpha^{-n}\left(\xi_n+\eta_n\right)\\
= & \alpha^{-(n\wedge N)}\left(\xi_{n\wedge N}+\eta_{n\wedge N}\right),
\end{aligned}$$

where the second last equality is due to the strong Markov property and in the last inequality, Corollary 10 is applied. By the standard properties of supermartingale,

$$\begin{aligned}
x+y=E^{(x,y)}\left[\xi_0+\eta_0\right] & \ge E^{(x,y)}\left[\alpha^{-(n\wedge N)}\left(\xi_{n\wedge N}+\eta_{n\wedge N}\right)\right]\\
& \ge E^{(x,y)}\left[\alpha^{-(n\wedge N)}\left(\xi_{n\wedge N}+\eta_{n\wedge N}\right)\mathbb{1}_{\{N>n\}}\right]\\
& =E^{(x,y)}\left[\alpha^{-n}\left(\xi_n+\eta_n\right)\mathbb{1}_{\{N>n\}}\right]\\
& \ge E^{(x,y)}\left[\alpha^{-n}\cdot B\cdot\mathbb{1}_{\{N>n\}}\right]\\
& =\alpha^{-n}B\cdot P^{(x,y)}\left(N>n\right),
\end{aligned}$$

where the last inequality follows by the fact that $\xi_n+\eta_n>B$ on the event $\{N>n\}$. Thus,

$$P^{(x,y)}\left(N>n\right)\le\frac{x+y}{B}\alpha^n.$$

Given that $\alpha\in[0,1)$, it follows by letting $n\to\infty$ that $N<\infty$ almost surely. Therefore, the jump process (ξ_n,η_n) inevitably hits the region $\partial\mathbb{R}^2_+\cap\mathcal{O}_B$. If we look back at the 2-d ROU process (X,Y), we can conclude that (X,Y) must hit $\partial\mathbb{R}^2_+\cap\mathcal{O}_B$ if it starts from a point on $\partial\mathbb{R}^2_+$.

Note that the first time when (X,Y) hits $\partial\mathbb{R}^2_+\cap\mathcal{O}_B$ is less than or equal to S_N. In the next step, we estimate the expectation of S_N and, consequently, the expected time for the process (X,Y) to visit $\partial\mathbb{R}^2_+\cap\mathcal{O}_B$. Indeed, by Lemma 8 and the strong Markov property,

$$\begin{aligned}
E^{(x,y)}\left[S_N\right] &= E^{(x,y)}\left[\sum_{n=0}^{N-1}\left(S_{n+1}-S_n\right)\right] \\
&= E^{(x,y)}\left[\sum_{n=0}^{\infty}\left(S_{n+1}-S_n\right)\mathbb{1}_{\{N>n\}}\right] \\
&= \sum_{n=0}^{\infty}E^{(x,y)}\left[\left(S_{n+1}-S_n\right)\mathbb{1}_{\{N>n\}}\right] \\
&= \sum_{n=0}^{\infty}E^{(x,y)}\left[\mathbb{1}_{\{N>n\}}\cdot E^{(X_{S_n},Y_{S_n})}\left[S\right]\right] \\
&\le \sum_{n=0}^{\infty}E^{(x,y)}\left[\mathbb{1}_{\{N>n\}}\cdot C_5\vee C_6\right] \\
&= C_5\vee C_6\cdot\sum_{n=0}^{\infty}P^{(x,y)}\left(N>n\right) \\
&\le C_5\vee C_6\cdot\sum_{n=0}^{\infty}\frac{x+y}{B}\alpha^n \\
&= C_5\vee C_6\cdot\frac{x+y}{(1-\alpha)B}.
\end{aligned}$$

Therefore, for the 2-d ROU process (X,Y) starting from a point (x,y) on $\partial\mathbb{R}^2_+$, it eventually reaches the region $\partial\mathbb{R}^2_+\cap\mathcal{O}_B$ with an expected time no greater than $(C_5\vee C_6)(x+y)/\left((1-\alpha)B\right)$. □

2.4. **Positive recurrence.** The purpose of this subsection is to present the main theorem (Theorem 12) of this section, which focuses on the positive recurrence of the 2-d ROU process (X,Y).

Recall that T is the first time when the process hits the neighborhood $\mathcal{N}$ of the origin.

Theorem 12. *There exists a constant C_{14} (see (2.19)) such that for any $(x,y)\in\mathbb{R}^2_+$,*

$$E^{(x,y)}\left[T\right]\le C_{14}\left(1+x+y\right).$$

Proof. Define

$$T_0:=\inf\{t:(X_t,Y_t)\in\partial\mathbb{R}^2_+\}$$

and

$$H:=\inf\{t\ge T_0:(X_t,Y_t)\in\partial\mathbb{R}^2_+\cap\mathcal{O}_B\}.$$

H is the first time that the process (X,Y) hits the area $\partial\mathbb{R}^2_+\cap\mathcal{O}_B$. We first prove

$$E^{(x,y)}\left[H\right]\le C_{15}(1+x+y), \tag{2.14}$$

where

$$C_{15} := \frac{C_5 \vee C_6}{(1-\alpha)B} \cdot \left(1 + \mu_1 + \mu_2 + (\sigma_1+\sigma_2)\left(\sqrt{C_1} + \theta_1^{-1/2}\right)\right) + C_1 + \frac{1}{\theta_1}.$$

Note that $T_0 \le \tau^X \wedge \tau^Y \le \tau^X$. Thus, by Lemma 4,

$$E^{(x,y)}\left[T_0\right] \le E^{(x,y)}\left[\tau^X\right] \le C_1 + \frac{1}{\theta_1}\log(x+1). \tag{2.15}$$

To bound $E^{(x,y)}[H]$, it remains to provide an upper bound for $E^{(x,y)}[H-T_0]$, which depends on the estimate for $E^{(x,y)}\left[X_{T_0}+Y_{T_0}\right]$. To achieve this, we note that for $t \le T_0$, (X_t, Y_t) behaves like a 2-d OU process. Using a similar argument as in the proof of Lemma 9, we conclude

$$\begin{aligned} & E^{(x,y)}\left[X_{T_0}+Y_{T_0}\right] \\ \le\ & xE^{(x,y)}\left[e^{-\theta_1 T_0}\right] + \mu_1 + \sigma_1\left\{E\left[T_0\right]\right\}^{\frac{1}{2}} + yE^{(x,y)}\left[e^{-\theta_1 T_0}\right] + \mu_2 + \sigma_2\left\{E\left[T_0\right]\right\}^{\frac{1}{2}} \\ \le\ & x + y + \mu_1 + \mu_2 + (\sigma_1+\sigma_2)\left(C_1 + \frac{1}{\theta_1}\log(x+1)\right)^{\frac{1}{2}}. \end{aligned} \tag{2.16}$$

Then, by the strong Markov property,

$$\begin{aligned} & E^{(x,y)}\left[H - T_0\right] \\ =\ & E^{(x,y)}\left[E^{(X_{T_0},Y_{T_0})}\left[\inf\left\{t \ge 0 : (X_t,Y_t) \in \partial\mathbb{R}_+^2 \cap \mathcal{O}_B\right\}\right]\right] \\ \le\ & E^{(x,y)}\left[E^{((X_{T_0},Y_{T_0}))}\left[S_N\right]\right] \\ \le\ & E^{(x,y)}\left[C_5 \vee C_6 \cdot \frac{X_{T_0}+Y_{T_0}}{(1-\alpha)B}\right] \\ \le\ & \frac{C_5 \vee C_6}{(1-\alpha)B} \cdot \left(x + y + \mu_1 + \mu_2 + (\sigma_1+\sigma_2)\left(C_1 + \frac{1}{\theta_1}\log(x+1)\right)^{\frac{1}{2}}\right), \end{aligned}$$

where the second last inequality is due to (2.13) and the last inequality follows from (2.16). Combining the last display with (2.15), after a straightforward algebra, (2.14) follows.

We have already proven that the process (X,Y), starting from any point on $\partial\mathbb{R}_+^2$, will reach $\partial\mathbb{R}_+^2 \cap \mathcal{O}_B$ within a time H whose mean is bounded by $1+x+y$ (up to a constant). We then define

$$R := 1 \wedge T \wedge \inf\{t \ge 0 : X_t + Y_t \ge 2B\}.$$

To finish the proof, we need the following result:

There is a constant $\epsilon > 0$ such that

$$P^{(x,y)}\left(R = T\right) \ge \epsilon, \tag{2.17}$$

for any $(x,y) \in \partial\mathbb{R}_+^2 \cap \mathcal{O}_B$.

The proof of this result relies on some standard properties of Brownian motion. By Lemma 1, we can construct a 2-d RBM $(\widetilde{X},\widetilde{Y})$ such that $X_t \le \widetilde{X}_t$ and $Y_t \le \widetilde{Y}_t$ for any $t \ge 0$. Define $\widetilde{T}$ and $\widetilde{R}$ for $(\widetilde{X},\widetilde{Y})$ similarly to T and R, respectively. By a standard property of Brownian motion, there exists $\epsilon > 0$ such that, for all $(x,y) \in \partial\mathbb{R}_+^2 \cap \mathcal{O}_B$, $P^{(x,y)}(\widetilde{R} = \widetilde{T}) \ge \epsilon$ (refer to [21]). Thus, we obtain

$$P^{(x,y)}\left(R = T\right) \ge P^{(x,y)}(\widetilde{R} = \widetilde{T}) \ge \epsilon.$$

We now return to complete the proof. We define

$$\begin{aligned} H_1 &= H, \\ R_1 &= H_1 + R \circ \theta(H_1), \end{aligned}$$

and define recursively for $n \geq 2$,

$$\begin{aligned} H_n &= R_{n-1} + H \circ \theta(R_{n-1}), \\ R_n &= H_n + R \circ \theta(H_n). \end{aligned}$$

It follows immediately from the definition of H and R that $(X_{H_n}, Y_{H_n}) \in \partial\mathbb{R}^2_+ \cap \mathcal{O}_B$ and $X_{R_n} + Y_{R_n} \leq 2B$ for $n \geq 1$. Considering the estimate (2.14) and applying the strong Markov property, we have for $n \geq 2$,

$$\begin{aligned} E^{(x,y)}\left[H_n - R_{n-1}\right] &= E^{(x,y)}\left[E^{(X_{R_{n-1}}, Y_{R_{n-1}})}\left[H\right]\right] \\ &\leq E^{(x,y)}\left[C_{15}(1 + X_{R_{n-1}} + Y_{R_{n-1}})\right] \\ &\leq C_{15}(1+2B). \end{aligned}$$

Thus, noting that $R \leq 1$, we obtain that for $n \geq 2$,

$$\begin{aligned} E^{(x,y)}\left[H_n - H_{n-1}\right] &= E^{(x,y)}\left[H_n - R_{n-1}\right] + E^{(x,y)}\left[R_{n-1} - H_{n-1}\right] \\ &\leq C_{15}(1+2B) + E^{(x,y)}\left[E^{(X_{H_{n-1}}, Y_{H_{n-1}})}\left[R\right]\right] \\ &\leq C_{15}(1+2B) + 1. \end{aligned}$$

From the above estimate, we conclude that for $n \geq 1$,

$$H_n - H_1 - \left(C_{15}(1+2B)+1\right)(n-1)$$

is a supermartingale with respect to the filtration $\{\mathcal{F}_{H_n}\}_{n\geq 1}$. Once again, $\mathcal{F}_{H_n}$ is the σ-field obtained by stopping $\{\mathcal{F}_t\}_{t\geq 0}$ at H_n (recall that $\{\mathcal{F}_t\}_{t\geq 0}$ is the natural filtration generated by (W_t, B_t)). Define

$$M := \inf\{n \geq 1 : (X_{R_n}, Y_{R_n}) \in \mathcal{N}\}.$$

Then, $M+1$ is a stopping time with respect to the filtration $\{\mathcal{F}_{H_n}\}_n$, since $R_{n-1} \leq H_n$. Applying the optional sampling theorem yields

$$\begin{aligned} E^{(x,y)}\left[H_{n\wedge(M+1)} - H_1\right] &\leq E^{(x,y)}\left[\left(C_{15}(1+2B)+1\right)(n \wedge (M+1) - 1)\right] \\ &= \left(C_{15}(1+2B)+1\right) E^{(x,y)}\left[(n-1)\wedge M\right]. \end{aligned}$$

Letting $n \to \infty$, and applying the monotone convergence theorem, we have

$$E^{(x,y)}\left[H_{M+1} - H_1\right] \leq \left(C_{15}(1+2B)+1\right) E^{(x,y)}\left[M\right].$$

Together with (2.14), we have

$$E^{(x,y)}\left[H_{M+1}\right] \leq C_{15}(1+x+y) + \left(C_{15}(1+2B)+1\right) E^{(x,y)}\left[M\right].$$

It follows immediately from the definition of M that $T \leq R_M \leq H_{M+1}$, thus,

$$E^{(x,y)}\left[T\right] \leq C_{15}(1+x+y) + \left(C_{15}(1+2B)+1\right) E^{(x,y)}\left[M\right]. \tag{2.18}$$

What remains is to estimate the expectation of M.

