Queueing models with random resetting

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Abstract. We introduce and study some queueing models with random resetting, including Markovian and non–Markovian models. The Markovian models include M/M/1, M/M/r and M/M/∞ queues with random resetting, in which a continuous-time Markov chain is formulated and the transition from each state includes a resetting to state zero in addition to the arrival and service transitions. Hence the chains are no longer a birth and death process as in the classical models. We explicitly characterize the stationary distributions of the queueing processes in these models. It is worth noting the distinction of the stability conditions from the standard models, that is, the positive recurrence of the Markov chains does not require the usual traffic intensity to be less than one.

The non–Markovian models include GI/GI/1, GI/GI/r and GI/GI/∞ queues with random resetting to state zero. For GI/GI/1 and GI/GI/r queues, we consider random resetting at arrival times, and introduce modified Lindley recursions and Kiefer–Wolfowitz recursions, respectively. Using an operator representation for these recursions, we characterize the stationary distributions via convergent series, as solutions to the modified Wiener–Hopf equations. For GI/GI/1 queues with random resetting, a particularly interesting case is when the difference of the service and interarrival times is positive, for which an explicit characterization of the stationary distribution of the delay/waiting time is provided via the associated characteristic functions. For GI/GI/∞ queues, we also consider random resettings at arrival times, by utilizing a version of the Kiefer–Wolfowitz recursion motivated from that for GI/GI/r queues, and also characterize the corresponding stationary distribution.

1. Introduction

In this paper we study some queueing models with random resetting, in which a queue clears when resettings occurs. Such models have many applications in service systems where machines or servers are subject to maintenance after some random time or periodically, or where disruptions occur due to power loss or breaking down. There exists a substantial literature in queueing theory to model such phenomena, such as queues in random environments, queues with disasters and so on. In this work we aim at developing a number of queueing models extending the standard ones in a unified manner to take into account random resettings and treating both Markovian and non–Markovian models.

We start with the Markovian models with random resetting, extending the standard M/M/1, M/M/r and M/M/∞ systems. The queueing process in these classical models is a birth–death process with jumps ± 1, whose stationary distribution is known explicitly. For

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M/M/1 and M/M/r queues, this leads to a stability condition: the traffic intensity per server is less than one. In a system with resetting, the queueing process is a continuous-time Markov chain (CTMC) which not only can jump by ± 1, but also can hop to state 0; consequently, the CTMC is no longer a birth–death process. Here the transitions are characterized by arrival, departure and resetting rates. It turns out that for the above models, the CTMC with resetting is always positive recurrent (even if the departure rate equals 0), and its stationary distribution can be written down explicitly. See Theorems 3.1, 2.1, and 4.1. Similar results have been obtained for stochastic clearing models of point processes studied in [24, 23, 28].

Next, we pass to non–Markovian models with random resetting, extending the standard GI/GI/1 and GI/GI/r systems under the FCFS discipline and the GI/GI/∞ system. For a standard GI/GI/1 queue, the Lindley recursion is fundamental in studying the delay/waiting-time behavior (see Section 5.1 for a brief review and [2, Chapter X.1]). Similarly, for a standard GI/GI/k queue, it is the Kiefer–Wolfowitz recursion that determines the delay/waiting–time behavior (see Section 6.1). We study the corresponding models with random resetting at customers' arrival times, which leads to modified Lindley and Kiefer–Wolfowitz recursions (see equations (5.4) and (6.7)).

For standard GI/GI/1 and GI/GI/r queues, the positive recurrence of the waiting–time process requires that the traffic intensity per server is less than one, and then the stationary distribution is characterized via the Wiener–Hopf equation. For the same models with random resetting, we show that the corresponding modified Lindley and the Kiefer–Wolfowitz recursions always generate positive recurrent waiting–time processes, regardless of whether their standard counterparts are positive recurrent or not. More importantly, the modified Lindley and the Kiefer–Wolfowitz recursions can be conveniently represented as an operator form (see equations (5.8) and (6.5)), from which we are able to express the stationary distribution as a convergent series (see equations (5.10) and (6.9)-(iv)). As a byproduct, we have an interesting finding for GI/GI/1 queues with random resetting when the difference of the service and interarrival times takes positive values (unlike the standard case assuming negative mean), that the stationary distribution for the waiting time/delay can be explicitly expressed via the characteristic functions of the difference variable mentioned above, see equation (5.14).

Finally, we consider GI/GI/∞ models with random resetting at arrival times. We first construct a recursion for the elapsed service times for the jobs in service in GI/GI/∞ queues, by adapting the Kiefer–Wolfowitz recursion for GI/GI/r queues, and then use it to formulate the recursion for the GI/GI/∞ queues with resetting. We show that the recursion is positive recurrent and also derive an explicit expression for the associated stationary distribution, similarly as the GI/GI/r queues with resetting (see Section 7).

A review of the literature. The models discussed in this paper are related to several streams of the existing literature. First, these models are related to the stochastic clearing models studied in [24, 23, 28], where a stochastic input process (such as the arrival process) is intermittently and instantaneously cleared. Various clearing policies have been studied, e.g., clearance when the input reaches a threshold, or at i.i.d. random times independent of the input process. Our Markovian models without service can be regarded as stochastic clearing models of Poisson arrivals at exponentially distributed random times, while our non–Markovian models without services can be regarded as stochastic clearing models of renewal arrivals at arrival times. However, the stochastic clearing models do not have output dynamics like our models.

Second, our models are related to the queueing models with disasters, see, e.g, [6, 10, 16, 3, 11, 19, 29, 7, 26]. For example, in [6], an M/G/1 queue with "disasters" has been considered, where disasters occur at certain random times including (a) deterministic equidistant times, (b) random times independent of the queueing process, and (c) at crossings of some pre-specified level. In these works, stationary distributions of the workload processes have been characterized via their Laplace transforms using certain modifications of the Lindley recursion.

For another example, the paper [16] considers an M/M/1 queue with catastrophes, where the server breaks downs at i.i.d. random times, independent of the service process. At the breakdown times, all jobs are lost, and it takes an exponential random time to repair the system. See also similar formulations of "catastrophes" or "clearing" in [10, 3, 11, 19, 26]. An associated (jump) diffusion approximation has been considered in [10, 9, 8]. In [13], a computational approach has been developed, for non-homogeneous Markovian single-server and infinite-server queueing models, whose formulation is like our Markovian models but with nonstationary transition rates. A more general birth-death process with catastrophes is studied in [9].

Our study clearly distinguishes from the existing literature of queues with disasters since the Markovian models have explicit stationary distributions, and the non–Markovian models with random resetting at arrival times are completely new. We also refer the readers to some recent studies of random walks, Brownian motions and diffusions with random resetting in [12, 25, 27, 1, 18].

Unlike the "clearing" phenomenon in the models described above, there are some recent studies of queues with random resetting in [4, 22], where the authors study a single server M/G/1 queue with service times being reset at random times whenever the service time is longer than a threshold. This concept of stochastic resetting is also exploited in random search problems, Cf. [5, 20].

Organization of the paper. The paper is organized as follows. The Markovian models with resetting are studied first, with $M/M/\infty$ queues in Section 2, M/M/1 queues considered in Section 3, and $M/M/r$ queues in Section 4. The non–Markovian models are studied next, with GI/GI/1 and $GI/GI/r$ queues with resettings at arrival times in Sections 5 and 6, respectively, and with infinite-server queues with resettings at arrival times in Section 7.

2. The $M/M/\infty$ queue with random resetting

The standard assumption in Sections 2–4 is that the jobs arrive in a Poisson process at rate $\lambda > 0$, the services times are i.i.d. exponential of rate $\mu \geq 0$, and all jobs in the system are cleared/reset after after subsequent i.i.d. exponential random times of rate $\kappa > 0$. As usual, we assume mutual independence of all the processes involved. The case of $\mu = 0$ means that no jobs are served; in this case we get a stochastic clearing model of a Poisson process, Cf. [24, 28].

