# Ergodic Risk Sensitive Control of Markovian Multiclass Many-Server Queues with Abandonment

SUMITH REDDY ANUGU AND GUODONG PANG

ABSTRACT. We study the optimal scheduling problem for a Markovian multiclass queueing network with abandonment in the Halfin–Whitt regime, under the long run average (ergodic) risk sensitive cost criterion. The objective is to prove asymptotic optimality for the optimal control arising from the corresponding ergodic risk sensitive control (ERSC) problem for the limiting diffusion. In particular, we show that the optimal ERSC value associated with the diffusion-scaled queueing process converges to that of the limiting diffusion in the asymptotic regime. The challenge that ERSC poses is that one cannot express the ERSC cost as an expectation over the mean empirical measure associated with the queueing process, unlike in the usual case of a long run average (ergodic) cost. We develop a novel approach by exploiting the variational representations of the limiting diffusion and the Poisson-driven queueing dynamics, which both involve certain auxiliary controls. The ERSC costs for both the diffusion-scaled queueing process and the limiting diffusion can be represented as the integrals of an extended running cost over an mean empirical measure associated with the corresponding extended processes using these auxiliary controls. For the lower bound proof, we exploit the connections of the ERSC problem for the limiting diffusion with a two-person zero-sum stochastic differential game. We also make use of the mean empirical measures associated with the extended limiting diffusion and diffusion-scaled processes with the auxiliary controls. One major technical challenge in both lower and upper bound proofs, is to establish the tightness of the aforementioned mean empirical measures for the extended processes. We identify nearly optimal controls appropriately in both cases so that the existing ergodicity properties of the limiting diffusion and diffusion-scaled queueing processes can be used.

## 1. Introduction

We study the dynamic scheduling problem for a Markovian multiclass queueing network with abandonment (in particular, the 'V' network) in the Halfin–Whitt regime, where the objective is to minimize a long-run average (ergodic) risk sensitive cost. Specifically, if $\hat{Q}^n$ is the diffusion-scaled queue-length process for different classes of customers (with scaling parameter $n$), the ergodic risk sensitive control (ERSC) problem is to minimize the cost

$$\limsup_{T\to\infty} \frac{1}{T} \log \mathbb{E}\left[e^{\int_0^T \widetilde{r}(\hat{Q}^n_t)dt}\right]$$

for a running cost function $\widetilde{r}$. The optimization is over all work-conserving scheduling policies, allowing preemption, that is, the allocation of service capacity to different classes at each time (no server will be idling if there is a job waiting in queue). In the Halfin–Whitt regime, when the arrival rates, service rates and abandonment rates (all class-dependent) and the number of servers are scaled properly (with growing number of servers and arrival rates), the queueing dynamics can be approximated by a limiting diffusion (see [11]). The goal of this paper is to show the asymptotic optimality of the solution to the ERSC problem for the limiting diffusion, in particular, the optimal

‡Department of Computational Applied Mathematics and Operations Research, George R. Brown School of Engineering, Rice University, Houston, TX 77005

*E-mail addresses*: anugu.reddy, gdpang@rice.edu.

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value for the ERSC problem for diffusion-scaled processes converges to that of the ERSC problem for the limiting diffusion.

The optimal scheduling problem for the 'V' network model has been studied under the infinite-horizon discounted cost in [11] and ergodic cost in [4]. These cost criteria do not take into account risk sensitivity in the decision making, in particular, the exponential cost criterion concerns all the moments of the queueing process, and particularly stresses the penalty when the queue lengths are large. Risk sensitive control of stochastic networks has been studied to a limited extent. For example, Atar et al. [10] studied a finite-horizon risk sensitive control problem for a Markovian multiclass parallel server model in the conventional heavy traffic regime (law of large numbers scaling), and solved the problem via a deterministic differential game. Atar and Cohen [9] studied a differential game arising from a finite-horizon risk sensitive control problem for a multiclass single-server queueing model in the moderate-deviation regime. Biswas [16] studied a similar problem in [9] for the network model with multiple servers in the moderate deviation regime, where the optimal control problem is also related to a differential game. Atar and Saha [12] studied the optimality of the generalized $c\mu$ rule for the multiclass single-server network under the finite-horizon risk-sensitive cost criterion. All these studies concern finite horizon control problems. As far as we know, our work is the first to study infinite-horizon ergodic risk sensitive control problems for stochastic networks. The techniques we use are also drastically different from the existing literature.

ERSC problems for Markov processes have been extensively studied, see, e.g., recent surveys [13, 18]. The works on diffusions in [5, 15, 17, 19, 27, 28, 29] under various conditions on the stability of controlled diffusions and the running cost function are most relevant. In [1], the ERSC problem for diffusions is studied under the near-monotone cost condition in addition to assuming that dynamics is recurrent, without any condition on the stability (like positive recurrence or exponential ergodicity) of the dynamics. In [5], it is further studied under certain stronger versions of exponential ergodicity and certain growth conditions on the running cost function. For our limiting diffusion, uniform exponential ergodicity under any stationary Markov control is proved in [7] (see Theorem 2.2) and the running cost function is chosen to satisfy appropriate growth condition. Therefore, by [5, Theorem 4.1], we obtain the existence and characterization for the optimal ERSC problem of our limiting diffusion (see Theorem 2.3).

The techniques we use to prove asymptotic optimality differ from those in [4, 8], where the classical ergodic control (CEC) problems for the 'V' network and multiclass multi-pool network were studied, respectively. It is worth stressing that this difference is a necessity, rather than simply a technicality. Under positive recurrence of the diffusion-scaled queueing process, it is clear that the CEC cost can be written as an integral of running cost function with respect to the mean empirical (occupation) measure of queueing process and the scheduling control policy (say $\mu_n$ with $n$ denoting the scaling parameter) corresponding to the diffusion-scaled queueing process. Therefore, proving that $\{\mu_n\}_{n\in\mathbb{N}}$ is tight, can ensure that along a subsequence, the integral of the running cost function with respect to $\mu_n$ converges to integral of running cost function with respect to $\mu^*$ (with $\mu^*$ being the ergodic occupation measure associated with the limiting diffusion). In contrast, even under the positive recurrence (or even exponential ergodicity) of the diffusion-scaled queueing process, the ERSC cost cannot be expressed as the integral with respect to the mean empirical measure $\mu_n$. Therefore, simply proving tightness of $\{\mu_n\}_{n\in\mathbb{N}}$ and using the weak convergence of $\{\mu_n\}_{n\in\mathbb{N}}$ (along a subsequence to a limit) do not ensure the convergence of the sequence of ERSC cost for diffusion-scaled process to that of the limiting diffusion.