For any $(x,y) \in \partial\mathbb{R}^2_+ \cap \mathcal{O}_B$, by the strong Markov property, we obtain

$$P^{(x,y)}(M > n) = E^{(x,y)}\left[\mathbb{1}_{\{M>n\}}\right] = E^{(x,y)}\left[\mathbb{1}_{\{R_1 \neq H_1 + T\circ\theta(H_1)\}} \cdot \mathbb{1}_{\{M>n\}}\right]$$

$$= E^{(x,y)}\left[\mathbb{1}_{\{R_1\neq H_1+T\circ\theta(H_1)\}}\cdot E^{(X_{H_2},Y_{H_2})}\left[\mathbb{1}_{\{M>n-1\}}\right]\right]$$
$$\leq E^{(x,y)}\left[\mathbb{1}_{\{R_1\neq H_1+T\circ\theta(H_1)\}}\cdot \sup_{(x,y)\in\partial\mathbb{R}^2_+\cap\mathcal{O}_B} P^{(x,y)}(M>n-1)\right]$$
$$= E^{(x,y)}\left[\mathbb{1}_{\{R_1\neq H_1+T\circ\theta(H_1)\}}\right]\cdot \sup_{(x,y)\in\partial\mathbb{R}^2_+\cap\mathcal{O}_B} P^{(x,y)}(M>n-1)$$
$$= E^{(x,y)}\left[E^{(X_{H_1},Y_{H_1})}\left[\mathbb{1}_{\{R\neq T\}}\right]\right]\cdot \sup_{(x,y)\in\partial\mathbb{R}^2_+\cap\mathcal{O}_B} P^{(x,y)}(M>n-1)$$
$$\leq (1-\epsilon)\cdot \sup_{(x,y)\in\partial\mathbb{R}^2_+\cap\mathcal{O}_B} P^{(x,y)}(M>n-1),$$

where the last inequality is due to (2.17). Taking the supremum over (x,y) gives

$$\sup_{(x,y)\in\partial\mathbb{R}^2_+\cap\mathcal{O}_B} P^{(x,y)}(M>n)\leq (1-\epsilon)\cdot \sup_{(x,y)\in\partial\mathbb{R}^2_+\cap\mathcal{O}_B} P^{(x,y)}(M>n-1).$$

Continuing the above procedure recursively, we have

$$\sup_{(x,y)\in\partial\mathbb{R}^2_+\cap\mathcal{O}_B} P^{(x,y)}(M>n)\leq (1-\epsilon)^n.$$

Since $(X_{H_1},Y_{H_1})\in\partial\mathbb{R}^2_+\cap\mathcal{O}_B$,

$$E^{(x,y)}[M]=E^{(x,y)}\left[E^{(X_{H_1},Y_{H_1})}[M]\right]=E^{(x,y)}\left[\sum_{n=0}^{\infty}P^{(X_{H_1},Y_{H_1})}(M>n)\right]$$
$$\leq E^{(x,y)}\left[\sum_{n=0}^{\infty}(1-\epsilon)^n\right]=\frac{1}{\epsilon}.$$

Plugging the above result into (2.18) gives

$$E^{(x,y)}[T]\leq C_{15}(1+x+y)+\left(C_{15}(1+2B)+1\right)/\epsilon\leq C_{14}(1+x+y),$$

where

$$C_{14}:=C_{15}+\frac{C_{15}(1+2B)+1}{\epsilon}. \tag{2.19}$$

We conclude the proof. □

3. Exponential rate of convergence

The section aims to investigate the rate of convergence to the stationary distribution of the 2-d ROU process. We will provide exponential rates of convergence in both the total variation distance and Wasserstein distance. Further, we will explicitly specify the exponent for the latter.

3.1. **A tail estimate for the first hitting time of a 1-d OU process.** Before addressing the rate of convergence, we first present a lemma (Lemma 13) regarding the tail estimate of the first hitting time for a 1-d OU process, which is the main ingredient in studying the rate of convergence in the total variation distance.

Lemma 13. *Let Z be a 1-d OU process driven by the SDE:*

$$dZ_t = \theta(\mu - Z_t)dt + \sigma dB_t$$

starting at z, where $\theta > 0$, $\mu \in \mathbb{R}$, $\sigma > 0$ and B is a standard Brownian motion. Let

$$\tau^Z := \inf\{t \geq 0 : Z_t = 0\}.$$

Then for any $t_* > 0$, there exist constants C_{16} and C_{17} depending on θ, μ, σ, and t_* such that for $t \geq t_*$,

$$P^z(\tau^Z > t) \leq C_{16} e^{-C_{17} t} + \frac{4}{\sqrt{\pi}} \frac{z\sigma}{\theta^2} e^{z\theta/\sigma^2} \cdot \frac{1}{\sqrt{t^3}} e^{-\frac{\theta^2}{4\sigma^2} t}.$$

Proof. We start by considering the case where $\mu \geq 0$. Our approach to estimating $P^z(\tau^Z > t)$ will be divided into three cases based on the value of z: (i) $z = \mu + 1$; (ii) $0 \leq z < \mu + 1$; (iii) $z > \mu + 1$.

(i) $\underline{z = \mu + 1}$. Using Corollary 3.2 in [1] (after an appropriate transformation), we have for any $t_* > 0$, there exist constants C_{16} and C_{17} depending on θ, μ, σ, and t_* such that for $t \geq t_*/2$,

$$P^{(\mu+1)}(\tau^Z > t) \leq C_{16} e^{-2C_{17} t}.$$

(ii) $\underline{0 \leq z < \mu + 1}$. By applying Remark 3, Z is bounded above by the 1-d OU process with the same parameters, but starting at $\mu + 1$. Hence, for $t \geq t_*/2$,

$$P^z(\tau^Z > t) \leq P^{(\mu+1)}(\tau^Z > t) \leq C_{16} e^{-2C_{17} t}.$$

(iii) $\underline{z > \mu + 1}$. We define

$$\kappa := \inf\{t \geq 0 : Z_t = \mu + 1\}.$$

Thus, on the event $\{t \leq \kappa\}$, we have $Z_t \geq \mu + 1$. Let $\widetilde{Z}$ be a Brownian motion with a drift driven by the SDE:

$$d\widetilde{Z}_t = -\theta dt + \sigma dB_t,$$

where it starts at z. Analogously, define

$$\tilde{\kappa} := \inf\{t \geq 0 : \widetilde{Z}_t = \mu + 1\}.$$

By a similar argument as that in the proof of Lemma 1, we have $Z_{t\wedge\kappa} \leq \widetilde{Z}_{t\wedge\kappa}$ for $t \geq 0$. Thus, $\kappa \leq \tilde{\kappa}$. Then, we have

$$P^z(\kappa > t) \leq P^z(\tilde{\kappa} > t) = \int_t^\infty \frac{z - (\mu+1)}{\sigma} \frac{\exp\left(-\frac{(z-(\mu+1)-\theta s)^2}{2\sigma^2 s}\right)}{\sqrt{2\pi s^3}} \, ds,$$

where the last equality follows by the standard result of the first hitting time for Brownian motion with a drift (cf. Section 5.6 of [29]). A routine calculation gives

$$\begin{aligned} &\int_t^\infty \frac{z - (\mu+1)}{\sigma} \frac{\exp\left(-\frac{(z-(\mu+1)-\theta s)^2}{2\sigma^2 s}\right)}{\sqrt{2\pi s^3}} \, ds \\ \leq\ & \frac{1}{\sqrt{2\pi}} \frac{z-(\mu+1)}{\sigma} e^{\frac{\theta(z-(\mu+1))}{\sigma^2}} \int_t^\infty \frac{e^{-\frac{\theta^2}{2\sigma^2} s}}{\sqrt{s^3}} \, ds \\ \leq\ & \frac{1}{\sqrt{2\pi}} \frac{z}{\sigma} e^{\frac{\theta z}{\sigma^2}} \int_t^\infty \frac{e^{-\frac{\theta^2}{2\sigma^2} s}}{\sqrt{s^3}} \, ds \leq \frac{1}{\sqrt{2\pi}} \frac{z}{\sigma} e^{\frac{\theta z}{\sigma^2}} \frac{1}{\sqrt{t^3}} \frac{2\sigma^2}{\theta^2} e^{-\frac{\theta^2}{2\sigma^2} t} \end{aligned}$$

$$= \sqrt{\frac{2}{\pi}}\frac{z\sigma}{\theta^2}e^{z\theta/\sigma^2}\cdot\frac{1}{\sqrt{t^3}}e^{-\frac{\theta^2}{2\sigma^2}t}.$$

Then, together with the strong Markov property, we have for $t \geq t_*$,

$$\begin{aligned} P^z(\tau^Z > t) &= P^z(\kappa + \tau^Z \circ \theta(\kappa) > t) \\ &\leq P^z(\kappa > t/2) + P^z(\tau^Z \circ \theta(\kappa) > t/2) \\ &\leq \frac{4}{\sqrt{\pi}}\frac{z\sigma}{\theta^2}e^{z\theta/\sigma^2}\cdot\frac{1}{\sqrt{t^3}}e^{-\frac{\theta^2}{4\sigma^2}t} + E^z\left[E^{Z_\kappa}\left[\mathbb{1}_{\{\tau^Z > t/2\}}\right]\right] \\ &\leq \frac{4}{\sqrt{\pi}}\frac{z\sigma}{\theta^2}e^{z\theta/\sigma^2}\cdot\frac{1}{\sqrt{t^3}}e^{-\frac{\theta^2}{4\sigma^2}t} + C_{16}e^{-C_{17}t}, \end{aligned}$$

where in the last inequality, we have invoked the result from case (i).

Combining the above three cases gives that for $z \geq 0$ and $t \geq t_*$,

$$P^z(\tau^Z > t) \leq C_{16}e^{-C_{17}t} + \frac{4}{\sqrt{\pi}}\frac{z\sigma}{\theta^2}e^{z\theta/\sigma^2}\cdot\frac{1}{\sqrt{t^3}}e^{-\frac{\theta^2}{4\sigma^2}t}.$$

What remains is the case that $\mu < 0$. Indeed, the desired result follows from the fact that Z_t is bounded by the 1-d OU process with $\mu = 0$. We conclude the proof. □

3.2. **Exponential rate of convergence in the total variation distance.** This subsection is devoted to the rate of convergence in the total variation distance. It begins by studying the total variation distance between two transition probability measures, followed by the total variation distance between the transition probability measure and the invariant measure.

Recall that $P_t\left((x,y),\cdot\right)$ is the transition probability measure for the process (X_t, Y_t) when it starts from (x,y) and π represents the stationary distribution.