This section focuses on an M/M/∞ queue with resetting. Denote by $X(t)$ the number of jobs in the system at time $t \geq 0$. Then $\{X(t) : t \geq 0\}$ is the continuous-time Markov chain (CTMC) on $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ with the transition rates

$$\begin{array}{llllll} i \geq 0 & \rightarrow & i+1 & \text{rate} & \lambda & \text{(arrival)}, \\ i \geq 1 & \rightarrow & i-1 & \text{rate} & i\mu & \text{(departure)}, \\ i \geq 1 & \rightarrow & 0 & \text{rate} & \kappa & \text{(resetting)}. \end{array} \tag{2.1}$$

The process $\{X(t)\}$ with $\mu > 0$ is dominated by the process with $\mu = 0$. Alternatively, $\{X(t)\}$ is dominated by the standard M/M/∞ queuing process, with the same μ and $\kappa = 0$. We denote by $\pi = \{\pi_i : i \in \mathbb{Z}_+\}$ the stationary distribution of process $\{X(t)\}$.

Theorem 2.1. *Assume that $\mu > 0$. The stationary distribution π for the* M/M/∞ *queue with resetting is given by*

$$\begin{aligned} \pi_0 &= \frac{1}{\lambda}\Big(\mu + \frac{\kappa}{\rho}(e^\rho - 1)\Big)\Big(\frac{\mu}{\lambda} + \frac{\kappa/\lambda + 1}{\rho}(e^\rho - 1)\Big)^{-1}, \\ \pi_i &= \Big(\frac{\mu}{\lambda} + \frac{\kappa/\lambda + 1}{\rho}(e^\rho - 1)\Big)^{-1}\frac{\rho^{i-1}}{i!}, \quad i \geq 1\,. \end{aligned} \tag{2.2}$$

Here $\rho > 0$ solves the equation:

$$\kappa(e^\rho - 1) = -\mu\rho^2 + (\lambda + \kappa)\rho. \tag{2.3}$$

Proof. To characterize the stationary distribution, we have the partial balance equations (PBEs):

$$\begin{aligned} \lambda\pi_0 &= \mu\pi_1 + \kappa\sum_{j\geq 1}\pi_j\,, \\ (\lambda + i\mu + \kappa)\pi_i &= \lambda\pi_{i-1} + (i+1)\mu\pi_{i+1}\,, \quad i \geq 1\,. \end{aligned}$$

We use the following *Ansatz*:

$$\pi_i = \pi_1\frac{\rho^{i-1}}{i!}, \quad \text{for} i \geq 1,$$

where ρ is to be determined.

By the above equation for π_0, we have

$$\lambda\pi_0 = \mu\pi_1 + \kappa\pi_1\sum_{j\geq 1}\frac{\rho^{j-1}}{j!} = \mu\pi_1 + \frac{\kappa}{\rho}\pi_1(e^\rho - 1)\,,$$

whence

$$\pi_0 = \frac{1}{\lambda}\Big(\mu + \frac{\kappa}{\rho}(e^\rho - 1)\Big)\pi_1\,. \tag{2.4}$$

By the *Ansatz*, $\pi_2 = \pi_1\rho/2$; we substitute it into the PBE for π_1. Together with the form of π_0, it leads to equation (2.3) which clearly has one positive solution.

Finally, equation (2.4) and the condition $\sum\limits_{i\geq 0}\pi_i = 1$ provide the explicit expression for π_1:

$$\pi_1 = \Big(\frac{\mu}{\lambda} + \frac{\kappa/\lambda + 1}{\rho}(e^\rho - 1)\Big)^{-1}. \tag{2.5}$$

Plugging this into (2.4), we obtain the claimed form of π. $\square$

Remarks. 2.1. The distribution π in (2.2) is a mixture of a Poisson distribution and the Dirac delta at 0:

$$\pi_i = \alpha \frac{\rho^i e^{-\rho}}{i!} + (1-\alpha)\delta_{i,0}, \quad i \ge 0, \tag{2.6}$$

where

$$\alpha = \frac{e^\rho}{\rho}\Big(\frac{\mu}{\lambda} + \frac{\kappa/\lambda + 1}{\rho}(e^\rho - 1)\Big)^{-1} \in (0,1). \tag{2.7}$$

When $\kappa = 0$, that is, the standard M/M/∞ queue, equation (2.3) reduces to $\rho = \lambda/\mu$, which implies $\pi_1 = \rho e^{-\rho}$, and then $\pi_0 = \rho^{-1}\pi_1 = e^{-\rho}$ and $\pi_i = e^{-\rho}\rho^i/i!$ for $i \ge 1$. That is, π follows the Poisson distribution of parameter ρ.

2.2. For $\mu = 0$ (the stochastic clearing model of Poisson process), the PBEs become

$$\lambda\pi_0 = \kappa \sum_{j\ge1} \pi_j, \quad (\lambda+\kappa)\pi_i = \lambda\pi_{i-1}\,, \quad i \ge 1\,. \tag{2.8}$$

It gives

$$\pi_i = \varrho^i(1-\varrho), \;\; i \ge 0 \quad \text{(geometric), where} \quad \varrho := \frac{\lambda}{\lambda+\kappa}. \tag{2.9}$$

3. The M/M/1 queue with random resetting

In this section we consider an M/M/1 queue under the first-come first-served (FCFS) discipline with random resetting. Let $X(t)$ be the number of jobs in the system at time t. The CTMC $\{X(t) : t \ge 0\}$ has the transition rates

$$\begin{array}{lllll} i \ge 0 & \to & i+1 & \text{rate } \lambda & \text{(arrival)}, \\ i \ge 1 & \to & i-1 & \text{rate } \mu & \text{(departure)}, \\ i \ge 1 & \to & 0 & \text{rate } \kappa & \text{(resetting)}. \end{array} \tag{3.1}$$

When $\mu = 0$, the CTMC $\{X(t)\}$ is positive recurrent $\forall\ \lambda, \kappa > 0$: see (2.8). Owing to the dominance, we obtain the following property.

Proposition 3.1. *For any $\mu \ge 0$ and $\lambda, \kappa > 0$, the* CTMC $\{X(t) : t \ge 0\}$ *is positive recurrent and has a unique stationary distribution.*

As before, let $\pi = \{\pi_i : i \in \mathbb{Z}_+\}$ be the stationary distribution, which is also equal to the limiting probability: $\pi_i = \lim_{t\to\infty} \mathbb{P}(X(t) = i)$ for $i \in \mathbb{Z}_+$.

Theorem 3.1. *Assume that $\mu > 0$. The stationary distribution π is given by*

$$\begin{aligned} \pi_0 &= \frac{1}{1-\beta}\Big(\frac{1}{1-\beta} + \frac{1}{1-\rho}\Big)^{-1}, \\ \pi_i &= \rho^{i-1}\Big(\frac{1}{1-\beta} + \frac{1}{1-\rho}\Big)^{-1}, \quad i \ge 1\,. \end{aligned} \tag{3.2}$$

Here $\beta \in (0,1)$ *and* $\rho \in (0,1)$ *are given by*

$$\beta = 1 - \Big(\frac{\mu}{\lambda} + \frac{\kappa}{\lambda}\frac{1}{1-\rho}\Big)^{-1}, \tag{3.3}$$

and

$$\rho = \frac{\lambda+\mu+\kappa}{2\mu} - \sqrt{\frac{(\lambda+\mu+\kappa)^2}{4\mu^2} - \frac{\lambda}{\mu}}\ . \tag{3.4}$$

Proof. We know that π must satisfy the partial balance equations (PBEs) and have $\sum_i \pi_i = 1$. It yields

$$\lambda\pi_0 = \mu\pi_1 + \kappa\sum_{j\geq 1}\pi_j\,,$$
$$(\lambda+\mu+\kappa)\pi_i = \lambda\pi_{i-1} + \mu\pi_{i+1}\,,\quad i \geq 1\,.$$

The *Ansatz* is now that $\pi_i = \pi_1\rho^{i-1}$ for some $\rho \in (0,1)$ and $i \geq 1$. Plugging it into the above equation for π_0, we obtain

$$\lambda\pi_0 = \mu\pi_1 + \kappa\pi_1\frac{1}{1-\rho} = \pi_1\Big(\mu + \frac{\kappa}{1-\rho}\Big)\,,$$

that is,

$$\pi_0 = \pi_1\Big(\frac{\mu}{\lambda} + \frac{\kappa}{\lambda}\frac{1}{1-\rho}\Big) = \pi_1(1-\beta)^{-1}\,.$$

The equation for π_i with $i \geq 2$ yields

$$(\lambda+\mu+\kappa)\rho = \lambda + \mu\rho^2.$$

From this equation, if $\mu = 0$, then $\rho = \dfrac{\lambda}{\lambda+\kappa}$ (cf. (2.9)), and if $\mu > 0$, then we have

$$\rho = \frac{\lambda+\mu+\kappa}{2\mu} - \sqrt{\frac{(\lambda+\mu+\kappa)^2}{4\mu^2} - \frac{\lambda}{\mu}}\,,$$

which can be easily checked to be in $(0,1)$. The other solution to the quadratic equation is disregarded since it is bigger than 1.