To overcome this difficulty, we make extensive use of certain variational representations (see Theorem 3.1 and 3.7) of functionals of Brownian motion and Poisson process. (We refer interested readers to [24] which presents many relevant techniques and results using these representations.) In particular, we first provide variational representations for the limiting diffusion and the Poisson-driven diffusion-scaled queueing processes using these results. These variational representations are in terms of some auxiliary controls, but the advantage is that these representations are linear in the running cost, and more importantly, the ERSC cost function for a given scheduling control policy can then be regarded as a CEC problem for an extended process under these auxiliary controls. As a consequence, one can rewrite the ERSC cost functions as the integrals of an extended running cost over the mean empirical measures associated with the corresponding extended processes using these auxiliary controls. Therefore, we can treat these ERSC problems as the corresponding CEC problems for the extended processes with the auxiliary controls by exploiting some of the existing techniques and results in the CEC theory (together with additional techniques).

We note that variational formulations have been used for risk sensitive control problems in the literature. In the case of a finite-horizon risk sensitive control problem, such a variational formulation was derived using the theory of large deviations in [30]. One of the first works studying the variational formulation of ERSC problem for diffusions is in [27, 28, 29], which was then followed by [2, 3, 5, 19]. In [29], the variational formulation was derived under restricted conditions of Markov controls that are continuous in their arguments and inf-compactness of the running cost function, whereas, in [28] the case of a linear-quadratic control problem is studied. However, all of these are in the context of diffusions under certain restrictive conditions, and not amenable to prove asymptotic optimality for our model.

We now explain the methodology of the proof of asymptotic optimality. One critical component in both lower and upper bound proofs concerns the ergodicity properties of the extended processes with the auxiliary controls arising from the variational representations. This cannot be deduced in a straightforward manner from the existing ergodicity properties of the limiting diffusion and the diffusion-scaled queueing processes. In the proof of the lower bound, we choose a nearly optimal scheduling policy and a careful choice of auxiliary control for the diffusion-scaled processes, under which the positive recurrence property of the extended diffusion-scaled process can be established. In the proof of the upper bound, we choose a nearly optimal control for the limiting diffusion control problem, and then construct a sequence of scheduling policies together with nearly optimal auxiliary controls. Then in order to prove tightness of the mean empirical measures associated with the extended diffusion-scaled processes, we prove tightness of a carefully chosen function of these nearly optimal controls.

For the proof of the lower bound, we make use of the connection of ERSC problems for diffusions with stochastic differential games. Such connections are recently studied in [3] (under near-monotonicity for running cost functions) and in [2] (under blanket/uniform stability for controlled diffusions). For earlier results, we refer to the references in [2, 3]. For our purpose, we first show that the optimal value of ERSC problem for diffusions is equal to that (where supremum and infimum operations are interchangeable) of a two-person zero-sum stochastic differential game (TP-ZS game, for short), where the optimizing criterion is a long time average of a certain extended running cost. (See Theorem 3.3. The closest work in terms of similarity of conditions on the dynamics and running cost is [2, Theorem 2.13], which is however given in terms of a Collatz–Wielandt formula.) Moreover, we show that there are compactly supported Markov strategies for the maximizing player that are nearly optimal. This uses a variant of spatial truncation technique (which was originally first introduced in [4, Section 4.1] in the context of CEC problems). In this technique, we show that the aforementioned TP-ZS game is a limit of a family of TP-ZS games where the allowed maximizing strategies are compactly supported. Using these results and choosing appropriately nearly optimal controls, on large enough compact sets, we make the aforementioned careful choice of auxiliary controls. This also helps us to prove the required positive recurrence of extended processes from the positive recurrence of the original process. This is because their respective infinitesimal generators coincide outside a large compact set. From this positive recurrence and the aforementioned interchangeability of supremum and infimum operations, we bound the optimal value of ERSC problem for the limiting diffusion from above by the integral of the aforementioned running cost with respect to any ergodic occupation measure corresponding to the nearly optimal compactly supported Markov strategies for the maximizing player. This leads to the lower bound for asymptotic optimality.

For the proof of the upper bound, as a consequence of the variational representation for the diffusion-scaled queueing process, the ERSC cost can be written in terms of the integral of a certain extended running cost function over the mean empirical (occupation) measure of the extended process with auxiliary controls. This extended running cost function is a sum of the original running cost function and an extra term that plays the role of a certain Radon-Nikodym derivative. Recall that the upper bound proof for the CEC problem of the same multiclass queueing model also employs such a strategy while the mean empirical measure only concerns the diffusion-scaled controlled queueing process itself (see Section 5.2 in [4]), for which tightness of these mean empirical measures as a result of uniform stability of the diffusion-scaled queueing process plays a crucial role. For the ERSC problem, despite the advantage of the representation using the mean empirical (occupation) measure mentioned above, we face the additional challenge to establish its tightness property, in particular, concerning the auxiliary controls. One difficulty comes from the fact that the nearly optimal auxiliary controls take values in a non-compact space and are parametrized by both a finite time and the scaling parameter $n$. We introduce a suitable topology in Section 3.3 that is appropriately weak to establish compactness of the set of auxiliary controls (see Lemma 3.4, 3.5 and Corollary 3.3). To be more elaborate, under this topology, the extra term in the extended cost behaves like an inf-compact function. In particular, bounding this term implies compactness under this topology. Using a Lyapunov function, we show in Lemma 5.4 that the family of the extended processes associated with any compact set of auxiliary controls (under this topology) is stable uniformly in $n$. Moreover, to prove the upper bound, we use appropriate truncations on both terms in the extended cost function in order to invoke compactness arguments and then take corresponding limits with these truncations in the appropriate topologies. Because of the exponential functional in the ERSC objective, the techniques used in taking the truncation to the limit resemble closely the techniques in the theory of large deviations [26] (see Lemmas 5.1 and 5.3, and Corollary 5.1). In contrast to this, the CEC problem does not require such a truncation of running cost function as all the relevant controls are usually compact space valued. These results give us the desired tightness of the mean empirical measures and connections between the ERSC objective function and that associated with the variational representation, and therefore complete the proof of the upper bound.