Theorem 14. *Fix $(x_1,y_1),(x_2,y_2) \in \mathbb{R}_+^2$. For any $t_* > 0$, there exist constants C_{18}, C_{19}, depending on $\theta_1, \mu_1, \sigma_1$ and t_*, and C_{20}, C_{21}, depending on $\theta_2, \mu_2, \sigma_2$ and t_*, such that for $t \geq t_*$,*

$$\begin{aligned} & d_{TV}\left(P_t\left((x_1,y_1),\cdot\right), P_t\left((x_2,y_2),\cdot\right)\right) \\ \leq\ & C_{18}e^{-C_{19}t} + C_{20}e^{-C_{21}t} + \frac{8}{\sqrt{\pi}}\frac{\sigma_1\max(x_1,x_2)}{\theta_1^2}e^{\frac{\theta_1\max(x_1,x_2)}{\sigma_1^2}}\cdot\frac{1}{\sqrt{t^3}}e^{-\frac{\theta_1^2}{4\sigma_1}t} \\ & + \frac{8}{\sqrt{\pi}}\frac{\sigma_2\max(y_1,y_2)}{\theta_2^2}e^{\frac{\theta_2\max(y_1,y_2)}{\sigma_2^2}}\cdot\frac{1}{\sqrt{t^3}}e^{-\frac{\theta_2^2}{4\sigma_2}t}. \end{aligned}$$

Proof. For $i = 1,2$, we use $(X_t^{(i)}, Y_t^{(i)})$ to denote the 2-d ROU process satisfying the SDEs:

$$\begin{aligned} dX_t^{(i)} &= \theta_1(\mu_1 - X_t^{(i)})dt + \sigma_1 dW_t + dL_t^{X^{(i)}} \\ dY_t^{(i)} &= \theta_2(\mu_2 - Y_t^{(i)})dt + \sigma_2 dB_t + dL_t^{Y^{(i)}} \end{aligned}$$

and starting from (x_i, y_i). Define for $i = 1,2$,

$$\begin{aligned} \tau^{X^{(i)}} &:= \inf\{t \geq 0 : X_t^{(i)} = 0\}, \\ \tau^{Y^{(i)}} &:= \inf\{t \geq 0 : Y_t^{(i)} = 0\}. \end{aligned}$$

Further, we define

$$\tau^X_{\max} := \tau^{X^{(1)}} \vee \tau^{X^{(2)}},$$
$$\tau^Y_{\max} := \tau^{Y^{(1)}} \vee \tau^{Y^{(2)}}.$$

We first claim that on the event $\{t \geq \tau^X_{\max}\}$, $X^{(1)}_t = X^{(2)}_t$. This will be demonstrated subsequently. Without loss of generality, we assume that $x_1 \leq x_2$. Applying Lemma 2, we have $X^{(1)}_t \leq X^{(2)}_t$ for $t \geq 0$. Thus, $\tau^{X^{(1)}} \leq \tau^{X^{(2)}}$ and $\tau^X_{\max} = \tau^{X^{(2)}}$. Since $X^{(1)}_t$ is non-negative, then we have

$$0 \leq X^{(1)}_{\tau^{X^{(2)}}} \leq X^{(2)}_{\tau^{X^{(2)}}} = 0,$$

which implies $X^{(1)}_{\tau^{X^{(2)}}} = X^{(2)}_{\tau^{X^{(2)}}} = 0$. By the strong Markov property, we have $X^{(1)}_t = X^{(2)}_t$ for $t \geq \tau^{X^{(2)}}$.

Similarly, on the event $\{t \geq \tau^Y_{\max}\}$, we have $Y^{(1)}_t = Y^{(2)}_t$.

With the above preparation, we proceed to study the total variation distance between $P_t\left((x_1, y_1), \cdot\right)$ and $P_t\left((x_2, y_2), \cdot\right)$. It follows from the definition of the total variation distance that

$$\begin{aligned}
& d_{TV}\left(P_t\left((x_1, y_1), \cdot\right), P_t\left((x_2, y_2), \cdot\right)\right) \\
= & \sup_{A \in \mathcal{B}(\mathbb{R}^2_+)} \left|P_t\left((x_1, y_1), A\right) - P_t\left((x_2, y_2), A\right)\right| \\
= & \sup_{A \in \mathcal{B}(\mathbb{R}^2_+)} \left|E\left[\mathbb{1}_A(X^{(1)}_t, Y^{(1)}_t)\right] - E\left[\mathbb{1}_A(X^{(2)}_t, Y^{(2)}_t)\right]\right| \\
\leq & \sup_{A \in \mathcal{B}(\mathbb{R}^2_+)} E\left[\left|\mathbb{1}_A(X^{(1)}_t, Y^{(1)}_t) - \mathbb{1}_A(X^{(2)}_t, Y^{(2)}_t)\right|\right].
\end{aligned}$$

Note that on the event $\{t \geq \tau^X_{\max} \vee \tau^Y_{\max}\}$, $(X^{(1)}_t, Y^{(1)}_t) = (X^{(2)}_t, Y^{(2)}_t)$. Thus,

$$\begin{aligned}
& \left|\mathbb{1}_A(X^{(1)}_t, Y^{(1)}_t) - \mathbb{1}_A(X^{(2)}_t, Y^{(2)}_t)\right| \\
= & \left|\mathbb{1}_A(X^{(1)}_t, Y^{(1)}_t) - \mathbb{1}_A(X^{(2)}_t, Y^{(2)}_t)\right| \cdot \mathbb{1}_{\{t < \tau^X_{\max} \vee \tau^Y_{\max}\}} \\
\leq & 2 \cdot \mathbb{1}_{\{t < \tau^X_{\max} \vee \tau^Y_{\max}\}} \\
\leq & 2 \cdot \mathbb{1}_{\{t < \tau^X_{\max}\}} + 2 \cdot \mathbb{1}_{\{t < \tau^Y_{\max}\}}.
\end{aligned}$$

Taking expectation on both sides of the last display and then taking supremum over A from $\mathcal{B}(\mathbb{R}^2_+)$, we have

$$d_{TV}\left(P_t\left((x_1, y_1), \cdot\right), P_t\left((x_2, y_2), \cdot\right)\right) \leq 2P(\tau^X_{\max} > t) + 2P(\tau^Y_{\max} > t). \tag{3.1}$$

What remains is to evaluate $P(\tau^X_{\max} > t)$ and $P(\tau^Y_{\max} > t)$. Without loss of generality, we assume $x_1 \leq x_2$, then $\tau^X_{\max} = \tau^{X^{(2)}}$ and

$$P(\tau^X_{\max} > t) = P(\tau^{X^{(2)}} > t). \tag{3.2}$$

On the event $\{t \leq \tau^{X^{(2)}}\}$, $X^{(2)}$ behaves like a 1-d OU process, hence, $\tau^{X^{(2)}}$ is the first hitting time to 0 for a 1-d OU process. By Lemma 13, for any t_*, there are constants C_{18} ans C_{19} depending on $\theta_1, \mu_1, \sigma_1$, and t_*, such that for $t \geq t_*$,

$$P(\tau^{X^{(2)}} > t) \leq \frac{1}{2} C_{18} e^{-C_{19} t} + \frac{4}{\sqrt{\pi}} \frac{\sigma_1 x_2}{\theta_1^2} e^{\frac{\theta_1 x_2}{\sigma_1^2}} \cdot \frac{1}{\sqrt{t^3}} e^{-\frac{\theta_1^2}{4\sigma_1} t}$$

$$= \frac{1}{2} C_{18} e^{-C_{19} t} + \frac{4}{\sqrt{\pi}} \frac{\sigma_1 \max(x_1, x_2)}{\theta_1^2} e^{\frac{\theta_1 \max(x_1,x_2)}{\sigma_1^2}} \cdot \frac{1}{\sqrt{t^3}} e^{-\frac{\theta_1^2}{4\sigma_1} t}.$$

Together with (3.2), we have

$$P(\tau^X_{\max} > t) = \frac{1}{2} C_{18} e^{-C_{19} t} + \frac{4}{\sqrt{\pi}} \frac{\sigma_1 \max(x_1, x_2)}{\theta_1^2} e^{\frac{\theta_1 \max(x_1,x_2)}{\sigma_1^2}} \cdot \frac{1}{\sqrt{t^3}} e^{-\frac{\theta_1^2}{4\sigma_1} t}. \tag{3.3}$$

Similarly, there exist constants C_{20} and C_{21} depending on $\theta_2, \mu_2, \sigma_2$ and t_*, such that for $t \geq t_*$,

$$P(\tau^Y_{\max} > t) = \frac{1}{2} C_{20} e^{-C_{21} t} + \frac{4}{\sqrt{\pi}} \frac{\sigma_2 \max(y_1, y_2)}{\theta_2^2} e^{\frac{\theta_2 \max(y_1,y_2)}{\sigma_2^2}} \cdot \frac{1}{\sqrt{t^3}} e^{-\frac{\theta_2^2}{4\sigma_2} t}. \tag{3.4}$$

The desired result follows from combining (3.1), (3.3), and (3.4). □

With Theorem 14 in hand, we are ready to study the rate of convergence to the stationary distribution in the total variation distance.

Theorem 15. *Fix $(x,y) \in \mathbb{R}_+^2$. For any $t_* > 0$, there exist constants C_{22}, depending on $\theta_1, \theta_2, \mu_1, \mu_2$,*
$\sigma_1, \sigma_2, t_*, x$ and y, and C_{23}, depending on $\theta_1, \theta_2, \mu_1, \mu_2, \sigma_1, \sigma_2$ and t_*, such that for $t \geq t_*$,

$$d_{TV}\left(P_t\left((x,y),\cdot\right),\pi\right) \leq C_{22} e^{-C_{23} t}.$$

Proof. It can be easily verified that

$$d_{TV}\left(P_t\left((x,y),\cdot\right),\pi\right) \leq \int_{\mathbb{R}_+^2} d_{TV}\left(P_t\left((x,y),\cdot\right), P_t\left((\tilde{x},\tilde{y}),\cdot\right)\right) \pi(d\tilde{x}, d\tilde{y}).$$

Plugging the result of Theorem 14 into the above display, after arrangement, we have for $t \geq t_*$,

$$\begin{aligned} & d_{TV}\left(P_t\left((x,y),\cdot\right),\pi\right) \\ \leq\ & C_{18} e^{-C_{19} t} + C_{20} e^{-C_{21} t} \\ & + \frac{8}{\sqrt{\pi\, t^3}} e^{-\frac{\theta_1^2}{4\sigma_1} t} \int_{\mathbb{R}_+^2} \frac{\sigma_1 \max(x,\tilde{x})}{\theta_1^2} e^{\frac{\theta_1 \max(x,\tilde{x})}{\sigma_1^2}} \pi(d\tilde{x}, d\tilde{y}) \\ & + \frac{8}{\sqrt{\pi\, t^3}} e^{-\frac{\theta_2^2}{4\sigma_2} t} \int_{\mathbb{R}_+^2} \frac{\sigma_2 \max(y,\tilde{y})}{\theta_2^2} e^{\frac{\theta_2 \max(y,\tilde{y})}{\sigma_2^2}} \pi(d\tilde{x}, d\tilde{y}). \end{aligned}$$

Note that the marginal measures of π are stationary distributions for 1-d ROU processes, hence, they have truncated Gaussian distributions (cf. Proposition 1 in [39]). Consequently,

$$\int_{\mathbb{R}_+^2} \frac{\sigma_1 \max(x,\tilde{x})}{\theta_1^2} e^{\frac{\theta_1 \max(x,\tilde{x})}{\sigma_1^2}} \pi(d\tilde{x}, d\tilde{y}) < \infty,$$
$$\int_{\mathbb{R}_+^2} \frac{\sigma_2 \max(y,\tilde{y})}{\theta_2^2} e^{\frac{\theta_2 \max(y,\tilde{y})}{\sigma_2^2}} \pi(d\tilde{x}, d\tilde{y}) < \infty.$$

Then the desired result follows. □

3.3. **Exponential rate of convergence in Wasserstein distance.** This subsection aims to demonstrate that the rate of convergence in Wasserstein distance is exponential, with the exponent being explicit. Again, we begin by examining the Wasserstein distance between two transition probability measures, and then turn to the Wasserstein distance between any transition probability measure and the stationary distribution.