Next, by $\sum_i \pi_i = 1$, we have

$$\pi_0 + \pi_1\frac{1}{1-\rho} = \pi_1\Big(\frac{1}{1-\beta} + \frac{1}{1-\rho}\Big) = 1\,,$$

which gives

$$\pi_1 = \Big(\frac{1}{1-\beta} + \frac{1}{1-\rho}\Big)^{-1}.$$

From here we obtain the expressions (2.1) and (2.2) as claimed.

Note that

$$\frac{1}{1-\beta} = \frac{\mu}{\lambda} + \frac{\kappa}{\lambda}\frac{1}{1-\rho} > 0\,,$$

which implies that $\beta \in (0,1)$. ∎

Remarks. 3.1. The distribution π in (3.2) is a mixture of a geometric distribution and the Dirac delta at 0:

$$\pi_i = \alpha\rho^i(1-\rho) \ + (1-\alpha)\delta_{i,0}, \quad i \geq 0. \tag{3.5}$$

Here, $\alpha = \dfrac{1-\beta}{\rho(1-\rho+1-\beta)} \in (0,1)$.

3.2. When $\mu = 0$, we again obtain π_i in the form of (2.9).

4. The M/M/r queue with random resetting

We now pass to an M/M/r queue under the FCFS discipline with random resetting. Letting $X(t)$ is the number of jobs in the system at time t, then $\{X(t) : t \geq 0\}$ is again a CTMC on $\mathbb{Z}_+$. Here the transition rates are:

$$\begin{array}{llllll} i \geq 0 & \to & i+1 & \text{rate} & \lambda & \text{(arrival)}, \\ i \geq 1 & \to & i-1 & \text{rate} & (i \wedge r)\mu & \text{(departure)}, \\ i \geq 1 & \to & 0 & \text{rate} & \kappa & \text{(resetting)}. \end{array} \tag{4.1}$$

It is evident that the CTMC $\{X(t)\}$ has a unique stationary distribution $\pi = \{\pi_i : i \in \mathbb{Z}_+\}$.

Theorem 4.1. *Assume that $\mu > 0$. The stationary distribution π for the* M/M/r *queue with random resetting is given by*

$$\begin{aligned} \pi_0 &= \frac{A/\lambda}{A/\lambda + B}, \\ \pi_i &= (A/\lambda + B)^{-1}\frac{\rho^{i-1}}{i!}, \quad 1 \leq i < r, \\ \pi_i &= (A/\lambda + B)^{-1}\frac{\rho^{i-1}}{r!r^{i-r}}, \quad i \geq r. \end{aligned} \tag{4.2}$$

Here $\rho \in (0,r)$ is given by the following:

$$\rho = \frac{(\lambda + r\mu + \kappa) - \sqrt{(\lambda + r\mu + \kappa)^2 - 4\mu\lambda r}}{2r\mu}, \tag{4.3}$$

whereas

$$A = \mu + \kappa B, \quad B = \sum_{i=1}^{r-1}\frac{\rho^{i-1}}{i!} + \sum_{i=r}^{\infty}\frac{\rho^{i-1}}{r!r^{i-r}}. \tag{4.4}$$

Proof. To characterize the stationary distribution, we again use the PBEs:

$$\begin{aligned} \lambda\pi_0 &= \mu\pi_1 + \kappa\sum_{j\geq 1}\pi_j, \\ (\lambda + i\mu + \kappa)\pi_i &= \lambda\pi_{i-1} + (i+1)\mu\pi_{i+1}, \quad 1 \leq i < r, \\ (\lambda + r\mu + \kappa)\pi_i &= \lambda\pi_{i-1} + r\mu\pi_{i+1}, \quad i \geq r. \end{aligned} \tag{4.5}$$

We now use the *Ansatz* in the following form:

$$\pi_i = \pi_1\frac{\rho^{i-1}}{i!}, \ \ 1 \leq i < r, \quad \text{and} \quad \pi_i = \pi_1\frac{\rho^{i-1}}{r!r^{i-r}}, \ \ i \geq r. \tag{4.6}$$

where $\rho \in (0, r)$ is a constant to be determined. From the top equation in (4.5), we obtain

$$\lambda\pi_0 = A\pi_1\,. \tag{4.7}$$

Then, from the equation for π_r, we get

$$(\lambda + r\mu + \kappa)\rho = \lambda r + r\mu\rho^2,$$

which yields ρ as in (4.3) (discarding the root > 1).

Next, the condition $\sum_{i\geq 0} \pi_i = 1$ gives an explicit form of π_1:

$$\pi_1 = (A/\lambda + B)^{-1}\,. \tag{4.8}$$

Then the expressions for π_0 and π_i are obtained from (4.7) and (4.8) whereas π_i for $i \geq 1$ are calculated from (4.6). □

Remarks. 4.1. The distribution π in (4.2) is a mixture of a stationary distribution for a sub-critical M/M/r queue and the Dirac delta at 0:

$$\pi_i = \alpha\widehat{\pi}_i \ + \ (1-\alpha)\delta_{i,0}, \quad i \geq 0. \tag{4.9}$$

Here,

$$\widehat{\pi}_i = C^{-1}\rho^i \times \begin{cases} 1/i!, & 0 \leq i < r, \\ 1/(r! r^{r-i}), & i \geq r, \end{cases} \qquad \text{where } C = \left(\sum_{i=0}^{r-1} \frac{\rho^i}{i!} + \frac{r^r}{r!}\frac{r}{r-\rho} \right)^{-1} \tag{4.10}$$

and $\alpha = \dfrac{B + 1/\rho}{A/\lambda + B} \in (0,1)$.

4.2. In the limit as $r \to \infty$, we have that $B \to (e^\rho - 1)/\rho$, and $A \to \mu + \kappa(e^\rho - 1)/\rho$, so that $A/\lambda + B \to \mu/\lambda + (1 + \kappa/\lambda)(e^\rho - 1)/\rho$. Hence, we recover the result for the M/M/∞ queue with resetting.

4.3. When $\kappa = 0$ and $\lambda < r\mu$, we get $\rho = \lambda/(r\mu) < 1$. Then $A = \mu$ and $B = C$, where C is as in (4.10).

5. The GI/GI/1 queues with random resetting at arrival times

In this section, we consider a GI/GI/1 queue under the FCFS discipline with random resetting at arrival times, particularly, focusing on the waiting times (delays) of jobs in the system. We will be using [2] as a main reference book; the original works containing related results can be traced via comments and the bibliography in [2].

5.1. *The Lindley recursion for* GI/GI/1 *queue.* The key ingredient of the GI/GI/1 model is a sequence of real random variables (RVs) X_n, $n \geq 0$, where $X_n = V_n - U_n$, U_n is the nth inter-arrival time and V_n the nth service time. It is assumed that the RVs X_n are IID, with a common cumulative distribution function (CDF) F_X. In all of Sections 5–7, we assume that F_X is a proper CDF on $\mathbb{R}$. The latter signifies that $\lim\limits_{x\to-\infty} F_X(x) = 0$, $\lim\limits_{x\to\infty} F_X(x) = 1$, i.e., that the RV X_n take finite values only.