1.1. **Organization of the paper.** In the rest of this section, we introduce the notation used in the paper. Section 2 presents the model description, ERSC problem formulation, and main results. Section 2.1 describes the network model in detail, and presents the ERSC formulation for the diffusion-scaled processes. Section 2.2 presents the ERSC problem for the limiting diffusion, and the characterization of the solution to the ERSC problem. Section 2.3 gives the main result on asymptotic optimality, and provides an overview of the main ideas for its proof. Sections 3.1 and 3.2 present the variational formulations for the limiting diffusion and the Poisson-driven diffusion-scaled queueing dynamics, respectively. Some important properties associated with these variational formulations are proved in these sections and Sections 3.3 and 3.4 in order to be used in the proofs for asymptotic optimality. Sections 4 and 5 prove the lower and upper bounds, respectively, for asymptotic optimality. Finally, we collect some auxiliary results and their proofs in the Appendix.

1.2. **Notation.** We use $(\Omega, \mathcal{F}, \mathbb{P})$ to denote the underlying abstract probability space with $\mathbb{E}$ as the associated expectation. $\mathbb{E}_x$ denotes the expectation when the underlying process starts at $x$. The standard Euclidean norm in $\mathbb{R}^d$ is denoted by $\|\cdot\|$, $x \cdot y$ denotes the inner product of $x, y \in \mathbb{R}^d$, and $x^{\mathsf{T}}$ denotes the transpose of $x \in \mathbb{R}^d$. The set of nonnegative real numbers (integers) is denoted by $\mathbb{R}_+$ ($\mathbb{Z}_+$), $\mathbb{N}$ stands for the set of natural numbers, and $\mathbb{1}_{\{\cdot \in A\}}$ denotes the indicator function corresponding to set $A$. The minimum (maximum) of two real numbers $a$ and $b$ is denoted by

$a \wedge b$ ($a \vee b$), respectively, and $a^{\pm} \doteq (\pm a) \vee 0$. The closure, boundary, and complement of a set $A \subset \mathbb{R}^d$ are denoted by $\bar{A}$, The term domain in $\mathbb{R}^d$ refers to a nonempty, connected open subset of $\mathbb{R}^d$. For a domain $D \subset \mathbb{R}^d$, the space $\mathcal{C}^k(D)$ ($\mathcal{C}^\infty(D)$, respectively), $k \geq 0$, refers to the class of all real-valued functions on $D$ whose partial derivatives up to order $k$ (any order, respectively) exist and are continuous. By $\mathcal{C}^{k,\alpha}(\mathbb{R}^d)$, we denote the set of functions that are $k$-times continuously differentiable and whose $k$-th derivatives are locally Hölder continuous with exponent $\alpha$. The space $L^p(D)$, $p \in [1, \infty)$, stands for the Banach space of (equivalence classes of) measurable functions $f$ satisfying $\int_D |f(x)|^p dx < \infty$, and $L^\infty(D)$ is the Banach space of functions that are essentially bounded in $D$. The standard Sobolev space of functions on $D$ whose generalized derivatives up to order $k$ are in $L^p(D)$, equipped with its natural norm, is denoted by $W^{k,p}(D)$, $k \geq 0, p \geq 1$. In general, if $\mathcal{X}$ is a space of real-valued functions on a set $Q$, $\mathcal{X}_{\text{loc}}$ consists of all functions $f$ such that $f\phi \in \mathcal{X}$ for every $\phi$ that is compactly supported smooth function on $Q$. Here, $f\phi$ is simply the scalar multiplication of the functions $f$ and $\phi$.

For $k \in \mathbb{N}$, we let $\mathfrak{D}^k \doteq D(\mathbb{R}_+, \mathbb{R}^k)$ ($\mathfrak{C}^k \doteq C(\mathbb{R}_+, \mathbb{R}^k)$, respectively) denote the space of $\mathbb{R}^k$ -valued càdlàg functions equipped with Skorohod topology (continuous functions equipped with locally uniform topology, respectively) on $\mathbb{R}_+$ ($\mathfrak{D}^1$ and $\mathfrak{C}^1$ are simply written as $\mathfrak{D}$ and $\mathfrak{C}$, respectively). Whenever the domain is $[0, T]$, we write $\mathfrak{D}^k_T$ or $\mathfrak{C}^k_T$ ($\mathfrak{D}_T$ or $\mathfrak{C}_T$, for $k = 1$). For a Polish space $\mathcal{X}$, $\mathcal{P}(\mathcal{X})$ is the set of Borel probability measures on $\mathcal{X}$ equipped with the topology of weak convergence. For a function $f : \mathbb{R}^d \to \mathbb{R}$,

$$\mathfrak{d}f(x; y) \doteq f(x + y) - f(y). \tag{1.1}$$

The identity function on the real line by $\mathfrak{e}$.