Theorem 16. *For* $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2_+$*, we have for* $t \geq 0$*,*

$$d_W\left(P_t\left((x_1,y_1),\cdot\right), P_t\left((x_2,y_2),\cdot\right)\right) \leq |x_1 - x_2|e^{-\theta_1 t} + |y_1 - y_2|e^{-\theta_2 t}.$$

Proof. We adopt the notation in the proof of Theorem 14. That is, $(X^{(i)}_t, Y^{(i)}_t)$ is the 2-d ROU process starting from (x_i, y_i) and $\tau^{X^{(i)}}$ and $\tau^{Y^{(i)}}$ are the first hitting times to 0.

By the definition of Wasserstein distance,

$$\begin{aligned}
& d_W\left(P_t\left((x_1,y_1),\cdot\right), P_t\left((x_2,y_2),\cdot\right)\right) \\
= & \sup_{f\in \mathrm{Lip}(1)} \left| \int f(\tilde{x},\tilde{y})\, P_t\left((x_1,y_1),(d\tilde{x},d\tilde{y})\right) - \int f(\tilde{x},\tilde{y})\, P_t\left((x_2,y_2),(d\tilde{x},d\tilde{y})\right) \right| \\
= & \sup_{f\in \mathrm{Lip}(1)} \left| E\left[f(X^{(1)}_t, Y^{(1)}_t)\right] - E\left[f(X^{(2)}_t, Y^{(2)}_t)\right] \right| \\
\leq & \sup_{f\in \mathrm{Lip}(1)} E\left[\left| f(X^{(1)}_t, Y^{(1)}_t) - f(X^{(2)}_t, Y^{(2)}_t)\right|\right].
\end{aligned}$$

Note that f is a Lipschitz function with Lipschitz constant ≤ 1. Thus,

$$\begin{aligned}
\left| f(X^{(1)}_t, Y^{(1)}_t) - f(X^{(2)}_t, Y^{(2)}_t)\right| &\leq \sqrt{\left(X^{(1)}_t - X^{(2)}_t\right)^2 + \left(Y^{(1)}_t - Y^{(2)}_t\right)^2} \\
&\leq \left|X^{(1)}_t - X^{(2)}_t\right| + \left|Y^{(1)}_t - Y^{(2)}_t\right|.
\end{aligned}$$

Taking expectation on both sides of the last display and then taking supremum over f in Lip(1), we have

$$\begin{aligned}
& d_W\left(P_t\left((x_1,y_1),\cdot\right), P_t\left((x_2,y_2),\cdot\right)\right) \\
\leq\ & E\left[\left|X^{(1)}_t - X^{(2)}_t\right|\right] + E\left[\left|Y^{(1)}_t - Y^{(2)}_t\right|\right].
\end{aligned} \tag{3.5}$$

In what follows, we will evaluate the expectations of $|X^{(1)}_t - X^{(2)}_t|$ and $|Y^{(1)}_t - Y^{(2)}_t|$. We begin with the expectation of $|X^{(1)}_t - X^{(2)}_t|$. Without loss of generality, we assume that $x_1 \leq x_2$. Let $\widetilde{X}^{(1)}_t$ be a 1-d OU process satisfying the SDE:

$$d\widetilde{X}^{(1)}_t = \theta_1(\mu_1 - \widetilde{X}^{(1)}_t)dt + \sigma_1 dW_t,$$

starting from x_1. Then, $\widetilde{X}^{(1)}_t$ has the representation

$$\widetilde{X}^{(1)}_t = x_1 e^{-\theta_1 t} + \mu_1\left(1 - e^{-\theta_1 t}\right) + \sigma_1 \int_0^t e^{-\theta_1(t-s)}\, dW_s. \tag{3.6}$$

Using Lemma 2 and Remark 3, we have for $t \geq 0$,

$$\widetilde{X}^{(1)}_t \leq X^{(1)}_t \leq X^{(2)}_t.$$

By a similar argument in the proof of Theorem 14, we conclude $X_t^{(1)} = X_t^{(2)}$ on the event $\{t \geq \tau^{X^{(2)}}\}$. Thus, combining these two results yields

$$\begin{aligned}\left|X_t^{(1)} - X_t^{(2)}\right| &= \left|X_t^{(1)} - X_t^{(2)}\right| \cdot \mathbb{1}_{\{t<\tau^{X^{(2)}}\}} = \left(X_t^{(2)} - X_t^{(1)}\right) \cdot \mathbb{1}_{\{t<\tau^{X^{(2)}}\}} \\ &\leq \left(X_t^{(2)} - \widetilde{X}_t^{(1)}\right) \cdot \mathbb{1}_{\{t<\tau^{X^{(2)}}\}}.\end{aligned} \tag{3.7}$$

On the event $\{t < \tau^{X^{(2)}}\}$, $X_t^{(2)}$ behaves like a 1-d OU process, and hence, has the representation

$$X_t^{(2)} = x_2 e^{-\theta_1 t} + \mu_1 \left(1 - e^{-\theta_1 t}\right) + \sigma_1 \int_0^t e^{-\theta_1 (t-s)}\, dW_s. \tag{3.8}$$

Then, plugging (3.6) and (3.8) into (3.7) gives

$$\left|X_t^{(1)} - X_t^{(2)}\right| \leq (x_2 - x_1) e^{-\theta_1 t} \cdot \mathbb{1}_{\{t<\tau^{X^{(2)}}\}} \leq (x_2 - x_1) e^{-\theta_1 t} = |x_2 - x_1| e^{-\theta_1 t}.$$

Thus,

$$E\left[\left|X_t^{(1)} - X_t^{(2)}\right|\right] \leq |x_2 - x_1| e^{-\theta_1 t}. \tag{3.9}$$

Similarly,

$$E\left[\left|Y_t^{(1)} - Y_t^{(2)}\right|\right] \leq |y_2 - y_1| e^{-\theta_2 t}. \tag{3.10}$$

Combining (3.5), (3.9) and (3.10), the desired result follows. ☐

Theorem 17. *For $(x, y) \in \mathbb{R}_+^2$, there exist constants C_{24}, depending on $\theta_1, \mu_1, \sigma_1$ and x, and C_{25}, depending on $\theta_2, \mu_2, \sigma_2$ and y, such that for $t \geq 0$,*

$$d_W\left(P_t\left((x, y), \cdot\right), \pi\right) \leq C_{24} e^{-\theta_1 t} + C_{25} e^{-\theta_2 t}.$$

Proof. It can be easily verified that

$$d_W\left(P_t\left((x, y), \cdot\right), \pi\right) \leq \int_{\mathbb{R}_+^2} d_W\left(P_t\left((x, y), \cdot\right), P_t\left((\tilde{x}, \tilde{y}), \cdot\right)\right) \pi(d\tilde{x}, d\tilde{y}).$$

Together with Theorem 16, we have

$$\begin{aligned} & d_W\left(P_t\left((x, y), \cdot\right), \pi\right) \\ \leq\ & \int_{\mathbb{R}_+^2} \left(|x - \tilde{x}| e^{-\theta_1 t} + |y - \tilde{y}| e^{-\theta_2 t}\right) \pi(d\tilde{x}, d\tilde{y}) \\ =\ & e^{-\theta_1 t} \int_{\mathbb{R}_+^2} |x - \tilde{x}|\, \pi(d\tilde{x}, d\tilde{y}) + e^{-\theta_2 t} \int_{\mathbb{R}_+^2} |y - \tilde{y}|\, \pi(d\tilde{x}, d\tilde{y}). \end{aligned}$$

Since the marginal measure $\int_{\mathbb{R}_+} \pi(\cdot, dy)$ has a truncated Gaussian distribution,

$$\int_{\mathbb{R}_+^2} |x - \tilde{x}|\, \pi(d\tilde{x}, d\tilde{y}) < \infty,$$

and its value depends on $\theta_1, \mu_1, \sigma_1$ and x. Similarly,

$$\int_{\mathbb{R}_+^2} |y - \tilde{y}|\, \pi(d\tilde{x}, d\tilde{y}) < \infty,$$

and its value depends on $\theta_2, \mu_2, \sigma_2$ and y. We conclude the proof. ☐

4. A numerical scheme for the stationary distribution

The purpose of this section is to provide a numerical scheme to approximate the stationary distribution for the 2-d ROU process. For diffusions, two common approaches for investigating the stationary distribution are the PDE method and the Monte-Carlo method. We begin by presenting the PDE that the density of the stationary distribution satisfies, along with the corresponding boundary conditions, provided the existence of the density. However, due to its complexity, we are unable to solve this PDE ourselves and defer it to the experts in PDE. Therefore, we turn to the Monte-Carlo method, which is partly inspired by [6]. Additionally, the Monte-Carlo method has an advantage in higher dimensions. The approach we present later can be naturally extended to high-dimensional ROU processes with normal reflections in each coordinate.

4.1. **Stationary distribution: a PDE perspective.** Assume that the stationary distribution, denoted by π, is absolutely continuous with density function $p(x, y)$. Indeed, we believe that the assumption can be justified by modifying the argument in Section 7 of [19]. However, as this is beyond the scope of the paper, we omit the detailed verification. In the following, we will characterize $p(x, y)$ by introducing the corresponding PDE.

For any smooth function f on $[0, \infty) \times [0, \infty)$ with compact support on $(0, \infty) \times (0, \infty)$. An application of Itô's lemma gives

$$\begin{aligned} f(X_t, Y_t) &= f(X_0, Y_0) + \int_0^t \mathcal{A}f(X_s, Y_s)\, ds + \int_0^t \frac{\partial f}{\partial x}(X_s, Y_s)\, dW_s \\ &\quad + \int_0^t \frac{\partial f}{\partial y}(X_s, Y_s)\, dB_s + \int_0^t \frac{\partial f}{\partial x}(X_s, Y_s)\, dL_s^X + \int_0^t \frac{\partial f}{\partial y}(X_s, Y_s)\, dL_s^Y, \end{aligned}$$

where $\mathcal{A}f$ is defined as

$$\mathcal{A}f := \frac{1}{2}\sigma_1^2 \frac{\partial^2 f}{\partial x^2} + \frac{1}{2}\sigma_2^2 \frac{\partial^2 f}{\partial y^2} + \rho\sigma_1\sigma_2 \frac{\partial^2 f}{\partial x \partial y} + \theta_1(\mu_1 - x)\frac{\partial f}{\partial x} + \theta_2(\mu_2 - y)\frac{\partial f}{\partial y}.$$

Noting that $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ are bounded and $\frac{\partial f}{\partial x}(0, y) = \frac{\partial f}{\partial y}(x, 0) = 0$, it follows by taking expectation that

$$E_\pi\left[f(X_t, Y_t)\right] = E_\pi\left[f(X_0, Y_0)\right] + E_\pi\left[\int_0^t \mathcal{A}f(X_s, Y_s)\, ds\right].$$

Since π is the stationary distribution for (X_t, Y_t), we have $E_\pi\left[f(X_t, Y_t)\right] = E_\pi\left[f(X_0, Y_0)\right]$. Then

$$E_\pi\left[\int_0^t \mathcal{A}f(X_s, Y_s)\, ds\right] = 0.$$

Hence,

$$E_\pi\left[\mathcal{A}f(X_t, Y_t)\right] = 0.$$

Since we assume that $p(x, y)$ is the density function of π, we have

$$\int_0^\infty \int_0^\infty \mathcal{A}f(x, y) p(x, y)\, dx dy = 0.$$

Using integration by parts yields

$$\int_0^\infty \int_0^\infty f\left(\frac{1}{2}\sigma_1^2 \frac{\partial^2 p}{\partial x^2} + \frac{1}{2}\sigma_2^2 \frac{\partial^2 p}{\partial y^2} + \rho\sigma_1\sigma_2 \frac{\partial^2 p}{\partial x \partial y}\right.$$

$$-\theta_1(\mu_1 - x)\frac{\partial p}{\partial x} - \theta_2(\mu_2 - y)\frac{\partial p}{\partial y} + (\theta_1 + \theta_2)p\bigg)\, dxdy = 0.$$