Let W_n be the nth waiting time. The Lindley recursive equation (originated in [17]) states:

$$W_{n+1} = (W_n + X_n)^+, \quad n \geq 0, \tag{5.1}$$

with some given RV $W_0 \geq 0$, assumed to be independent of $\{X_n\}$. Here and below, we set $Y^+ = 0 \vee Y$. Then $\{W_n,\ n \geq 0\}$ is a discrete-time Markov chain (DTMC) on $\mathbb{R}_+ = [0, \infty)$.

It is known (see, e.g., [2, Chapter X.1]) that if $\mathbb{E}[X] < 0$, there exists a unique stationary distribution of DTMC $\{W_n\}$. (In fact, for $\mathbb{E}[X] < 0$, the DTMC $\{W_n\}$ is Harris ergodic.) The stationary distribution is characterized by a proper CDF F_W on $\mathbb{R}_+$ determined as a unique solution to the stationary Wiener–Hopf (WH) equation

$$F_W(t) = (F_X * F_W)(t)\mathbf{1}_{\mathbb{R}_+}(t), \quad t \in \mathbb{R}. \tag{5.2}$$

Here and below, $G_1 * G_2$ means the convolution of CDFs G_i:

$$G_1 * G_2(t) = \int_{\mathbb{R}} G_1(t-y)dG_2(y) = G_2 * G_1(t), \quad t \in \mathbb{R}.$$

When $\mathbb{E}[X] \geq 0$, (5.2) has no solution among proper CDFs (again, see [2, Chapter X.1]).

A stochastic version of equation (5.2) reads

$$W \overset{\mathrm{d}}{=} (W + X)^+ . \tag{5.3}$$

Here X and W are 'generic' RVs with CDFs F_X and F_W, respectively, independent of each other, and $\overset{\mathrm{d}}{=}$ means "equality in distribution".

5.2. *The modified Lindley recursion for a* GI/GI/1 *queue with resetting.* We consider a GI/GI/1 model where random resettings occur independently at arrival times. That is, the $(n+1)$st reset waiting time $W^{\mathtt{R}}_{n+1}$ either continues as in (5.1) with probability $q \in (0,1)$, or is set to be 0 with probability $1-q$, independently of $(X_k, W^{\mathtt{R}}_k)$ with $0 \leq k \leq n$. Recursively, it can be expressed as follows:

$$W^{\mathtt{R}}_{n+1} = Z_{n+1}(W^{\mathtt{R}}_n + X_n)^+, \quad n \geq 0. \tag{5.4}$$

Here $\{Z_n : n \geq 1\}$ is a sequence of IID Bernoulli RVs with probability $\mathbb{P}(Z_n = 0) = q = 1 - \mathbb{P}(Z_n = 1)$, independent of $\{X_n\}$. Equivalently, we can write

$$W^{\mathtt{R}}_{n+1} = \begin{cases} 0, & \text{with probability } q, \\ (W^{\mathtt{R}}_n + X_n)^+, & \text{with probability } 1-q, \end{cases} \tag{5.5}$$

independently of $(X_k, W^{\mathtt{R}}_k)$ with $0 \leq k \leq n$.

Equations (5.4) and (5.5) are referred to as a modified Lindley recursion with resetting. The sequence $\{W^{\mathtt{R}}_n\}$ forms a DTMC on $\mathbb{R}_+$. We show that it is Harris ergodic: this implies that the DTMC $\{W_n^{\mathtt{R}}\}$ has a unique stationary distribution, and the corresponding CDF, denoted by $F_{W^{\mathtt{R}}}$, is proper on $\mathbb{R}_+$ and has $F_{W^{\mathtt{R}}}(0)>0$.

Proposition 5.1. *For any $q\in(0,1)$ and a sequence of* IID RV*s* $\{X_n\}$*, the* DTMC $\{W_n^{\mathtt{R}}\}$ *is Harris ergodic.*

Proof. Observe that the process is regenerative (possibly after the first cycle in case the system starts from $W_0^{\mathtt{R}}>0$), with the cycles $(W_1^{\mathtt{R}},\dots,W_T^{\mathtt{R}})$ where $W_1^{\mathtt{R}}=0$ and $W_T^{\mathtt{R}}$ for T being geometric of parameter q. That is, the state 0 forms a regeneration set. Hence, the time for the chain to return to state 0 has a finite mean. Then, Harris ergodicity is straightforward. □

The stationary CDF $F_{W^{\mathtt{R}}}$ is identified as a solution to a stationary WH equation with resetting

$$F_{W^{\mathtt{R}}}(t)=\Big[q+(1-q)(F_{W^{\mathtt{R}}}*F_X)(t)\Big]\mathbf{1}_{\mathbb{R}_+}(t),\quad t\in\mathbb{R},\tag{5.6}$$

or to its stochastic analog

$$W^{\mathtt{R}}\stackrel{\mathrm{d}}{=}Z(W^{\mathtt{R}}+X)^+\,.\tag{5.7}$$

Here X and $W^{\mathtt{R}}$ are 'generic' RVs with CDFs F_X and $F_{W^{\mathtt{R}}}$, respectively, and Z is a Bernoulli RV with $\mathbb{P}(Z=0)=q=1-\mathbb{P}(Z=1)$. Furthermore, the RVs X, $W^{\mathtt{R}}$ and Z are independent, and, as before, $\stackrel{\mathrm{d}}{=}$ means "equality in distribution".

5.3. *The operator calculus for a* GI/GI/1 *queue with resetting.* It is convenient to write equation (5.6) in an operator form:

$$\begin{gathered}F_{W^{\mathtt{R}}}=q\mathbf{1}_{\mathbb{R}_+}+(1-q)\mathtt{K}F_{W^{\mathtt{R}}}\\ \text{or, equivalently,}\quad \big(\mathtt{I}-(1-q)\mathtt{K}\big)F_{W^{\mathtt{R}}}=q\mathbf{1}_{\mathbb{R}_+}.\end{gathered}\tag{5.8}$$

Here the operator $\mathtt{K}$ acts on a CDF H by the convolution

$$(\mathtt{K}H)(t)\ =\ (H*F_X)(t)\mathbf{1}_{\mathbb{R}_+}(t),\quad t\in\mathbb{R},\tag{5.9}$$

and $\mathtt{I}$ stands for the unit map. [1] In other words, if Y is an RV with CDF H, then $\mathtt{K}H$ is the CDF of $(X+Y)^+$ where X and Y are taken to be independent.

Equation (5.8) is solved by the series

$$\begin{aligned}F_{W^{\mathtt{R}}}&=\big(\mathtt{I}-(1-q)\mathtt{K}\big)^{-1}(q\mathbf{1}_{\mathbb{R}_+})=q\sum_{j\geq0}(1-q)^j\mathtt{K}^j\mathbf{1}_{\mathbb{R}_+}\\ &=q\Big[\mathbf{1}_{\mathbb{R}_+}+(1-q)F_X\mathbf{1}_{\mathbb{R}_+}+(1-q)^2(F_X*(F_X\mathbf{1}_{\mathbb{R}_+}))\mathbf{1}_{\mathbb{R}_+}\\ &\qquad+(1-q)^3(F_X*((F_X*(F_X\mathbf{1}_{\mathbb{R}_+}))\mathbf{1}_{\mathbb{R}_+}))\mathbf{1}_{\mathbb{R}_+}+\dots\Big].\end{aligned}\tag{5.10}$$

If the series on the RHS of (5.10) converges, say, point-wise, and the limit is a proper CDF with support in $\mathbb{R}_+$, then we get a workable representation of $F_{W^{\mathtt{R}}}$.

[1]Operator $\mathtt{K}$ is linear in the sense that $\mathtt{K}\left(\sum\limits_{i=1}^n\alpha_iG_i\right)=\sum\limits_{i=1}^n\alpha_i\mathtt{K}G_i$. for any $\alpha_1,\dots,\alpha_n\geq0$ with $\sum\limits_{i=1}^n\alpha_i=1$ and CDFs G_1, …, G_n. In this paper we do not explore further properties of $\mathtt{K}$, including a specification of the domain in a linear space and the issue of its boundedness/continuity/compactness of $\mathtt{K}$. A similar view will be adopted in Sects 6 and 7.