## 2. Model and Results

2.1. **ERSC for Multiclass $M/M/n + M$ queues.** We study a multiclass Markovian queueing model with $d$ classes of jobs/customers and one pool of $n$ parallel servers. Each class has an independent Poisson arrival process of rate $\lambda^n_i > 0$, $i = 1, \dots, d$. The service times for class $i$ jobs are i.i.d. exponential with rate $\mu^n_i > 0$. Jobs of each class form their own queue and are served in the first-come first-served (FCFS) discipline. Jobs can abandon while waiting in the queue, and class $i$ jobs have i.i.d. exponential patience times with rate $\gamma^n_i > 0$. Let $r^n_i = \lambda^n_i / \mu^n_i$ be the mean offered load of class $i$, and then the traffic intensity is given by $\rho^n = n^{-1} \sum_{i=1}^d r^n_i$. We assume that the system is operating in the Halfin–Whitt regime, in which the parameters are assumed to satisfy the following conditions: as $n \to \infty$,

$$\begin{gathered} \frac{\lambda^n_i}{n} \to \lambda_i > 0, \quad \frac{\lambda^n_i - n\lambda_i}{\sqrt{n}} \to \hat{\lambda}_i \in \mathbb{R}, \\ \mu^n_i \to \mu_i > 0, \quad \sqrt{n}(\mu^n_i - \mu_i) \to \hat{\mu}_i, \quad \gamma^n_i \to \gamma_i > 0, \end{gathered} \tag{2.1}$$

and

$$\tilde{\rho}^n \doteq \sqrt{n}(1 - \rho^n) \to \hat{\rho} = \sum_{i=1}^d \frac{\rho_i \hat{\mu}_i - \hat{\lambda}_i}{\mu_i} \in \mathbb{R}\,, \tag{2.2}$$

where $\rho_i = \frac{\lambda_i}{\mu_i} < 1$ satisfies $\sum_{i=1}^d \rho_i = 1$. It is clear that $n^{-1} r^n_i \to \rho_i$ for each $i$. In addition, for each $i = 1, \dots, d$, as $n \to \infty$,

$$\ell^n_i = \frac{\lambda^n_i - n\lambda_i}{\sqrt{n}} - \rho_i \sqrt{n}(\mu^n_i - \mu_i) \to \ell_i = \frac{\hat{\lambda}_i - \rho_i \hat{\mu}_i}{\mu_i}\,.$$

Denote $\ell = (\ell_1, \dots, \ell_d)^{\mathsf{T}}$.

Let $\{X^n_{i,t}\}_{t \geq 0}$, $\{Q^n_{i,t}\}_{t \geq 0}$ and $\{Z^n_{i,t}\}_{t \geq 0}$ be the total numbers of class $i$ jobs in the system, in the queues, and in service at each time, respectively. Write $X^n = (X^n_1, \dots, X^n_d)^{\mathsf{T}}$ as the $d$-dimensional processes, and similarly, for $Q^n, Z^n$ and so on. The processes $\{Z^n_{i,t}\}_{t\geq 0}$ also represent the server allocation at each time, and hence they are regarded as "scheduling control policies" (SCPs). We will consider work-conserving policies that are non-anticipative and allow preemption, which require that

$$e\cdot Z^n_t = (e\cdot X^n_t)\wedge n, \quad t\geq 0. \tag{2.3}$$

The action set $\mathbb{A}^n(x)$ is given by

$$\mathbb{A}^n(x) = \{z\in\mathbb{Z}^d_+ : z\leq x \text{ and } e\cdot z = (e\cdot x)\wedge n\}.$$

Let $\mathbb{U}^n$ be the set of all admissible control policies $Z^n\in\mathbb{A}^n(x)$ for any given $x\in\mathbb{Z}^d_+$. Here, the notion of admissibility is adopted from [4, Pg. 3520], given the balance equation

$$X^n_{i,t} = Q^n_{i,t} + Z^n_{i,t}, \quad t\geq 0, \quad i=1,\dots,d,$$

under a work-conserving control policy $Z^n$ satisfying (2.3), the process $X^n$ can be described by the following equation:

$$X^n_{i,t} = X^n_{i,0} + A^n_i(\lambda^n_i t) - S^n_i\left(\mu^n_i\int_0^t Z^n_{i,s}ds\right) - R^n_i\left(\gamma^n_i\int_0^t Q^n_{i,s}ds\right), \tag{2.4}$$

where $A^n_i, S^n_i, R^n_i$, $i=1,\dots,d$, are mutually independent standard Poisson processes. Let $\hat{X}^n$, $\hat{Q}^n$ and $\hat{Z}^n$ be the diffusion-scaled processes defined by

$$\hat{X}^n_{i,t} = \frac{1}{\sqrt{n}}(X^n_{i,t} - n\rho_i)\,, \quad \hat{Q}^n_{i,t} = \frac{1}{\sqrt{n}}Q^n_{i,t}\,, \quad \hat{Z}^n_{i,t} = \frac{1}{\sqrt{n}}(Z^n_{i,t} - n\rho_i)\,.$$

We now introduce a process $U^n$ which from now on shall be interpreted as control. $U^n$ is defined as follows: for $t\geq 0$

$$U^n_t \doteq \begin{cases} \frac{\hat{X}^n_t - \hat{Z}^n_t}{(e\cdot\hat{X}^n_t)^+}, \text{whenever } (e\cdot\hat{X}^n_t)^+ > 0, \\ e_d, \text{ otherwise.}\end{cases} \tag{2.5}$$

Here, $e_d = (0,\dots,1)\in\mathbb{R}^d$. Observe that $U^n\in\mathbb{U} = \{u\in\mathbb{R}^d_+ : e\cdot u = 1\}$. With slight abuse to notation, we denote the set of controls $U^n$ that correspond to admissible SCPs also by $\mathbb{U}^n$. Evidently, for the diffusion-scaled processes, in terms of the control $U^n$, we have

$$\hat{Q}^n_t = (e\cdot\hat{X}^n_t)^+ U^n_t, \quad \hat{Z}^n_t = \hat{X}^n_t - (e\cdot\hat{X}^n_t)^+ U^n_t\,. \tag{2.6}$$