Here $\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial^2 p}{\partial x^2}, \ldots$ can be understood as weak derivatives. By the arbitrariness of f, we conclude

$$\frac{1}{2}\sigma_1^2\frac{\partial^2 p}{\partial x^2} + \frac{1}{2}\sigma_2^2\frac{\partial^2 p}{\partial y^2} + \rho\sigma_1\sigma_2\frac{\partial^2 p}{\partial x \partial y} - \theta_1(\mu_1 - x)\frac{\partial p}{\partial x} - \theta_2(\mu_2 - y)\frac{\partial p}{\partial y} + (\theta_1 + \theta_2)p = 0.$$

We now turn to the boundary conditions for $p(x, y)$. Since X_t and Y_t are 1-d ROU processes, the marginal distributions of π correspond to the stationary distributions of X_t and Y_t, that is,

$$\int_0^\infty p(x, y)\, dy = p_1(x) := \sqrt{\frac{2\theta_1}{\sigma_1^2}}\frac{\phi\left(\sqrt{2\theta_1/\sigma_1^2}(x - \mu_1)\right)}{1 - \Phi\left(-\sqrt{2\theta_1\mu_1^2/\sigma_1^2}\right)},$$

$$\int_0^\infty p(x, y)\, dx = p_2(y) := \sqrt{\frac{2\theta_2}{\sigma_2^2}}\frac{\phi\left(\sqrt{2\theta_2/\sigma_2^2}(y - \mu_2)\right)}{1 - \Phi\left(-\sqrt{2\theta_2\mu_2^2/\sigma_2^2}\right)},$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ are the density and distribution functions of a standard normal random variable. The explicit representations of $p_1(x)$ and $p_2(y)$ come from [39].

In summary, $p(x, y)$ satisfies the following PDE

$$\frac{1}{2}\sigma_1^2\frac{\partial^2 p}{\partial x^2} + \frac{1}{2}\sigma_2^2\frac{\partial^2 p}{\partial y^2} + \rho\sigma_1\sigma_2\frac{\partial^2 p}{\partial x \partial y} - \theta_1(\mu_1 - x)\frac{\partial p}{\partial x} - \theta_2(\mu_2 - y)\frac{\partial p}{\partial y} + (\theta_1 + \theta_2)p = 0$$

subject to the boundary conditions

$$\int_0^\infty p(x, y)\, dy = p_1(x),$$

$$\int_0^\infty p(x, y)\, dx = p_2(y).$$

We are unable to solve the above PDE, as its boundary conditions do not fall into standard categories such as Dirichlet, Neumann, or Robin. So we entrust the task to experts in PDE.

4.2. **Numerical scheme and its convergence.** In this subsection, we will develop a convergent numerical procedure to approximate the stationary distribution of our 2-d ROU process. Consistently, the stationary distribution is denoted by π. We begin by constructing a discrete-time process that serves as an approximation to the ROU process.

Let $\{\lambda_k\}_{k=1}^\infty$ be a sequence of real numbers satisfying the following conditions:

(a) $0 < \lambda_k < 1/\max\{\theta_1, \theta_2\}$ for all $k \in \mathbb{N}_+$;

(b) $\lambda_k \to 0$ as $k \to \infty$;

(c) $\Lambda_n := \sum_{k=1}^n \lambda_k \to \infty$ as $n \to \infty$.

Here, $\mathbb{N}_+$ is the set of all positive integers. Note that the above conditions are satisfied if $\lambda_k = 1/(\max\{\theta_1, \theta_2\}(k + 1)^\nu)$ with $\nu \in (0, 1]$. Let $\{(G_{k,1}, G_{k,2})\}_{k=1}^\infty$ be a sequence of i.i.d random vectors in $\mathbb{R}^2$ with mean $\mathbf{0}$ and covariance matrix $\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$, with $\rho \in (-1, 1)$. Furthermore, we impose the following assumption on $G_{k,i}$'s:

$$E\left[e^{\lambda G_{k,i}}\right] \leq e^{\beta\lambda^2}, \tag{4.1}$$

for some $\beta \in (0,\infty)$ and for all $k \in \mathbb{N}_+$, $i = 1,2$, and $\lambda \in \mathbb{R}$. The above assumption is satisfied when $(G_{k,1}, G_{k,2})$ has a joint Gaussian distribution. As usual, we define $\mathcal{G}_k = \sigma(G_{1,1}, G_{1,2}, \dots, G_{k,1}, G_{k,2})$ for $k \in \mathbb{N}_+$ and $\mathcal{G}_0 = \{\emptyset, \Omega\}$.

With the above preparation, we are ready to iteratively construct a discrete-time process in $\mathbb{R}^2_+$. For $(x_0, y_0) \in \mathbb{R}^2_+$, define

$$\begin{cases} U_0 = x_0, \\ V_0 = y_0, \\ U_{k+1} = \left[U_k + \theta_1(\mu_1 - U_k)\lambda_{k+1} + \sigma_1\sqrt{\lambda_{k+1}}G_{k+1,1}\right]^+, \\ V_{k+1} = \left[V_k + \theta_2(\mu_2 - V_k)\lambda_{k+1} + \sigma_2\sqrt{\lambda_{k+1}}G_{k+1,2}\right]^+, \end{cases}$$

where $[z]^+ := \max\{z, 0\}$. Obviously, $\{(U_k, V_k)\}_{k=0}^\infty$ is a sequence of random vectors with values in $\mathbb{R}^2_+$. Employing these random vectors, we construct a sequence of random measures on $\mathbb{R}^2_+$ that approximate the stationary distribution π as follows:

$$\pi_n = \frac{1}{\Lambda_n}\sum_{k=1}^n \lambda_k\, \delta_{(U_{k-1}, V_{k-1})}, \qquad \text{for } n \in \mathbb{N}_+,$$

where $\delta_{(x,y)}$ is the Dirac measure on $\mathbb{R}^2$. In particular, these random measures yield an approximation for any integral of the form $\int_{\mathbb{R}^2_+} f(x,y)\, d\pi(x,y)$ through the corresponding weighted averages:

$$\frac{1}{\Lambda_n}\sum_{k=1}^n \lambda_k\, f(U_{k-1}, V_{k-1}).$$

With these random measures in hand, we are now prepared to introduce the main result of this section.

Theorem 18. *As $n \to \infty$, π_n converges weakly to π, almost surely.*

The proof of Theorem 18 is relegated to the following two subsections.

4.3. **Tightness.** The aim of this subsection is to establish the tightness of $\{\pi_n\}_{n=1}^\infty$ almost surely. To this end, we prepare the following lemmas.

Lemma 19. *For non-negative integers n, l, we have*

$$\begin{aligned} U_{n+l} \;\le\; & U_n \prod_{j=n+1}^{n+l}(1-\theta_1\lambda_j) + \mu_1^+ \\ & + \sigma_1 \max_{0\le i\le l} \sum_{m=1}^{i} \prod_{j=n+l+2-m}^{n+l} (1-\theta_1\lambda_j)\sqrt{\lambda_{n+l+1-m}}\, G_{n+l+1-m,1}\,, \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} V_{n+l} \;\le\; & V_n \prod_{j=n+1}^{n+l}(1-\theta_2\lambda_j) + \mu_2^+ \\ & + \sigma_2 \max_{0\le i\le l} \sum_{m=1}^{i} \prod_{j=n+l+2-m}^{n+l} (1-\theta_2\lambda_j)\sqrt{\lambda_{n+l+1-m}}\, G_{n+l+1-m,2}\,. \end{aligned} \tag{4.3}$$

Proof. We only prove (4.2) and (4.3) follows similarly.

Define

$$L_{k+1} = -\min\left\{U_k + \theta_1(\mu_1 - U_k)\lambda_{k+1} + \sigma_1\sqrt{\lambda_{k+1}}G_{k+1,1}, 0\right\}.$$

Then

$$\begin{aligned} U_{k+1} &= [U_k + \theta_1(\mu_1 - U_k)\lambda_{k+1} + \sigma_1\sqrt{\lambda_{k+1}}G_{k+1,1}]^+ \\ &= U_k + \theta_1(\mu_1 - U_k)\lambda_{k+1} + \sigma_1\sqrt{\lambda_{k+1}}G_{k+1,1} + L_{k+1} \\ &= (1-\theta_1\lambda_{k+1})U_k + \theta_1\mu_1\lambda_{k+1} + \sigma_1\sqrt{\lambda_{k+1}}G_{k+1,1} + L_{k+1}. \end{aligned}$$

Dividing by $\prod_{j=1}^{k+1}(1-\theta_1\lambda_j)$ on both sides yields

$$\begin{aligned} \frac{U_{k+1}}{\prod_{j=1}^{k+1}(1-\theta_1\lambda_j)} &= \frac{U_k}{\prod_{j=1}^{k}(1-\theta_1\lambda_j)} + \frac{\theta_1\mu_1\lambda_{k+1}}{\prod_{j=1}^{k+1}(1-\theta_1\lambda_j)} \\ &\quad + \frac{\sigma_1\sqrt{\lambda_{k+1}}G_{k+1,1}}{\prod_{j=1}^{k+1}(1-\theta_1\lambda_j)} + \frac{L_{k+1}}{\prod_{j=1}^{k+1}(1-\theta_1\lambda_j)}. \end{aligned}$$

In the last display, by letting $k = n, n+1, \ldots, n+l-1$, and summing up these l equations, we have

$$\begin{aligned} \frac{U_{n+l}}{\prod_{j=1}^{n+l}(1-\theta_1\lambda_j)} &= \frac{U_n}{\prod_{j=1}^{n}(1-\theta_1\lambda_j)} + \sum_{k=n}^{n+l-1}\frac{\theta_1\mu_1\lambda_{k+1}}{\prod_{j=1}^{k+1}(1-\theta_1\lambda_j)} \\ &\quad + \sum_{k=n}^{n+l-1}\frac{\sigma_1\sqrt{\lambda_{k+1}}G_{k+1,1}}{\prod_{j=1}^{k+1}(1-\theta_1\lambda_j)} + \sum_{k=n}^{n+l-1}\frac{L_{k+1}}{\prod_{j=1}^{k+1}(1-\theta_1\lambda_j)}. \end{aligned}$$

For simplicity of notation, we define

$$F_{n:l} := \frac{U_n}{\prod_{j=1}^{n}(1-\theta_1\lambda_j)} + \sum_{k=n}^{n+l-1}\frac{\theta_1\mu_1\lambda_{k+1}}{\prod_{j=1}^{k+1}(1-\theta_1\lambda_j)} + \sum_{k=n}^{n+l-1}\frac{\sigma_1\sqrt{\lambda_{k+1}}G_{k+1,1}}{\prod_{j=1}^{k+1}(1-\theta_1\lambda_j)}$$

and

$$\widetilde{L}_{n:l} := \sum_{k=n}^{n+l-1}\frac{L_{k+1}}{\prod_{j=1}^{k+1}(1-\theta_1\lambda_j)}.$$

Thus,

$$\frac{U_{n+l}}{\prod_{j=1}^{n+l}(1-\theta_1\lambda_j)} = F_{n:l} + \widetilde{L}_{n:l}.$$