In fact, we have the point-wise bounds involving F_{X^+}, the CDF of RV $X^+ = X \vee 0$:

$$\mathbf{1}_{\mathbb{R}_+} \geq F_{W^{\mathtt{R}}} \geq q\Big[\mathbf{1}_{\mathbb{R}_+} + (1-q)F_{X^+} + (1-q)^2 F_{X^+} * F_{X^+} + (1-q)^3 F_{X^+} * F_{X^+} * F_{X^+} + \dots\Big] = \big(\mathtt{I} - (1-q)\mathtt{K}^+\big)^{-1}(q\mathbf{1}_{\mathbb{R}_+}) \tag{5.11}$$

where $\mathtt{K}^+$ acts on a CDF H by

$$(\mathtt{K}^+ H)(t) = (H * F_{X^+})(t), \quad t \in \mathbb{R}. \tag{5.12}$$

This is a consequence of stochastic ordering where $W^{\mathtt{R}} \geq 0$ and $Z(W^{\mathtt{R}} + X)^+ \leq_{\mathrm{so}} Z(W^{\mathtt{R}} + X^+)$. The upper bound in (5.11) makes sure that the series in both (5.10) and (5.11) converge point-wise on $\mathbb{R}$ (the non-trivial part is convergence on $\mathbb{R}_+$).

For methodological reasons, we will connect the series in (5.10) and (5.11) in the following Theorems 5.1 and 5.2.

Theorem 5.1. *Fix $q \in (0,1)$ and suppose that the series in (5.11) gives a proper* CDF *on $\mathbb{R}_+$. Then the series in (5.10) determines a proper* CDF *satisfying (5.6). Furthermore, for any $q \in (0,1)$, the equation (5.6) has a unique bounded solution, and this solution is given by the series in (5.10).*

Proof. The assertions of series convergence and validity of equation (5.8) follow from the construction and bounds in (5.11). It remains to check uniqueness. Let $G_1 : \mathbb{R} \to \mathbb{R}$ and $G_2 : \mathbb{R} \to \mathbb{R}$ be two bounded functions such that $G_i(t) = \Big[q + (1-q)(G_i * F_X)(t)\Big]\mathbf{1}(t \geq 0)$, for $t \in \mathbb{R}$, $i = 1, 2$. Set: $G := G_1 - G_2$, then

$$G(t) = (1-q)(G * F_X)(t)\mathbf{1}_{\mathbb{R}_+}(t), \quad t \in \mathbb{R}.$$

It implies that for the value $\gamma := \sup_{t \in \mathbb{R}} |G(t)|$, we get $\gamma \in (0, \infty)$ and

$$\gamma \leq (1-q)\gamma \quad \text{or, iterating ,} \quad \gamma \leq (1-q)^k, \quad \forall \quad k \geq 1.$$

As $k \to \infty$, it yields $\gamma = 0$. □

We now turn to the case where RV $X \geq 0$: $F_{W^{\mathtt{R}}}$ is determined by the series in (5.11), that is,

$$F_{W^{\mathtt{R}}} = q\Big[\mathbf{1}_{\mathbb{R}_+} + (1-q)F_X + (1-q)^2 F_X * F_X + (1-q)^3 F_X * F_X * F_X + \dots\Big]. \tag{5.13}$$

Let $\phi(\theta) = \mathbb{E}[e^{\mathrm{i}\theta X}]$ and $\psi(\theta) = \mathbb{E}[e^{\mathrm{i}\theta W^{\mathtt{R}}}]$ be the characteristic functions (CFs) of X and $W^{\mathtt{R}}$, respectively, for $\theta \in \mathbb{R}$. Then (5.13) is equivalent to

$$\psi(\theta) = q\sum_{k \geq 0}(1-q)^k(\phi(\theta))^k = \frac{q}{1-(1-q)\phi(\theta)}. \tag{5.14}$$

We obtain the following assertion.

Theorem 5.2. *For any $q \in (0,1)$ and* RV *$X \geq 0$, the* RV *$W^{\mathtt{R}}$ with* CDF *$F_{W^{\mathtt{R}}}$ and* CF *ψ as in (5.13) and (5.14) is a proper* RV *with values in $[0, \infty)$.*

As an example, we next consider the case X taking integer values.

Example 5.1. We start with the simplest scenario: $X = 1$. For simplicity, suppose that $W_0 = 0$. In the model without resetting, the waiting time $W_n = n \to \infty$ as $n \to \infty$. Cf. (5.1). In the presence of resetting, we have a DTMC $\{W_n^{\mathtt{R}}\}$ on the state space $c\mathbb{Z}_+$, with transition probabilities $P = (P_{ij})$: $P_{i,0} = q$ and $P_{i,i+1} = 1 - q$ for $i \geq 0$. The stationarity condition $\pi P = \pi$ gives $q \sum\limits_{i\geq 1} \pi_i = \pi_0$ and $(1-q)\pi_i = \pi_{i+1}$ for $i \geq 0$. Thus, we obtain that $\pi_i = q(1-q)^i$, $i \geq 0$, i.e., the RV $W^{\mathtt{R}}$ is geometric. It is straightforward that the CDF $F_{W^{\mathtt{R}}}$ and CF ψ satisfy (5.13) and (5.14).

Example 5.2. Next, assume that X takes values $k = 1, 2, \ldots$, with probabilities p_k and take again $W_0 = 0$. Then the DTMC $\{W_n^{\mathtt{R}}\}$ has transition probabilities $P_{i,0} = q$ for $i \geq 0$, and $P_{i,j} = (1-q)p_{j-i}$ for $i \geq 0$ and $j \geq i+1$; the remaining entries equal to zero. Let $\pi = (\pi_0, \pi_1, \ldots)$ be the stationary distribution. Then we get $q \sum\limits_{i=1}^{\infty} \pi_i = \pi_0$, and $(1-q) \sum\limits_{j=0}^{i} \pi_j p_{i-j} = \pi_i$ for $i \geq 1$. From this we obtain the equation for the characteristic functions: $\psi(\theta) = 1 + (1-q)\psi(\theta)\phi(\theta)$, which results in (5.14).

6. The GI/GI/r queue with random resetting at arrival times

6.1. *The Kiefer–Wolfowitz recursion for a* GI/GI/r *queue.* In a standard GI/GI/r queue with $r > 1$ servers and under the FCFS discipline, we operate with a collection of random vectors $\{\underline{W}_n\}$ where $\underline{W}_n = (W_{n1}, \ldots, W_{nr})$ and $0 \leq W_{n1} \leq \ldots \leq W_{nr}$. In other words, $\underline{W}_n$ takes values in the simplex $\mathbb{S}_{\leq}^{+} \subset \mathbb{R}^r$ where $\mathbb{S}_{\leq}^{+} = \Big\{ \underline{x} = (x_1, \ldots, x_r) :\ 0 \leq x_1 \leq \ldots \leq x_r \Big\}$. Pictorially, $\underline{W}_n$ represents the residual workload vector at the time of arrival of the nth job, and its smallest entry, W_{n1}, gives the waiting time for the nth job.

The recursion that generates the sequence $\{\underline{W}_n\}$ is due to Kiefer and Wolfowitz [15]:

$$\underline{W}_{n+1} = \Big[\mathcal{R}\big(\underline{W}_n + V_n\, \underline{e}^{(1)} \big) - U_n\, \underline{1}\ \Big]_+ \tag{6.1}$$

with the following ingredients on the RHS:

(i) V_n is the service time of the nth arrival, and U_n is the time between the nth and $(n+1)$st arrival. It is assumed that the pairs (U_n, V_n), $n = 0, 1, \ldots$, form an IID sequence. The joint CDF for (U_n, V_n) is denoted by G:

$$G(u, v) = \mathbb{P}(U_n \leq u, V_n \leq v). \tag{6.2}$$

We will assume that CDF G is proper, i.e., RVs U_n and V_n take finite values only.