Given the initial state $X^n_0$ and a work-conserving control policy $Z^n\in\mathbb{U}^n$ (equivalently, $U^n\in\mathbb{U}^n$, which we use instead), the ergodic risk sensitive cost function for the diffusion-scaled state process $\hat{X}^n$ is given by

$$J^n(\hat{X}^n_0, U^n) \doteq \limsup_{T\to\infty}\frac{1}{T}\log\mathbb{E}\left[e^{\int_0^T \kappa\cdot[(e\cdot\hat{X}^n_t)^+ U^n_t]dt}\right] \tag{2.7}$$

for some constant $\kappa\in\mathbb{R}^d_+$. Here the objective function penalizes the queueing process in the diffusion scale (represented by the diffusion-scaled queueing process $\hat{Q}^n$). Recall (2.6). For notational convenience, we write $r(\hat{X}^n_t, U^n_t) = \kappa\cdot[(e\cdot\hat{X}^n_t)^+ U_t]$, or $r(x,u) = \kappa\cdot[(e\cdot x)^+ u]$. (See also Remark 2.1 for the choice of this running cost function.) Whenever there is no confusion in the scheduling policy, we write $r(\hat{X}^n_t, U^n_t) = \widetilde{r}(\hat{Q}^n_t) \doteq \kappa\cdot\hat{Q}^n_t$.

The associated cost minimization problem is

$$\hat{\Lambda}^n(\hat{X}^n_0) \doteq \inf_{U^n\in\mathbb{U}^n} J^n(\hat{X}^n_0, U^n)\,. \tag{2.8}$$

We refer to $\hat{\Lambda}^n(\hat{X}^n_0)$ as the diffusion-scaled ergodic risk-sensitive value given the initial state $\hat{X}^n_0$. Before we proceed further, we show that the above minimization problem is well-defined. Define

$$\hat{x}^n(x) \doteq \Big(\frac{x_1 - n\rho_1}{\sqrt{n}}, \frac{x_2 - n\rho_2}{\sqrt{n}}, \dots, \frac{x_d - n\rho_d}{\sqrt{n}}\Big). \tag{2.9}$$

Whenever there is no confusion, we simply write $\hat{x}^n$. The generator of the diffusion-scaled queueing process $\hat{X}^n$ under a constant SCP $u \in \mathbb{U}^n$ is given by

$$\mathcal{L}^{n,u} f(\hat{x}^n) \doteq \sum_{i=1}^{d} \Big(\lambda^n_i \mathfrak{d} f(x; e_i) + \big(\mu^n_i z_i + \gamma^n_i q_i(x,z)\big)\mathfrak{d} f(x, -e_i)\Big) \tag{2.10}$$

with $q_i(x,z) = x_i - z_i$ and the notation $\mathfrak{d} f$ defined in (1.1). The following exponential ergodicity result is taken from [7, Theorem 3.4] which helps us in showing that the ERSC problem in (2.8) is well-defined.

**Theorem 2.1.** *There exist positive constants $\hat{C}_0$ and $\hat{C}_1$ (independent of $n$), an inf-compact $\mathcal{C}^2$ function $\mathcal{V}^n$ such that $\mathcal{V}^n \geq 1$ and*

$$\mathcal{L}^{n,u}\mathcal{V}^n(\hat{x}^n) \leq (\hat{C}_0 - \hat{C}_1\|\hat{x}^n\|)\mathcal{V}^n(\hat{x}^n), \tag{2.11}$$

*for sufficiently large $n$.*

**Assumption 2.1.** $\sqrt{d}\max_{1\leq i\leq d}\kappa_i < \hat{C}_1$, where $\kappa$ is as in the definition of $r$ and $\hat{C}_1$ is the constant from Theorem 2.1.

An immediate consequence of the above theorem and assumption is that $\hat{\Lambda}^n(\hat{X}^n_0) < \infty$. In other words, there is at least one $U^n \in \mathbb{U}^n$ such that $J^n(\hat{X}^n_0, U^n) < \infty$. Moreover, we have the following result.

**Lemma 2.1.** *Under Assumption 2.1, for every $U^n \in \mathbb{U}^n$,*

$$J(\hat{X}^n_0, U^n) < \infty. \tag{2.12}$$

*Proof.* Recall that $r(x,u) = \kappa \cdot [(e\cdot x)^+ u]$. From Assumption 2.1, $l_r(x) \doteq l(x) - \max_{u\in\mathbb{U}^n(x)} r(x,u)$ is inf-compact with $l(x) \doteq \hat{C}_1\|x\|$.

Using Theorem 2.1, $\mathcal{L}^{n,u}\mathcal{V}^n(\hat{x}^n) \leq (\hat{C}_0 - l(\hat{x}^n))\mathcal{V}^n(\hat{x}^n)$ and applying Itô's formula to $e^{\int_0^t l(\hat{X}^n_s)ds}\mathcal{V}^n(\hat{X}^n_t)$ and using the fact that $\mathcal{V}^n \geq 1$, we have

$$J^n_l \doteq \limsup_{T\to\infty} \frac{1}{T}\log\mathbb{E}\Big[e^{\int_0^T l(\hat{X}^n_t)dt}\Big] \leq \hat{C}_0 < \infty.$$

For any $U^n \in \mathbb{U}^n$, using the inf-compactness of $l_r$, we have

$$e^{\int_0^t r(\hat{X}^n_t, U^n_t)dt} \leq e^{\int_0^t l(\hat{X}^n_t)dt + \hat{C}_2 t},$$

for some large enough $\hat{C}_2 > 0$. This implies that $\hat{\Lambda}^n(\hat{X}^n_0) \leq J^n(\hat{X}^n_0, U^n) \leq J^n_l + \hat{C}_2 < \infty$, from the finiteness of $J^n_l$. This proves the result. □

The following corollary of Theorem 2.1 is required in the proof of Theorem 5.1 (in particular, in the proof of Lemma 5.1).