At this stage, we fix n and allow l to vary. Then $U_{n+l}/\prod_{j=1}^{n+l}(1-\theta_1\lambda_j)$ is non-negative for all $l \in \mathbb{N}_+$. $\widetilde{L}_{n:l}$, viewed as a function of l, is non-decreasing. Furthermore, $\widetilde{L}_{n:l}$ increases at l only if $L_{n+l} > 0$ which occurs only if $U_{n+l} = 0$ according to the definition of L_{n+l}. Combining the above observations, together with the uniqueness of the Skorokhod map, it follows that

$$\widetilde{L}_{n:l} = \max\{0, -F_{n:1}, -F_{n:2}, \ldots, -F_{n:l}\}.$$

Thus,

$$\begin{aligned} \frac{U_{n+l}}{\prod_{j=1}^{n+l}(1-\theta_1\lambda_j)} &= F_{n:l} + \max\{0, -F_{n:1}, -F_{n:2}, \ldots, -F_{n:l}\} \\ &= \max\{F_{n:l}, F_{n:l} - F_{n:1}, F_{n:l} - F_{n:2}, \ldots, F_{n:l} - F_{n:(l-1)}, 0\}. \end{aligned}$$

Note that for $1 \le i \le l-1$,

$$\begin{aligned} F_{n:l} - F_{n:i} &= \sum_{k=n+i}^{n+l-1} \frac{\theta_1 \mu_1 \lambda_{k+1}}{\prod_{j=1}^{k+1}(1-\theta_1\lambda_j)} + \sum_{k=n+i}^{n+l-1} \frac{\sigma_1 \sqrt{\lambda_{k+1}} G_{k+1,1}}{\prod_{j=1}^{k+1}(1-\theta_1\lambda_j)} \\ &\le \sum_{k=n+i}^{n+l-1} \frac{\theta_1 \mu_1^+ \lambda_{k+1}}{\prod_{j=1}^{k+1}(1-\theta_1\lambda_j)} + \sum_{k=n+i}^{n+l-1} \frac{\sigma_1 \sqrt{\lambda_{k+1}} G_{k+1,1}}{\prod_{j=1}^{k+1}(1-\theta_1\lambda_j)} \\ &= \sum_{k=n+i}^{n+l-1} \frac{\theta_1 \mu_1^+ \lambda_{k+1}}{\prod_{j=1}^{k+1}(1-\theta_1\lambda_j)} + \sum_{m=1}^{l-i} \frac{\sigma_1 \sqrt{\lambda_{n+l+1-m}} G_{n+l+1-m,1}}{\prod_{j=1}^{n+l+1-m}(1-\theta_1\lambda_j)}, \end{aligned}$$

where in the last equality we substitute m for $n+l-k$. Furthermore,

$$\begin{aligned} F_{n:l} &\le \frac{U_n}{\prod_{j=1}^{n}(1-\theta_1\lambda_j)} + \sum_{k=n}^{n+l-1} \frac{\theta_1 \mu_1^+ \lambda_{k+1}}{\prod_{j=1}^{k+1}(1-\theta_1\lambda_j)} + \sum_{k=n}^{n+l-1} \frac{\sigma_1 \sqrt{\lambda_{k+1}} G_{k+1,1}}{\prod_{j=1}^{k+1}(1-\theta_1\lambda_j)} \\ &= \frac{U_n}{\prod_{j=1}^{n}(1-\theta_1\lambda_j)} + \sum_{k=n}^{n+l-1} \frac{\theta_1 \mu_1^+ \lambda_{k+1}}{\prod_{j=1}^{k+1}(1-\theta_1\lambda_j)} + \sum_{m=1}^{l} \frac{\sigma_1 \sqrt{\lambda_{n+l+1-m}} G_{n+l+1-m,1}}{\prod_{j=1}^{n+l+1-m}(1-\theta_1\lambda_j)}. \end{aligned}$$

Combining the last three displays, together with the nonnegativity of $\theta_1\mu_1^+\lambda_{k+1}$, we have

$$\begin{aligned} \frac{U_{n+l}}{\prod_{j=1}^{n+l}(1-\theta_1\lambda_j)} \;\; &\le \;\; \frac{U_n}{\prod_{j=1}^{n}(1-\theta_1\lambda_j)} + \sum_{k=n}^{n+l-1} \frac{\theta_1 \mu_1^+ \lambda_{k+1}}{\prod_{j=1}^{k+1}(1-\theta_1\lambda_j)} \\ &\quad + 0 \vee \max_{0\le i\le l-1} \sum_{m=1}^{l-i} \frac{\sigma_1 \sqrt{\lambda_{n+l+1-m}} G_{n+l+1-m,1}}{\prod_{j=1}^{n+l+1-m}(1-\theta_1\lambda_j)} \\ &= \;\; \frac{U_n}{\prod_{j=1}^{n}(1-\theta_1\lambda_j)} + \sum_{k=n}^{n+l-1} \frac{\theta_1 \mu_1^+ \lambda_{k+1}}{\prod_{j=1}^{k+1}(1-\theta_1\lambda_j)} \\ &\quad + 0 \vee \max_{1\le i\le l} \sum_{m=1}^{i} \frac{\sigma_1 \sqrt{\lambda_{n+l+1-m}} G_{n+l+1-m,1}}{\prod_{j=1}^{n+l+1-m}(1-\theta_1\lambda_j)} \\ &= \;\; \frac{U_n}{\prod_{j=1}^{n}(1-\theta_1\lambda_j)} + \sum_{k=n}^{n+l-1} \frac{\theta_1 \mu_1^+ \lambda_{k+1}}{\prod_{j=1}^{k+1}(1-\theta_1\lambda_j)} \\ &\quad + \max_{0\le i\le l} \sum_{m=1}^{i} \frac{\sigma_1 \sqrt{\lambda_{n+l+1-m}} G_{n+l+1-m,1}}{\prod_{j=1}^{n+l+1-m}(1-\theta_1\lambda_j)}, \end{aligned}$$

where in the second last equality we substitute i for $l-i$ and the last equality follows by the convention that $\sum_{m=1}^{0} x_m = 0$. In the last display, multiplying by $\prod_{j=1}^{n+l}(1-\theta_1\lambda_j)$ on both sides, after changing the dummy variables, we have

$$\begin{aligned} U_{n+l} \;\; &\le \;\; U_n \prod_{j=n+1}^{n+l} (1-\theta_1\lambda_j) + \sum_{k=n+1}^{n+l} \theta_1 \mu_1^+ \prod_{j=k+1}^{n+l} (1-\theta_1\lambda_j)\lambda_k \\ &\quad + \max_{0\le i\le l} \sum_{m=1}^{i} \sigma_1 \prod_{j=n+l+2-m}^{n+l} (1-\theta_1\lambda_j) \sqrt{\lambda_{n+l+1-m}} G_{n+l+1-m,1}. \end{aligned}$$

To prove (4.2), it remains to prove

$$\sum_{k=n+1}^{n+l}\prod_{j=k+1}^{n+l}(1-\theta_1\lambda_j)\lambda_k \le \frac{1}{\theta_1}. \tag{4.4}$$

Indeed, noting that $\lambda_n \le 1/\theta_1$ for $\forall n \in \mathbb{N}_+$, it follows that

$$\begin{aligned}
&\sum_{k=n+1}^{n+l}\prod_{j=k+1}^{n+l}(1-\theta_1\lambda_j)\lambda_k \\
=\;& \sum_{k=n+2}^{n+l}\prod_{j=k+1}^{n+l}(1-\theta_1\lambda_j)\lambda_k + \prod_{j=n+2}^{n+l}(1-\theta_1\lambda_j)\lambda_{n+1} \\
\le\;& \sum_{k=n+2}^{n+l}\prod_{j=k+1}^{n+l}(1-\theta_1\lambda_j)\lambda_k + \prod_{j=n+2}^{n+l}(1-\theta_1\lambda_j)\frac{1}{\theta_1} && (4.5)\\
=\;& \sum_{k=n+3}^{n+l}\prod_{j=k+1}^{n+l}(1-\theta_1\lambda_j)\lambda_k + \prod_{j=n+3}^{n+l}(1-\theta_1\lambda_j)\lambda_{n+2} + \prod_{j=n+3}^{n+l}(1-\theta_1\lambda_j)\left(\frac{1}{\theta_1}-\lambda_{n+2}\right) \\
=\;& \sum_{k=n+3}^{n+l}\prod_{j=k+1}^{n+l}(1-\theta_1\lambda_j)\lambda_k + \prod_{j=n+3}^{n+l}(1-\theta_1\lambda_j)\frac{1}{\theta_1}. && (4.6)
\end{aligned}$$

Comparing (4.5) and (4.6), and repeating this procedure, we have

$$\begin{aligned}
&\sum_{k=n+2}^{n+l}\prod_{j=k+1}^{n+l}(1-\theta_1\lambda_j)\lambda_k + \prod_{j=n+2}^{n+l}(1-\theta_1\lambda_j)\frac{1}{\theta_1} \\
=\;& \sum_{k=n+l}^{n+l}\prod_{j=k+1}^{n+l}(1-\theta_1\lambda_j)\lambda_k + \prod_{j=n+l}^{n+l}(1-\theta_1\lambda_j)\frac{1}{\theta_1} \\
=\;& \lambda_{n+l} + (1-\theta_1\lambda_{n+l})\frac{1}{\theta_1} \\
=\;& \frac{1}{\theta_1}.
\end{aligned}$$

Combining the last two displays yields (4.4). We conclude the proof. □

Lemma 20. *For non-negative integers n, l and any $r \in \mathbb{R}_+$, we have*

$$E\left[\exp\left(r\sigma_1 \max_{0\le i\le l}\sum_{m=1}^{i}\prod_{j=n+l+2-m}^{n+l}(1-\theta_1\lambda_j)\sqrt{\lambda_{n+l+1-m}}\,G_{n+l+1-m,1}\right)\right] \le 4e^{\beta r^2\sigma_1^2/\theta_1} \tag{4.7}$$

and

$$E\left[\exp\left(r\sigma_2 \max_{0\le i\le l}\sum_{m=1}^{i}\prod_{j=n+l+2-m}^{n+l}(1-\theta_2\lambda_j)\sqrt{\lambda_{n+l+1-m}}\,G_{n+l+1-m,2}\right)\right] \le 4e^{\beta r^2\sigma_2^2/\theta_2}.$$

Consequently, it follows by Hölder's inequality that

$$\begin{aligned}
&E\Bigg[\exp\left(r\sigma_1 \max_{0\le i\le l}\sum_{m=1}^{i}\prod_{j=n+l+2-m}^{n+l}(1-\theta_1\lambda_j)\sqrt{\lambda_{n+l+1-m}}\,G_{n+l+1-m,1}\right)\\
&\qquad\times\exp\left(r\sigma_2 \max_{0\le i\le l}\sum_{m=1}^{i}\prod_{j=n+l+2-m}^{n+l}(1-\theta_2\lambda_j)\sqrt{\lambda_{n+l+1-m}}\,G_{n+l+1-m,2}\right)\Bigg]\\
\le\;& 4e^{2\beta r^2\sigma_1^2/\theta_1+2\beta r^2\sigma_2^2/\theta_2}.
\end{aligned}$$

Proof. By symmetry, we only prove (4.7).