(ii) $\underline{e}^{(1)} = (1, 0, \ldots, 0) \in \mathbb{Z}_+^r$ and $\underline{1} = (1, \ldots, 1) \in \mathbb{Z}_+^r$ are r-dimensional $0,1$-vectors.

(iii) $\mathcal{R}\big(\underline{W}_n + V_n\,\underline{e}^{(1)}\big) \in \mathbb{S}^+_{\leq}$ is the result of the re-arrangement operation $\mathcal{R}$ applied to the vector $\underline{W}_n + V_n\,\underline{e}^{(1)} \in \mathbb{R}^r_+$: the vector $\mathcal{R}\big(\underline{W}_n + V_n\,\underline{e}^{(1)}\big)$ has the same collection of entries as $\underline{W}_n + V_n\,\underline{e}^{(1)}$ re-arranged in the non-decreasing order.

(iv) $\Big[\mathcal{R}\big(\underline{W}_n + V_n\,\underline{e}^{(1)}\big) - U_k\,\underline{1}\,\Big]^+ \in \mathbb{S}^+_{\leq}$ is the vector obtained when the negative entries in $\mathcal{R}\big(\underline{W}_n + V_n\,\underline{e}^{(1)}\big) - U_k\,\underline{1}$ are replaced with zeros and non-negative entries are left intact.

Equation (6.1) generates a DTMC $\{\underline{W}_n,\ n = 0, 1, \ldots\}$ on $\mathbb{S}^+_{\leq}$. It can be re-written in terms of the r-dimensional CDFs $F_n(\underline{x}) = \mathbb{P}(\underline{W}_n \leq \underline{x})$, $n \geq 0$, as follows:

$$F_{n+1}(\underline{x}) = \int_{\mathbb{R}^2}\int_{\mathbb{S}^r} \mathbf{1}\Big(\underline{w} \in \mathbb{A}(\underline{x}, u, v)\Big) \mathrm{d}F_n(\underline{w})\mathrm{d}G(u, v), \quad \underline{x} \in \mathbb{R}^r, \tag{6.3}$$

where the set $\mathbb{A}(\underline{x}, u, v)) \subset \mathbb{S}^+_{\leq}$ is given by

$$\mathbb{A}(\underline{x}, u, v) = \left\{\underline{w} \in \mathbb{S}^+_{\leq} :\ \Big[\mathcal{R}(\underline{w} + v\underline{e}^{(1)}) - u\,\underline{i}\,\Big]^+ \leq \underline{x}\right\} \tag{6.4}$$

and $G(u, v)$ is given in (6.2). Here and below, the inequality between vectors means the inequality between their respective entries.

As before, it is instructive to write equation (6.3) in an operator form

$$\begin{aligned} &F_{n+1} = \mathtt{K}F_n \quad \text{where operator } \mathtt{K} \text{ acts on a CDF } H \text{ by} \\ &(\mathtt{K}H)(\underline{x}) = \int_{\mathbb{R}^2_+}\int_{\mathbb{S}^+_{\leq}} \mathbf{1}\Big(\underline{w} \in \mathbb{A}(\underline{x}, u, v)\Big) \mathrm{d}H(\underline{w})\mathrm{d}G(u, v), \quad \underline{x} \in \mathbb{R}^r. \end{aligned} \tag{6.5}$$

It is known that if the traffic intensity $\rho := \dfrac{\mathbb{E}[V]}{r\mathbb{E}[U]} < 1$, then the stationary Kiefer–Wolfowitz equation

$$\begin{aligned} &\underline{W} \simeq \Big[\mathcal{R}\big(\underline{W} + V\,\underline{e}^{(1)}\big) - U\,\underline{i}\,\Big]_+ \quad \text{or, equivalently, } F = \mathtt{K}F, \text{ i.e.,} \\ &F(\underline{x}) = \int_{\mathbb{R}^2}\int_{\mathbb{S}^r} \mathbf{1}(\underline{w} \in A(\underline{x}, u, v))\mathrm{d}F(\underline{w})\mathrm{d}G(u, v) \end{aligned} \tag{6.6}$$

has a unique solution giving a proper CDF F on $\mathbb{R}^r$. In fact, for $\rho < 1$, the DTMC $\{\underline{W}_n\}$ is Harris ergodic. On the other hand, when $\rho \geq 1$, there is no proper CDF F on $\mathbb{R}^r$ satisfying equation (6.6), Cf. [2, Chapter XII.2].

6.2. *The modified Kiefer–Wolfowitz recursion for a* GI/GI/r *queue with resetting.* The model with random resetting at arrival times again involves the parameter $q \in [0, 1)$. Set $\underline{0} = (0, \ldots, 0)$. Equations (6.1) and (6.3) are replaced with

$$\underline{W}^{\mathtt{R}}_{n+1} = \begin{cases} \underline{0}, & \text{with probability } q, \\ \Big[\mathcal{R}\big(\underline{W}^{\mathtt{R}}_n + V_n\,\underline{e}^{(1)}\big) - U_n\,\underline{1}\,\Big]_+, & \text{with probability } 1 - q, \end{cases} \tag{6.7}$$

and

$$F^{\mathtt{R}}_{n+1}(\underline{x}) = q\mathbf{1}_{\mathbb{S}^+_{\leq}}(\underline{x}) + (1 - q)\int_{\mathbb{R}^2}\int_{\mathbb{S}^+_{\leq}} \mathbf{1}\Big(\underline{w} \in \mathbb{A}(\underline{x}, u, v)\Big) \mathrm{d}F^{\mathtt{R}}_n(\underline{w})\mathrm{d}G(u, v), \tag{6.8}$$

respectively, with $F^{\mathtt{R}}_n(\underline{x}) = \mathbb{P}(\underline{W}_n \le \underline{x})$. As above, equation (6.7) determines a DTMC $\{\underline{W}^{\mathtt{R}}_n,\ n = 0, 1, \ldots\}$ on $\mathbb{S}^+_{\le}$. Again, the vector $\underline{W}^{\mathtt{R}}_n$ represents the residual workloads at the servers at the nth arrival time.

Accordingly, the stationary equations for $\{\underline{W}^{\mathtt{R}}_n\}$ take the following equivalent forms:

$$\begin{aligned}
&\text{(i)}\quad \underline{W}^{\mathtt{R}} \simeq \begin{cases} \underline{0}, & \text{with probability } q,\\ \Big[\mathcal{R}\big(\underline{W}^{\mathtt{R}} + V\,\underline{e}^{(1)}\big) - U\,\underline{1}\,\Big]_+, & \text{with probability } 1-q, \end{cases} \quad \text{or}\\
&\text{(ii)}\quad F^{\mathtt{R}}(\underline{x}) = q\mathbf{1}_{\mathbb{S}^+_{\le}}(\underline{x})\\
&\qquad\qquad +(1-q)\int_{\mathbb{R}^2}\int_{\mathbb{S}^+_{\le}} \mathbf{1}\Big(\underline{w} \in \mathbb{A}(\underline{x}, u, v)\Big)\mathrm{d}F_{W^{\mathtt{R}}_n}(\underline{w})\mathrm{d}G(u, v), \quad \text{or}\\
&\text{(iii)}\quad F^{\mathtt{R}} = q\mathbf{1}_{\mathbb{S}^+_{\le}} + (1-q)\mathtt{K}F^{\mathtt{R}} \quad\Longleftrightarrow\quad \big(\mathtt{I} - (1-q)\mathtt{K}\big)F = q\mathbf{1}_{\mathbb{S}^+_{\le}}\\
&\qquad \text{solved by}\\
&\text{(iv)}\quad F = q\big(\mathtt{I} - (1-q)\mathtt{K}\big)^{-1}\mathbf{1}_{\mathbb{S}^+_{\le}}\\
&\qquad\qquad = q\mathbf{1}_{\mathbb{S}^+_{\le}} + q(1-q)\mathtt{K}\mathbf{1}_{\mathbb{S}^+_{\le}} + q(1-q)^2\mathtt{K}^2\mathbf{1}_{\mathbb{S}^+_{\le}} + \ldots .
\end{aligned} \tag{6.9}$$