**Corollary 2.1.** *For $0 < \hat{C}_3 < \hat{C}_1$ and under any admissible control $U^n \in \mathbb{U}^n$,*

$$\limsup_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\log\mathbb{E}\Big[e^{\hat{C}_3\int_0^T \|\hat{X}^n_t\|dt}\Big] < \infty.$$

*Proof.* First note that $(\hat{C}_1 - \hat{C}_3)\|x\|$ is still inf-compact. Hence, following the arguments in the proof of Lemma 2.1, we get the result. □

2.2. **ERSC for the limiting diffusion.** It is shown in [11] that under the conditions in (2.1)–(2.2) and under the work-conserving control policies, if

$$n^{-\frac{1}{2}}\mathbb{Z}^d \ni \hat{X}_0^n \Rightarrow X_0 \in \mathbb{R}^d \tag{2.13}$$

as $n \to \infty$, $\hat{X}^n \Rightarrow X$ in $\mathfrak{D}^d$ as $n \to \infty$, where $X$ is a $d$-dimensional controlled diffusion given as a solution to

$$X_t = X_0 + \int_0^t b(X_s, U_s)ds + \Sigma W_t, \tag{2.14}$$

where

$$b(x,u) = \ell - R(x - (e\cdot x)^+ u) - (e \cdot x)^+ \Gamma u \tag{2.15}$$

with

$$R = \operatorname{diag}(\mu_1, \dots, \mu_d), \quad \Gamma = \operatorname{diag}(\gamma_1, \dots, \gamma_d), \quad \Sigma\Sigma^{\mathsf{T}} = \operatorname{diag}(2\lambda_1, \dots, 2\lambda_d).$$

In the rest of the paper, we assume that (2.13) holds with $X_0 = x$ being a deterministic constant. The process $U$ (referred to as control) is assumed to take values in $\mathbb{U} \doteq \{u \in \mathbb{R}^d_+ : e \cdot u = 1\}$.

**Definition 2.1.** A $\mathbb{U}$–valued process $U$ is called admissible if it satisfies the following: if $U_t = U_t(\omega)$ is jointly measurable in $(t,\omega) \in \mathbb{R}^+ \times \Omega$ and for every $0 \le s < t$, $W_t - W_s$ is independent of the completed filtration (with respect to $(\mathcal{F}, \mathbb{P})$) generated by $\{X_0, U_r, W_r : r \le s\}$. The set of all such controls is denoted by $\mathfrak{U}$.

Let $\mathfrak{U}_{\mathrm{SM}}$ denote the set of stationary Markov controls. In order to study the convergence of stationary Markov controls or existence of optimal stationary Markov controls, it is useful to consider a weaker notion of a stationary Markov control, *viz.,* relaxed control - the control is defined in the sense of distribution. To be more precise, a control $v$ is said to be a relaxed Markov control if $v = v(\cdot)$ is a Borel measurable map from $\mathbb{R}^d$ to $\mathcal{P}(\mathbb{U})$. In this case, we write $v_*(du|x)$ to distinguish the relaxed Markov control $v$ from other Markov controls which are referred to as precise Markov controls. Clearly, the set of relaxed Markov controls contains $\mathfrak{U}_{\mathrm{SM}}$. But with slight abuse of notation, we represent the set of relaxed Markov controls also by $\mathfrak{U}_{\mathrm{SM}}$. Under $v \in \mathfrak{U}_{\mathrm{SM}}$, the controlled diffusion $X$ in (2.14) has a unique solution [6, Theorem 2.2.4]. Moreover, under $v \in \mathfrak{U}_{\mathrm{SM}}$, $X$ is strongly Markov and the transition probabilities are locally Hölder continuous.

We denote the generator $\mathcal{L}^u : \mathcal{C}^2(\mathbb{R}^d) \mapsto \mathcal{C}(\mathbb{R}^d)$ of the controlled diffusion $X$ as

$$\mathcal{L}^u f(x) = \sum_{i=1}^d b_i(x,u)\frac{\partial}{\partial x_i} f(x) + \sum_{i=1}^d \lambda_i \frac{\partial^2}{\partial x_i^2} f(x)\,.$$

We write the generator $\mathcal{L}^u$ as $\mathcal{L}^v$ under $v \in \mathfrak{U}_{\mathrm{SM}}$. $\mathcal{L}^v$ is the generator of a strongly-continuous semigroup on $\mathcal{C}_b(\mathbb{R}^d)$, which is strong Feller. We denote by $\mathfrak{U}_{\mathrm{SSM}}$ the subset of $\mathfrak{U}_{\mathrm{SM}}$ that consists of *stable stationary Markov controls* (*i.e.,* stationary Markov controls under which $X$ is positive recurrent) and by $\mu_v$ the invariant probability measure of the process under the control $v \in \mathfrak{U}_{\mathrm{SSM}}$. For our model, $\mathfrak{U}_{\mathrm{SSM}} = \mathfrak{U}_{\mathrm{SM}}$ since the controlled diffusion $X$ is uniformly exponentially ergodic under all stationary Markov controls (see Theorem 2.2).

For the limiting diffusion $X$ in (2.14) with $X_0 = x$ and control $u \in \mathfrak{U}$, the ergodic risk sensitive cost function is given by

$$J(x,u) \doteq \limsup_{T\to\infty} \frac{1}{T} \log \mathbb{E}\left[e^{\int_0^T r(X_t,u_t)dt}\right]. \tag{2.16}$$

The associated cost minimization problem is

$$\Lambda(x) \doteq \inf_{u\in\mathfrak{U}} J(x,u) \quad \text{and} \quad \Lambda \doteq \inf_{x\in\mathbb{R}^d} \Lambda(x). \tag{2.17}$$

This is the optimal value for the ERSC problem for limiting diffusion given the initial state $x$. In addition, let

$$\Lambda_{\mathrm{SM}}(x) \doteq \inf_{u\in\mathfrak{U}_{\mathrm{SM}}} J(x,u)\,, \tag{2.18}$$

be the the optimal value over all stationary Markov controls given the initial state $x$. We will show that the optimal value is independent of the initial value. Denote

$$\Lambda_{\mathrm{SM}} = \inf_{x\in\mathbb{R}^d} \Lambda_{\mathrm{SM}}(x) \quad \text{and} \quad \Lambda = \inf_{x\in\mathbb{R}^d} \Lambda(x)\,.$$

For notational convenience, let

$$\Lambda_v(x) \doteq J(x, v(\cdot)) \quad \text{for} \quad v\in\mathfrak{U}_{\mathrm{SM}}.$$

We state the uniform stability result that is proved in [7] as it is vital to invoke the existing results on ERSC problems for diffusions (see [1, 5]). To that end, we have the following result [7, Theorem 2.1] for the limiting diffusion.