Note that

$$\begin{aligned}
&E\left[\exp\left(r\sigma_1 \max_{0\le i\le l}\sum_{m=1}^{i}\prod_{j=n+l+2-m}^{n+l}(1-\theta_1\lambda_j)\sqrt{\lambda_{n+l+1-m}}\,G_{n+l+1-m,1}\right)\right]\\
=\;&E\left[\max_{0\le i\le l}\exp\left(r\sigma_1 \sum_{m=1}^{i}\prod_{j=n+l+2-m}^{n+l}(1-\theta_1\lambda_j)\sqrt{\lambda_{n+l+1-m}}\,G_{n+l+1-m,1}\right)\right].
\end{aligned}$$

For fixed n and l, since $G_{n+1,1}, G_{n+2,1}, \dots, G_{n+l,1}$ are independent with a zero mean,

$$\sum_{m=1}^{i}\prod_{j=n+l+2-m}^{n+l}(1-\theta_1\lambda_j)\sqrt{\lambda_{n+l+1-m}}\,G_{n+l+1-m,1}$$

forms a martingale with respect to $\{\mathcal{G}_i\}$. Thus,

$$\exp\left(r\sigma_1 \sum_{m=1}^{i}\prod_{j=n+l+2-m}^{n+l}(1-\theta_1\lambda_j)\sqrt{\lambda_{n+l+1-m}}\,G_{n+l+1-m,1}\right)$$

is a submartingale. From Doob's maximal inequality for submartingales, we have

$$\begin{aligned}
&E\left[\max_{0\le i\le l}\exp\left(r\sigma_1 \sum_{m=1}^{i}\prod_{j=n+l+2-m}^{n+l}(1-\theta_1\lambda_j)\sqrt{\lambda_{n+l+1-m}}\,G_{n+l+1-m,1}\right)\right]\\
\le\;& 4E\left[\exp\left(r\sigma_1 \sum_{m=1}^{l}\prod_{j=n+l+2-m}^{n+l}(1-\theta_1\lambda_j)\sqrt{\lambda_{n+l+1-m}}\,G_{n+l+1-m,1}\right)\right]\\
=\;& 4\prod_{m=1}^{l}E\left[\exp\left(r\sigma_1 \prod_{j=n+l+2-m}^{n+l}(1-\theta_1\lambda_j)\sqrt{\lambda_{n+l+1-m}}\,G_{n+l+1-m,1}\right)\right]\\
\le\;& 4\prod_{m=1}^{l}\exp\left(\beta r^2\sigma_1^2 \prod_{j=n+l+2-m}^{n+l}(1-\theta_1\lambda_j)^2\lambda_{n+l+1-m}\right)\\
\le\;& 4\prod_{m=1}^{l}\exp\left(\beta r^2\sigma_1^2 \prod_{j=n+l+2-m}^{n+l}(1-\theta_1\lambda_j)\lambda_{n+l+1-m}\right)\\
=\;& 4\exp\left(\beta r^2\sigma_1^2 \sum_{m=1}^{l}\prod_{j=n+l+2-m}^{n+l}(1-\theta_1\lambda_j)\lambda_{n+l+1-m}\right)
\end{aligned}$$

$$
\begin{aligned}
&= 4\exp\left(\beta r^2\sigma_1^2 \sum_{k=n+1}^{n+l}\prod_{j=k+1}^{n+l}(1-\theta_1\lambda_j)\lambda_k\right)\\
&\leq 4e^{\beta r^2\sigma_1^2/\theta_1},
\end{aligned}
$$

where the second inequality follows by the assumption (4.1) on $G_{k,i}$, in the last equality we have substituted k for $n+l+1-m$, and in the last inequality we have invoked (4.4). We complete the proof. □

With these two lemmas in hand, we can establish a uniform upper bound for the expectations of $\exp(rU_l + rV_l)$. This result is summarized in the following lemma.

Lemma 21. *For any $r \in \mathbb{R}_+$, we have*

$$
\sup_{0\leq l<\infty} E\left[e^{rU_l+rV_l}\right] \leq 4e^{r(x_0+y_0+\mu_1^++\mu_2^+)+2\beta r^2\sigma_1^2/\theta_1+2\beta r^2\sigma_2^2/\theta_2}.
$$

Proof. Letting $n=0$ in Lemma 19 and noting that $U_0 = x_0$ yields

$$
\begin{aligned}
U_l &\leq U_0\prod_{j=1}^{l}(1-\theta_1\lambda_j) + \mu_1^+ + \sigma_1 \max_{0\leq i\leq l}\sum_{m=1}^{i}\prod_{j=l+2-m}^{n+l}(1-\theta_1\lambda_j)\sqrt{\lambda_{l+1-m}}\,G_{l+1-m,1}\\
&\leq x_0 + \mu_1^+ + \sigma_1 \max_{0\leq i\leq l}\sum_{m=1}^{i}\prod_{j=l+2-m}^{n+l}(1-\theta_1\lambda_j)\sqrt{\lambda_{l+1-m}}\,G_{l+1-m,1}.
\end{aligned}
$$

Similarly,

$$
V_l \leq y_0 + \mu_2^+ + \sigma_2 \max_{0\leq i\leq l}\sum_{m=1}^{i}\prod_{j=l+2-m}^{n+l}(1-\theta_2\lambda_j)\sqrt{\lambda_{l+1-m}}\,G_{l+1-m,2}.
$$

Combining the last two displays with Lemma 20, the desired result follows. □

Before presenting the next result, we introduce some necessary notation. Define $\lambda : [0,\infty) \to [0,\infty)$ and $\iota : [0,\infty) \to \mathbb{N}$ as

$$
\lambda(s) = \Lambda_k; \quad \iota(s) = k, \quad \text{if } \Lambda_k \leq s < \Lambda_{k+1},\ k \in \mathbb{N},
$$

where we define $\Lambda_0 = 0$. Furthermore, define

$$
\lambda_0 = \max_{1\leq i<\infty} \lambda_i.
$$

For any $t > 0$, it can be easily verified that

$$
t - \lambda_0 \leq \lambda(s+t) - \lambda(s) \leq t + \lambda_0.
$$

Lemma 22. *For any $r \in \mathbb{R}_+$, define*

$$
\begin{aligned}
M(r) &:= r\mu_1^+ + r\mu_2^+ + 2\beta r^2\sigma_1^2/\theta_1 + 2\beta r^2\sigma_2^2/\theta_2 + \ln 4,\\
\Delta &:= \lambda_0 + \max\{\ln 2/\theta_1, \ln 2/\theta_2\}.
\end{aligned}
$$

Then for any $t \geq 0$, we have

$$
E\left[e^{rU_{\iota(t+\Delta)}+rV_{\iota(t+\Delta)}} \mid \mathcal{G}_{\iota(t)}\right] \leq e^{-1}e^{rU_{\iota(t)}+rV_{\iota(t)}} + e^{2M(r)+1}.
$$

Proof. Applying Lemmas 19 and 20, together with the independence of the sequence $\{(G_{n,1}, G_{n,2})\}_{n=1}^{\infty}$, we have

$$
\begin{aligned}
& E\left[e^{rU_{\iota(t+\Delta)}+rV_{\iota(t+\Delta)}} \mid \mathcal{G}_{\iota(t)}\right] \\
\leq\ & \exp\left(rU_{\iota(t)} \prod_{j=\iota(t)+1}^{\iota(t+\Delta)} (1-\theta_1\lambda_j) + rV_{\iota(t)} \prod_{j=\iota(t)+1}^{\iota(t+\Delta)} (1-\theta_2\lambda_j)\right) \times e^{r\mu_1^+ + r\mu_2^+} \\
& \times E\Bigg[\exp\left(r\sigma_1 \max_{0\leq i\leq \iota(t+\Delta)-\iota(t)} \sum_{m=1}^{i} \prod_{j=\iota(t+\Delta)+2-m}^{\iota(t+\Delta)} (1-\theta_1\lambda_j)\sqrt{\lambda_{\iota(t+\Delta)+1-m}}\, G_{\iota(t+\Delta)+1-m,1}\right) \\
& \qquad \times \exp\left(r\sigma_2 \max_{0\leq i\leq \iota(t+\Delta)-\iota(t)} \sum_{m=1}^{i} \prod_{j=\iota(t+\Delta)+2-m}^{\iota(t+\Delta)} (1-\theta_2\lambda_j)\sqrt{\lambda_{\iota(t+\Delta)+1-m}}\, G_{\iota(t+\Delta)+1-m,2}\right)\Bigg] \\
\leq\ & \exp\left(rU_{\iota(t)} \prod_{j=\iota(t)+1}^{\iota(t+\Delta)} (1-\theta_1\lambda_j) + rV_{\iota(t)} \prod_{j=\iota(t)+1}^{\iota(t+\Delta)} (1-\theta_2\lambda_j)\right) \\
& \quad \times 4e^{r\mu_1^+ + r\mu_2^+ + 2\beta r^2\sigma_1^2/\theta_1 + 2\beta r^2\sigma_2^2/\theta_2} \\
=\ & e^{M(r)} \cdot \exp\left(rU_{\iota(t)} \prod_{j=\iota(t)+1}^{\iota(t+\Delta)} (1-\theta_1\lambda_j) + rV_{\iota(t)} \prod_{j=\iota(t)+1}^{\iota(t+\Delta)} (1-\theta_2\lambda_j)\right),
\end{aligned} \tag{4.8}
$$

where the last equality follows by the definition of $M(r)$. We then claim

$$\sum_{j=\iota(t)+1}^{\iota(t+\Delta)} \lambda_j = \lambda(t+\Delta) - \lambda(t).$$

In fact, if $\iota(t) = k$ and $\iota(t+\Delta) = l$, then $\lambda(t) = \Lambda_k$ and $\lambda(t+\Delta) = \Lambda_l$. Therefore,

$$\lambda(t+\Delta) - \lambda(t) = \Lambda_l - \Lambda_k = \sum_{j=k+1}^{l} \lambda_j = \sum_{j=\iota(t)+1}^{\iota(t+\Delta)} \lambda_j.$$

Note that $\lambda(t+\Delta) - \lambda(t) \geq \Delta - \lambda_0$. It follows that

$$
\begin{aligned}
\prod_{j=\iota(t)+1}^{\iota(t+\Delta)} (1-\theta_1\lambda_j) &\leq \prod_{j=\iota(t)+1}^{\iota(t+\Delta)} e^{-\theta_1\lambda_j} = \exp\Bigg(-\theta_1 \sum_{j=\iota(t)+1}^{\iota(t+\Delta)} \lambda_j\Bigg) \\
&= e^{-\theta_1(\lambda(t+\Delta)-\lambda(t))} \leq e^{-\theta_1(\Delta-\lambda_0)} \leq e^{-\theta_1(\ln 2/\theta_1 + \lambda_0 - \lambda_0)} = \frac{1}{2}.
\end{aligned}
$$

Similarly,

$$\prod_{j=\iota(t)+1}^{\iota(t+\Delta)} (1-\theta_2\lambda_j) \leq \frac{1}{2}.$$

Combining the last two displays with (4.8), we have

$$
\begin{aligned}
& E\left[e^{rU_{\iota(t+\Delta)}+rV_{\iota(t+\Delta)}} \mid \mathcal{G}_{\iota(t)}\right] \\
\leq\ & e^{M(r)} \cdot e^{rU_{\iota(t)}/2 + rV_{\iota(t)}/2} \\
=\ & e^{M(r)} e^{rU_{\iota(t)}/2 + rV_{\iota(t)}/2} \mathbb{1}_{\{U_{\iota(t)}+V_{\iota(t)} > 2(M(r)+1)/r\}}
\end{aligned}
$$

$$
\begin{aligned}
&\quad + e^{M(r)} e^{rU_{\iota(t)}/2 + rV_{\iota(t)}/2} \mathbb{1}_{\{U_{\iota(t)}+V_{\iota(t)} \le 2(M(r)+1)/r\}} \\
\le &\quad e^{M(r)} e^{rU_{\iota(t)}+rV_{\iota(t)} - \frac{r}{2}\times 2(M(r)+1)/r} \mathbb{1}_{\{U_{\iota(t)}+V_{\iota(t)} > 2(M(r)+1)/r\}} \\
&\quad + e^{M(r)} e^{\frac{r}{2}\times 2(M(r)+1)/r} \mathbb{1}_{\{U_{\iota(t)}+V_{\iota(t)} \le 2(M(r)+1)/r\}} \\
\le &\quad e^{-1} e^{rU_{\iota(t)}+rV_{\iota(t)}} + e^{2M(r)+1}.
\end{aligned}
$$

We conclude the proof. □

With the above preparation, we arrive at our primary result in this subsection, indicating the almost sure tightness of the sequence of random measures $\{\pi_n\}_{n=1}^{\infty}$.