Proposition 6.1. *For any $q \in (0, 1)$ and a sequence of* IID RV *pairs* $\{(U_n, V_n)\}$, *the* DTMC $\{\underline{W}^{\mathtt{R}}_n\}$ *is Harris ergodic.*

Proof. We will only give here a sketch of the (rather tedious) proof as it does not contain serious novel elements. It is reduced to a repetition of arguments from [2, Chapters XII.1, XII.2]. The crux of the matter is Theorem 1.2 on page 432 in [2, Chapter XII.1] and Theorem 2.2 on page 345 in [2, Chapter XII.2] rewritten in a modified form for the DTMC with resetting $\{W^{\mathtt{R}}_n\}$. In turn, the proof of the modified theorems is based on analogs of Lemma 1.3 and Lemmas 2.3 and 2.4 in [2, Chapters XII.1, XII.2]. Such analogs connect the DTMC $\{W^{\mathtt{R}}_n\}$ with the majorizing Markov chain $\{\widetilde{W}^{\mathtt{R}}_n\}$ where arriving jobs are directed to servers in the cyclic order with probability $1-q$ and trigger resetting of the whole vector of residual workloads to $\underline{0}$ with probability q. The analysis of the majorizing DTMC $\{\widetilde{W}^{\mathtt{R}}_n\}$ is essentially reduced to the GI/GI/1 model with resetting which leads to the assertion of Proposition 6.1. □

The above construction then leads to the following result.

Theorem 6.1. *For any $q \in (0, 1)$, the series in* (6.9)(iv) *determines a proper* CDF *satisfying* (6.9)(ii)*. Furthermore, equation* (6.9)(ii) *has a unique bounded solution, and this solution is given by the series in* (6.9)(iv)*.*

Proof. As in the case of the model GI/GI/1 with resetting, the fact that the series in (6.9)(iv) gives a solution to (6.9)(ii) follows from the construction with the help of Proposition 6.1. Uniqueness is also established by the same argument as for the GI/GI/1 model. □

7. The GI/GI/∞ queue with random resetting at arrival times

7.1. *The recursion for a* GI/GI/∞ *queue.* A standard GI/GI/∞ can be described via a sequence of random vectors $\underline{W}_n = (W_{n1}, \ldots, W_{ns})$, $n = 0, 1, \ldots$, of a variable dimension $s = 0, 1, \ldots$, with entries $W_{n1} \geq \ldots \geq W_{ns} > 0$ for $s \geq 1$; for $s = 0$, one formally sets $\underline{W}_n = \underline{0}$. Pictorially, $\underline{W}_n$ represents the residual workload vector at the time of arrival of the nth job, and its largest entry, W_{n1}, gives the time needed for clearing the system of jobs entered before the nth arrival time. The equality $\underline{W}_n = \underline{0}$ means that the nth job finds an empty queue at its arrival. Formally, $\underline{W}_n$ takes values in the union $\underline{\mathbb{O}}_+(\geq) := \bigcup_{s\geq 0} \mathbb{O}_+^s(\geq)$ of simplexes $\mathbb{O}_+^s(\geq) = \big\{\underline{x} = (x_1, \ldots, x_s) : \ x_1 \geq \ldots \geq x_s > 0\big\}$ of varying dimension $s \geq 1$, and a single-state set $\mathbb{O}_+^0(\geq) = \{\underline{0}\}$ for $s = 0$.

The recursion for the residual workload vector $\{\underline{W}_n\}$ in the GI/GI/∞ model is given by

$$\underline{W}_{n+1} = \mathcal{R}\Big(\mathcal{S}\left\{\big[\mathcal{P}(V_n, \underline{W}_n) - U_n \underline{1}_{s(n)+1}\big]_+\right\}\Big), \quad n = 0, 1, \ldots. \tag{7.1}$$

Here the RHS contains the following components:

(i) As in Section 6, V_n is the service time of the nth arrival, and U_n is the time between the nth and $(n+1)$st arrival. It is assumed that the pairs (U_n, V_n), $n = 0, 1, \ldots$, form an IID sequence. The joint CDF for (U_n, V_n) is again denoted by G and assumed to be proper.

(ii) $s(n)$ is the dimension of $\underline{W}_n$ and the vector $\underline{1}_{s(n)+1} = (1, \ldots, 1) \in \mathbb{Z}_+^{s(n)+1}$ has all entries 1.

(iii) $\mathcal{P}(V_n, \underline{W}_n) \in \mathbb{R}^{s(n)+1}$ is the result of concatenation of the value V_n and the vector $\underline{W}_n$.

(iv) $\big[\mathcal{P}(V_n, \underline{W}_n) - U_n \underline{1}_{s(n)+1}\big]_+$ is the vector obtained from $\mathcal{P}(V_n, \underline{W}_n) - U_n \underline{1}_{s(n)+1}$ when negative entries are replaced with zeros and non-negative entries are left intact.

(v) $\mathcal{S}\left\{\big[\mathcal{P}(V_n, \underline{W}_n) - U_n \underline{1}_{s(n)+1}\big]_+\right\}$ is the result of shortening vector $\big[\mathcal{P}(V_n, \underline{W}_n) - U_n \underline{1}_{s(n)+1}\big]_+$ by removing the zero entries. The dimension of $\mathcal{S}\Big\{\big[\mathcal{P}(V_n, \underline{W}_n) - U_n \underline{1}_{s(n)+1}\big]_+\Big\}$ equals $s(n+1)$.

(iv) $\mathcal{R}\Big(\mathcal{S}\left\{\big[\mathcal{P}(V_n, \underline{W}_n) - U_n \underline{1}_{s(n)+1}\big]_+\right\}\Big)$ is the result of the re-arrangement applied to the vector $\mathcal{S}\left\{\big[\mathcal{P}(V_n, \underline{W}_n) - U_n \underline{1}_{s(n)+1}\big]_+\right\}$. The vector $\mathcal{R}\Big(\mathcal{S}\left\{\big[\mathcal{P}(V_n, \underline{W}_n) - U_n \underline{1}_{s(n)+1}\big]_+\right\}\Big)$ has the same collection of entries as $\mathcal{S}\left\{\big[\mathcal{P}(V_n, \underline{W}_n) - U_n \underline{1}_{s(n)+1}\big]_+\right\}$ re-arranged in the non-increasing order.

Equation (7.1) generates a DTMC $\{\underline{W}_n, \ n = 0, 1, \ldots\}$ on $\underline{\mathbb{O}}_+(\geq)$. The probability distribution of $\underline{W}_n$ on $\underline{\mathbb{O}}_+(\geq)$ is described by a sequence $\underline{F}_n = (F_{n,0}, F_{n,1}, F_{n,2}, \ldots)$ of marginal (non-normalized) CDFs $F_{n,k}$, where

$$F_{n,0} = \mathbb{P}\big(\underline{W}_n = \underline{0}\big), \quad F_{n,k}(\underline{x}^{(k)}) = \mathbb{P}\big(s(n) = k, \underline{W}_n \leq \underline{x}^{(k)}\big), \quad \underline{x}^{(k)} \in \mathbb{R}^k, \quad k \geq 1. \tag{7.2}$$

For convenience, we will still refer to $F_{n,k}$ as a CDF.