**Theorem 2.2.** *[7, Theorem 2.2] There exist a $\mathcal{C}^2(\mathbb{R}^d)$ inf-compact function $\mathcal{V}$ with $\mathcal{V}\ge 1$ and constants $C_0, C_1 > 0$ such that for every $u\in\mathbb{U}$,*

$$\mathcal{L}^u\mathcal{V}(x) \le \bigl(C_0 - C_1\|x\|^2\bigr)\mathcal{V}(x) \quad \text{for } x\in\mathbb{R}^d. \tag{2.19}$$

To keep the notation concise, whenever $v\in\mathfrak{U}_{\mathrm{SM}}$, we write $r(x,v(x))$ as $r^v(x)$. Whenever there is no confusion, we also write $r(x,u)$ as $r^u(x)$ for $u\in\mathbb{U}$. The following theorem gives the well-posedness and characterization of optimal stationary Markov controls, which follows directly from [5, Theorem 4.1].

**Theorem 2.3.** *Under Assumption 2.1, the following hold.*

(i) *The HJB equation*

$$\min_{u\in\mathbb{U}}\bigl[\mathcal{L}^u V(x) + r^u(x)\,V(x)\bigr] \;=\; \Lambda\,V(x) \qquad \forall\, x\in\mathbb{R}^d \tag{2.20}$$

*has a unique solution $V\in\mathcal{C}^2(\mathbb{R}^d)$, satisfying $V(0)=1$ and $\inf_{x\in\mathbb{R}^d} V(x) > 0$.*

(ii) *Any $v\in\mathfrak{U}_{\mathrm{SM}}$ that satisfies*

$$\mathcal{L}^v V(x) + r^v(x)\,V(x) \;=\; \min_{u\in\mathbb{U}}\,\bigl[\mathcal{L}^u V(x) + r^u(x)\,V(x)\bigr] \quad \text{for a.e. } x\in\mathbb{R}^d \tag{2.21}$$

*is stable, and is optimal in the class $\mathfrak{U}_{\mathrm{SM}}$, i.e., $\Lambda_v(y) = \Lambda$ for all $y\in\mathbb{R}^d$.*

(iii) *Every optimal stationary Markov control satisfies (2.21).*

*Remark* 2.1. We have chosen $r(x,u)$ to be in the form $\kappa\cdot[(e\cdot x)^+u]$. The reason for not choosing a more general $r(x,u)$ is as follows. The main contribution of this work is proving the asymptotic optimality, where we show that the ERSC problem for the diffusion-scaled queueing process can be approximated by that of the limiting diffusion as $n\to\infty$ (in an appropriate sense). To address this, we have to show that ERSC problem for the diffusion-scaled process is well-defined for the chosen running cost function for large enough $n$.

From Theorem 2.1 and the arguments in the proof of Lemma 2.1, it is clear that the ERSC problem for the diffusion-scaled queueing process is well-defined for large enough $n$ as long as

$$\hat{C}_1\|x\| - \max_{u\in\mathbb{U}^n(x)} r(x,u) \text{ is inf-compact.}$$

On the other hand, to invoke [5, Theorem 4.1] (and thereby proving that the ERSC problem for the limiting diffusion is well-defined), the conditions on $r(x,u)$ can be relaxed as long as

$$C_1\|x\|^2 - \max_{u\in\mathbb{U}} r(x,u) \text{ is inf-compact.}$$

Here, $C_1$ is the constant from Theorem 2.2. We therefore have chosen $r(x,u)$ to be in the form $\kappa\cdot[(e\cdot x)^+u]$. □

2.3. **Asymptotic optimality.** We are now in a position to state the main result of this paper.

**Theorem 2.4.** *Under Assumption 2.1,*

$$\lim_{n\to\infty} \hat{\Lambda}^n(\hat{X}_0^n) = \Lambda.$$

The proof is given in the later sections. In the following, we give a brief overview of the proof which is divided into two parts: In the first part, we show the lower bound

$$\liminf_{n\to\infty} \hat{\Lambda}^n(\hat{X}_0^n) \geq \Lambda \tag{2.22}$$

in Section 4 and in the second part, we show the upper bound

$$\limsup_{n\to\infty} \hat{\Lambda}^n(\hat{X}_0^n) \leq \Lambda \tag{2.23}$$

in Section 5. We now illustrate the ideas to prove these two parts below. We do not explicitly give definitions of the processes and sets of controls here and refer the reader to the later sections. We begin by viewing the ERSC problem (for the limiting diffusion) from a different perspective, using the following variational representation: for $v \in \mathfrak{U}_{\mathrm{SM}}$,

$$\limsup_{T\to\infty} \frac{1}{T} \log \mathbb{E}\Big[e^{\int_0^T r^v(X_t)dt}\Big] = \limsup_{T\to\infty} \sup_{w\in\mathcal{A}} \mathbb{E}\left[\frac{1}{T}\int_0^T \Big(r^v(X_t^{*,v,w}) \ - \frac{1}{2}\|w_t\|^2\Big)dt\right]. \tag{2.24}$$

This implies

$$\Lambda = \inf_{v\in\mathfrak{U}_{\mathrm{SM}}} \limsup_{T\to\infty} \sup_{w\in\mathcal{A}} \mathbb{E}\left[\frac{1}{T}\int_0^T \Big(r^v(X_t^{*,v,w}) \ - \frac{1}{2}\|w_t\|^2\Big)dt\right]. \tag{2.25}$$