Proposition 23. *For any $r \in \mathbb{R}_+$, we have*

$$\sup_{1\le n<\infty} \frac{1}{\Lambda_n} \sum_{k=1}^{n} \lambda_k e^{rU_{k-1}+rV_{k-1}} < \infty$$

almost surely. In other words,

$$\sup_{1\le n<\infty} \int_{\mathbb{R}_+^2} e^{rx+ry}\, \pi_n(dx,dy) < \infty$$

almost surely. Consequently, the sequence $\{\pi_n\}_{n=1}^{\infty}$ is tight almost surely.

Proof. The proof is similar to that of Lemma 6 in [6]; therefore, we omit the specific details. □

4.4. **Identification of the limit.** To complete the proof of Theorem 18, we must demonstrate that for almost every ω, any weak limit of $\pi_n(\omega)$ is π. To this end, we follow the method in Section 2.2 of [6]. It is worth noting that all the conditions necessary for the deduction in Section 2.2 of [6] are satisfied, except for the requirement of boundedness in $|b(x)|$, which corresponds to the boundedness of $|\mu_1 - x|$ and $|\mu_2 - y|$ in our case. However, $|\mu_1 - x| \le |\mu_1| + |x|$ and $|\mu_2 - y| \le |\mu_2| + |y|$. When modifying the proof in Section 2.2 of [6], it only requires the following results:

(R1) $\frac{1}{\Lambda_n}\sum_{k=1}^{n} \lambda_k^2 (U_{k-1}+V_{k-1})^2 \to 0$ almost surely as $n\to\infty$;

(R2) For any non-negative integer j,

$$\sup_{1\le n<\infty} \frac{1}{\Lambda_n} \sum_{k=1}^{n} \lambda_k (U_{k-1}+V_{k-1})^j \mathbb{1}_{\{U_{k-1}+V_{k-1}>M\}} \to 0$$

almost surely as $M\to\infty$;

(R3)

$$\sum_{k=1}^{\infty} \frac{\lambda_{k+1}^2 E\left[(U_k+V_k)^2\right]}{\Lambda_k^2} < \infty.$$

All the results can be derived from Lemma 21 and Proposition 23. Indeed, note that

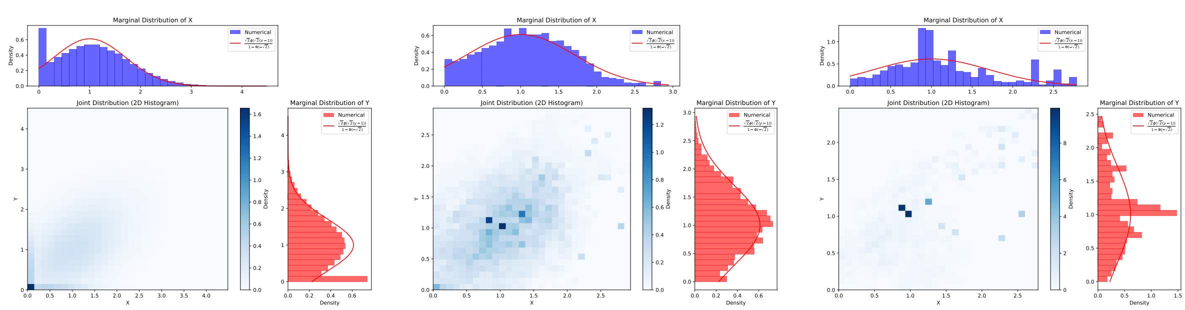
$$\frac{1}{\Lambda_n}\sum_{k=1}^{n} \lambda_k^2 (U_{k-1}+V_{k-1})^2 \le \frac{2}{\Lambda_n}\sum_{k=1}^{n} \lambda_k^2 e^{U_{k-1}+V_{k-1}}.$$

Therefore, (R1) follows immediately from Proposition 23 and the fact that $\lambda_n \to 0$ and $\Lambda_n \to \infty$. Furthermore, (R2) follows similarly, since

$$
\begin{aligned}
& \sup_{1\le n<\infty} \frac{1}{\Lambda_n} \sum_{k=1}^{n} \lambda_k (U_{k-1}+V_{k-1})^j \mathbb{1}_{\{U_{k-1}+V_{k-1}>M\}} \\
\le\ & \sup_{1\le n<\infty} \frac{1}{\Lambda_n} \sum_{k=1}^{n} \lambda_k (U_{k-1}+V_{k-1})^j \times \frac{U_{k-1}+V_{k-1}}{M} \\
\le\ & \frac{(j+1)!}{M} \sup_{1\le n<\infty} \frac{1}{\Lambda_n} \sum_{k=1}^{n} \lambda_k e^{U_{k-1}+V_{k-1}}.
\end{aligned}
$$

Finally, (R3) follows from the uniform boundedness of $E[(U_k+V_k)^2]$, which is a direct result of Lemma 21. This completes the convergence analysis of the numerical scheme.

4.5. **Numerical results.** In this subsection, we conduct simulation experiments to illustrate our numerical scheme to approximate the stationary distribution of the 2-d ROU process. We first illustrate the selection of the step size, under the parameter setting $\theta_1 = \theta_2 = \mu_1 = \mu_2 = \sigma_1 = \sigma_2 = 1$ and the initial conditions $x_0 = 1$ and $y_0 = 1$. The resulting numerical joint and marginal distributions are computed using three different schemes: (i) $\lambda_k = (1+k)^{-0.1}$, (ii) $\lambda_k = (1+k)^{-0.7}$, and (iii) $\lambda_k = (1+k)^{-1}$. The number of iterations is set to $n = 20000$. Additionally, the true marginal distributions, with density functions given by $\sqrt{2}\phi(\sqrt{2}(x-1))/(1-\Phi(-\sqrt{2}))$ and $\sqrt{2}\phi(\sqrt{2}(y-1))/(1-\Phi(-\sqrt{2}))$, are plotted as red curves to enable comparison and evaluate the accuracy of the numerical approximations. Here, $\phi(x)$ and $\Phi(x)$ denote the probability density function and cumulative distribution function of the standard normal distribution, respectively.



(a) $\lambda_k = (1+k)^{-0.1}$ (b) $\lambda_k = (1+k)^{-0.7}$ (c) $\lambda_k = (1+k)^{-1}$

FIGURE 2. The numerical results for the joint distribution and marginal distributions when $\theta_1 = \theta_2 = \mu_1 = \mu_2 = \sigma_1 = \sigma_2 = 1$, $x_0 = y_0 = 1$, and $n = 20000$. From left to right, the diagrams correspond to the following schemes: (i) $\lambda_k = (1+k)^{-0.1}$, (ii) $\lambda_k = (1+k)^{-0.7}$, and (iii) $\lambda_k = (1+k)^{-1}$, respectively.

When the number of iterations is relatively small, the step size should be chosen carefully to balance accuracy and stability. If the step size is too large, such as $\lambda_k = (1+k)^{-0.1}$, the discretized process (U_n, V_n), as defined in Subsection 4.2, may frequently visit $(0,0)$, resulting in an approximating distribution that is overly concentrated at $(0,0)$. Moreover, as illustrated in Figure 2(a), the marginal distributions exhibit a good fit except at 0. Conversely, if the step size is too small, for instance, $\lambda_k = (1+k)^{-1}$, the discretized process (U_n, V_n) tends to remain near the initial position $(1,1)$, leading to an approximating distribution concentrated at $(1,1)$ and thus producing an inaccurate result (see Figure 2(c)). In contrast, when the step size is moderate, the approximating distribution is more dispersed. As suggested by the marginal distribution diagrams in Figure 2(b), we believe this choice provides a closer approximation to the true stationary distribution.

We next use our numerical scheme to illustrate how the correlation of the 2-d ROU process under different parameter settings changes with respect to the correlation of the driving Brownian motion. As before, the initial position is set to $(x_0, y_0) = (1,1)$, and the number of iterations is $n = 20000$, and we choose the step size $\lambda_k = (1+k)^{-0.7}$. Since the means and variances of the marginal distributions can be explicitly computed from their density functions, we omit their numerical results. The numerical results for the correlation are summarized in Tables 1, 2, 3, and 4, as well as in Figure 3. These results strongly suggest that the correlation of the 2-d ROU process is non-decreasing with respect to the correlation of the Brownian motion terms. An analytical investigation of this property is left for future work.

ρ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
ρ_{ROU}	0.0418	0.2164	0.1753	0.3079	0.5332	0.5777	0.6734	0.7603	0.8900
ρ	-0.1	-0.2	-0.3	-0.4	-0.5	-0.6	-0.7	-0.8	-0.9
ρ_{ROU}	-0.0023	-0.1475	-0.2464	-0.4193	-0.4744	-0.5402	-0.6105	-0.7403	-0.8265

Table 1. Numerical results for the correlation of the 2-d ROU process for different values of ρ, when $\theta_1 = \theta_2 = 1$, $\mu_1 = \mu_2 = 1$, $\sigma_1 = \sigma_2 = 1$, $\lambda_k = (1+k)^{-0.7}$ and $n = 20000$.

ρ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
ρ_{ROU}	0.1245	0.1321	0.2275	0.3704	0.4188	0.5192	0.5969	0.7785	0.8696
ρ	-0.1	-0.2	-0.3	-0.4	-0.5	-0.6	-0.7	-0.8	-0.9
ρ_{ROU}	-0.0650	-0.0819	-0.0689	-0.2009	-0.2321	-0.2138	-0.3121	-0.3374	-0.4103

Table 2. Numerical results for the correlation of the 2-d ROU process for different values of ρ, when $\theta_1 = \theta_2 = 1$, $\mu_1 = \mu_2 = -1$, $\sigma_1 = \sigma_2 = 1$, $\lambda_k = (1+k)^{-0.7}$ and $n = 20000$.

ρ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
ρ_{ROU}	0.0642	0.1475	0.1543	0.2531	0.3766	0.4446	0.4741	0.6062	0.6823
ρ	-0.1	-0.2	-0.3	-0.4	-0.5	-0.6	-0.7	-0.8	-0.9
ρ_{ROU}	-0.0586	-0.1079	-0.1896	-0.3056	-0.3685	-0.3285	-0.4681	-0.5157	-0.5567

Table 3. Numerical results for the correlation of the 2-d ROU process for different values of ρ, when $\theta_1 = \theta_2 = 1$, $\mu_1 = 1$, $\mu_2 = -1$, $\sigma_1 = 1$, $\sigma_2 = \sqrt{2}$, $\lambda_k = (1+k)^{-0.7}$ and $n = 20000$.

ρ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
ρ_{ROU}	0.0975	0.1036	0.1952	0.3467	0.3913	0.4959	0.5873	0.7614	0.8504
ρ	-0.1	-0.2	-0.3	-0.4	-0.5	-0.6	-0.7	-0.8	-0.9
ρ_{ROU}	-0.0909	-0.1098	-0.0943	-0.2434	-0.2672	-0.2525	-0.3485	-0.3766	-0.4726

TABLE 4. Numerical results for the weighted correlation of the 2-d ROU process for different values of p, when $\theta_1 = \theta_2 = 1$, $\mu_1 = \mu_2 = -1$, $\sigma_1 = 1$, $\sigma_2 = \sqrt{2}$, $\lambda_k = (1+k)^{-0.7}$ and $n = 20000$.

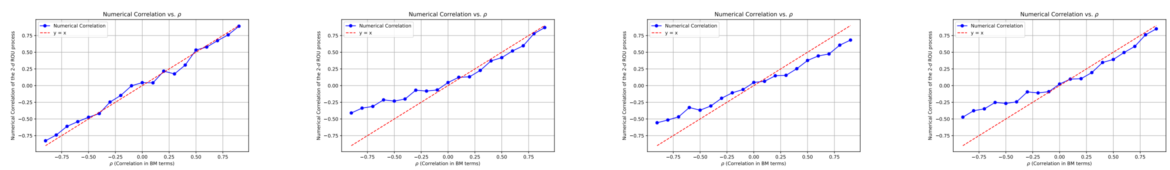


FIGURE 3. Numerical Correlation of the 2-d ROU process vs. correlation of the Brownian motion terms. From left to right, the diagrams correspond to Tables 1, 2, 3, and 4, respectively.

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