Equation (7.1) can be re-written in terms of the sequences $\underline{F}_n$, as follows:

$$\begin{aligned} F_{n+1,0} &= \sum_l \int_{\mathbb{R}_+^2} \int_{\mathbb{O}_+^l(\geq)} \mathbf{1}\Big(\underline{w} \in \mathbb{A}_{0,l}(u,v)\Big) \mathrm{d}F_{n,l}(\underline{w}) \mathrm{d}G(u,v), \\ F_{n+1,k}(\underline{x}^{(k)}) &= \sum_l \int_{\mathbb{R}_+^2} \int_{\mathbb{O}_+^l(\geq)} \mathbf{1}\Big(\underline{w} \in \mathbb{A}_{k,l}(\underline{x}^{(k)},u,v)\Big) \mathrm{d}F_{n,l}(\underline{w}) \mathrm{d}G(u,v), \\ &\qquad\qquad\qquad\qquad \underline{x}^{(k)} \in \mathbb{R}^k, \quad k \geq 1. \end{aligned} \tag{7.3}$$

Here, the sets $\mathbb{A}_{0,l}(u,v), \mathbb{A}_{k,l}(\underline{x}^{(k)},u,v)) \subset \mathbb{O}_+^l(\geq)$ are given by

$$\begin{aligned} \mathbb{A}_{0,l}(u,v) &= \Big\{\underline{w}^{(l)} \in \mathbb{O}_+^l(\geq) : \mathcal{R}\Big(\mathcal{S}\Big\{\big[\mathcal{P}(v,\underline{w}^{(l)}) - u\underline{1}_{l+1}\big]_+\Big\}\Big) = \underline{0}\Big\}, \\ \mathbb{A}_{k,l}(\underline{x}^{(k)},u,v) &= \Big\{\underline{w}^{(l)} \in \mathbb{O}_+^l(\geq) : \mathcal{R}\Big(\mathcal{S}\Big\{\big[\mathcal{P}(v,\underline{w}^{(l)}) - u\underline{1}_{l+1}\big]_+\Big\}\Big) \in \mathbb{O}_+^k(\geq), \\ &\qquad \mathcal{R}\Big(\mathcal{S}\Big\{\big[\mathcal{P}(v,\underline{w}^{(l)}) - u\underline{1}_{l+1}\big]_+\Big\}\Big) \leq \underline{x}^{(k)}\Big\}, \quad k \geq 1. \end{aligned} \tag{7.4}$$

As before, it is instructive to write equation (7.3) in an operator form

$$\begin{aligned} F_{n+1,k} &= \sum_l \mathtt{K}_{k,l} F_{n,l}, \quad \text{where operator } \mathtt{K}_{k,l} \text{ acts on a CDF } H_l \text{ on } \mathbb{R}^l \text{ by} \\ \mathtt{K}_{0,l} H_l &= \int_{\mathbb{R}_+^2} \int_{\mathbb{O}_+^l(\geq)} \mathbf{1}\Big(\underline{w}^{(l)} \in \mathbb{A}_{0,l}(u,v)\Big) \mathrm{d}H(\underline{w}^{(l)}) \mathrm{d}G(u,v), \\ (\mathtt{K}_{k,l} H_l)(\underline{x}^{(k)}) &= \int_{\mathbb{R}_+^2} \int_{\mathbb{O}_+^l(\geq)} \mathbf{1}\Big(\underline{w}^{(l)} \in \mathbb{A}_{k,l}(\underline{x}^{(k)},u,v)\Big) \mathrm{d}H(\underline{w}^{(l)}) \mathrm{d}G(u,v), \\ &\qquad\qquad\qquad \underline{x}^{(k)} \in \mathbb{R}^k, \; k \geq 1. \end{aligned} \tag{7.5}$$

For future use, it is convenient to introduce the operator $\underline{\mathtt{K}} = (\mathtt{K}_{k,l})$ with blocks $\mathtt{K}_{k,l}$ acting on the CDF sequences $\underline{H} = (H_0, H_1, H_2, \ldots)$:

$$\begin{aligned} \underline{\mathtt{K}}\,\underline{H} &= \Big((\underline{\mathtt{K}}\,\underline{H})_0, (\underline{\mathtt{K}}\,\underline{H})_1, (\underline{\mathtt{K}}\,\underline{H})_2, \ldots \Big) \\ &\text{where} \quad (\underline{\mathtt{K}}\,\underline{H})_k = \sum_l \mathtt{K}_{k,l} H_l, \quad k = 0, 1, \ldots. \end{aligned} \tag{7.6}$$

7.2. *The recursion for a* GI/GI/∞ *queue with resetting.* The recursion for a GI/GI/∞ model with resetting takes the form

$$\underline{W}^{\mathtt{R}}_{n+1} = \begin{cases} \underline{0}, & \text{with probability } q, \\ \mathcal{R}\Big(\mathcal{S}\Big\{\big[\mathcal{P}(V_n,\underline{W}_n) - U_n\underline{1}_{s(n)+1}\big]_+\Big\}\Big), & \text{with probability } 1-q. \end{cases} \tag{7.7}$$

It generates a DTMC $\{\underline{W}^{\mathtt{R}}_n, \; n = 0, 1, \ldots\}$ on $\underline{\mathbb{O}}$.

As above, we rewrite equation (7.7) in terms of the CDF $\underline{F}^{\mathtt{R}}_n = (F^{\mathtt{R}}_{n,0}, F^{\mathtt{R}}_{n,1}, \ldots)$, where

$$F^{\mathtt{R}}_{n,k} = \mathbb{P}\Big(\underline{W}^{\mathtt{R}}_n \in \mathbb{O}_+^k(\geq), \; \underline{W}_n \leq \underline{x}^{(k)}\Big).$$

Then, we have

$$\begin{aligned} &\underline{F}^{\mathtt{R}}_{n+1} = q\,\underline{\mathbf{1}}_{\,\underline{\mathbb{O}}_+(\geq)} + (1-q)\,\underline{\mathtt{K}}\,\underline{F}^{\mathtt{R}}_n, \quad \text{or, entry-wise,} \\ &F^{\mathtt{R}}_{n+1,k}(\underline{x}^{(k)}) = q\mathbf{1}_{\mathbb{O}^k_+(\geq)}(\underline{x}^{(k)}) + (1-q)\sum_l (\mathtt{K}_{k,l}F^{\mathtt{R}}_{n,l})(\underline{x}^{(k)}), \\ &\qquad\qquad\qquad\qquad\qquad\qquad \underline{x}^{(k)} \in \mathbb{R}^k, \quad k = 0, 1, \ldots. \end{aligned} \tag{7.8}$$

The stationary version becomes

$$\begin{aligned} &\underline{F}^{\mathtt{R}} = q\,\underline{\mathbf{1}}_{\,\underline{\mathbb{O}}_+(\geq)} + (1-q)\,\underline{\mathtt{K}}\,\underline{F}^{\mathtt{R}} \Longleftrightarrow (\underline{\mathtt{I}} - (1-q)\underline{\mathtt{K}})\underline{F}^{\mathtt{R}} = q\,\underline{\mathbf{1}}_{\,\underline{\mathbb{O}}_+(\geq)} \\ &\text{or, entry-wise,} \\ &F^{\mathtt{R}}_{n+1,k}(\underline{x}^{(k)}) = q\mathbf{1}_{\mathbb{O}^k_+(\geq)}(\underline{x}^{(k)}) + (1-q)\sum_l (\mathtt{K}_{k,l}F^{\mathtt{R}}_l)(\underline{x}^{(k)}), \quad k = 0, 1, \ldots. \end{aligned} \tag{7.9}$$

Equation (7.9) is solved by

$$\underline{F}^{\mathtt{R}} = q\Big(\underline{\mathbf{1}}_{\,\underline{\mathbb{O}}_+(\geq)} + (1-q)\underline{\mathtt{K}}\,\underline{\mathbf{1}}_{\,\underline{\mathbb{O}}_+(\geq)} + (1-q)^2\underline{\mathtt{K}}^2\,\underline{\mathbf{1}}_{\,\underline{\mathbb{O}}_+(\geq)} + \ldots\Big) \tag{7.10}$$

Therefore we conclude the following result.

Theorem 7.1. *The DTMC $\{\underline{W}^{\mathtt{R}}_n\}$ has a stationary probability distribution characterized by equation* (7.10).

8. Concluding Remarks

In this paper, we have considered the standard queueing models with random resettings. Several extensions are possible future works. First, an immediate extension would be to consider more general Markov chains with random resettings. It would be interesting to identify conditions under which an explicit stationary distribution could be derived. Some efforts in this direction are made in our forthcoming paper [21]. Second, for non–Markovian queues, it would be interesting to consider different forms of resettings other than those at arrival times. Third, diffusions have been established to approximate the performances of queues in heavy traffic. It would be also worth considering such diffusion models with random resetting, particularly, their ergodic properties and characterization of stationary distributions.

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