Here, $w$ is a new auxiliary control, and $X^{*,v,w}$ is an extended process (see equation (3.4)) associated to the limiting diffusion $X$ and that auxiliary control. (When $w \equiv 0$, $X = X^{*,v,w}$.) In Section 3.1, we show that

$$\Lambda = \sup_{w\in\mathfrak{W}_{\mathrm{SM}}} \inf_{v\in\mathfrak{U}_{\mathrm{SM}}} \limsup_{T\to\infty} \mathbb{E}\left[\frac{1}{T}\int_0^T \Big(r^v(X_t^{*,v,w}) \ - \frac{1}{2}\|w_t\|^2\Big)dt\right]. \tag{2.26}$$

Here, $\mathfrak{W}_{\mathrm{SM}} \subset \mathcal{A}$ is an appropriate set of stationary Markov controls. Note the switch between $\sup_{w\in\mathfrak{W}_{\mathrm{SM}}}$ and $\inf_{v\in\mathfrak{U}_{\mathrm{SM}}}$.

Next, we re-express the ERSC cost for diffusion-scaled queueing process in the following way. For a given SCP $Z^n$ (with $\hat{Q}_t^n$ depending on $Z^n$), we show the following representation to hold in Section 3.2:

$$\limsup_{T\to\infty} \frac{1}{T} \log \mathbb{E}\Big[e^{\int_0^T \widetilde{r}(\hat{Q}_t^n)dt}\Big] = \limsup_{T\to\infty} \sup_{\psi\in\mathcal{E}^n} \mathbb{E}\left[\frac{1}{T}\int_0^T \Big(\widetilde{r}(\hat{Q}_t^{n,\psi}) - \lambda^n \varkappa(\phi_t) - n\mu^n\varkappa(\psi_t) - n\gamma^n\varkappa(\varphi_t)\Big)dt\right]. \tag{2.27}$$

Here, $\psi = (\phi, \psi, \varphi)$ and $\hat{Q}^{n,\psi}$ are a new auxiliary control and an extended process of $\hat{Q}^n$ associated to that control, respectively. When $\psi \equiv (e, e, e)$, then $\hat{Q}^n \equiv \hat{Q}^{n,\psi}$. When $\psi = \psi^n$, we write $\hat{Q}^{n,\psi^n}$. The process $\hat{X}^{n,\psi}$ is defined similarly with $\hat{X}^{n,\psi} \equiv \hat{X}^n$, whenever $\psi \equiv (e, e, e)$. See (3.31). Recall that $e = (1, 1, \dots, 1)^{\mathsf{T}} \in \mathbb{R}^d$.

These variational representations are crucial in the proofs of the lower and upper bounds. We illustrate how they are used below.

**Sketch proof of the lower bound in (2.22) (Theorem 4.1):**

For every $n$, we choose a nearly optimal SCP $Z^n$ for $\hat{\Lambda}^n(\hat{X}_0^n)$ and for such an SCP, we apply (2.27).

We then make a particular choice of $\psi^n = (\phi^n, \psi^n, \varphi^n)$ which is a priori sub-optimal (with respect to the supremum in (2.27)). This choice (which is motivated from Theorem 3.5) is

$$\psi^n_t = \left(e - \frac{w^*(\hat{X}^{n,\psi}_t)}{\sqrt{n}}, e - \frac{w^*(\hat{X}^{n,\psi}_t)}{\sqrt{n}}, e\right)$$

with $w^*$ being an appropriate nearly optimal (maximizing) $w$ in (2.26) such that

$$\Lambda \le \int_{\mathbb{R}^d} \left(r^v(x) \ - \frac{1}{2}\|w^*(x)\|^2\right)\widetilde{\mu}^*_{v,w}(dx) + \delta, \tag{2.28}$$

for the ergodic occupation measure $\widetilde{\mu}^*_{v,w}$ of $X^{*,v,w^*}$, for any $v \in \mathfrak{U}_{\mathrm{SM}}$ and for small $\delta > 0$. The main hurdle in proving the above display is to prove positive recurrence of $X^{*,v,w^*}$ for the ergodic occupation measure to be well-defined. We achieve this by showing that there are nearly optimal controls $w^*$ that vanish outside a large enough ball. This ensures that stability properties of $X$ can be borrowed by $X^{*,v,w^*}$ as their respective infinitesimal generators coincide outside a large ball. Finally, we show that

$$\limsup_{T\to\infty} \sup_{\psi\in\mathcal{E}^n} \mathbb{E}\left[\frac{1}{T}\int_0^T \left(\widetilde{r}(\hat{Q}^{n,\psi^n}_t) - \lambda^n \varkappa(\phi^n_t) - n\mu^n \varkappa(\psi^n_t) - n\gamma^n \varkappa(\varphi^n_t)\right)dt\right]$$

converges as $n \to \infty$ (or at least along a subsequence) to the right hand side of (2.28). From (2.28), this is bounded from below by $\Lambda - \delta$. This proves (2.22).

**Sketch proof of the upper bound in** (2.23) **(Theorem 5.1):**

We begin by choosing a nearly optimal control (for the limiting diffusion) that is stationary, Markov and continuous (Lemma A.1 guarantees that such a control exists). Using this control we explicitly construct a SCP (which is a priori sub-optimal for the ERSC problem for the diffusion-scaled queueing process) using the construction in [4]. For this constructed SCP, we apply (2.27) and choose $\psi^n = (\phi^n, \psi^n, \varphi^n)$ that is nearly optimal (with respect to the supremum in (2.27)). This will then give us

$$\hat{\Lambda}(\hat{X}^n_0) \le \limsup_{T\to\infty} \mathbb{E}\left[\frac{1}{T}\int_0^T \left(\widetilde{r}(\hat{Q}^{n,\psi^n}_t) - \lambda^n \varkappa(\phi^n_t) - n\mu^n \varkappa(\psi^n_t) - n\gamma^n \varkappa(\varphi^n_t)\right)dt\right] + \delta$$

for small $\delta > 0$. It then remains to identify the limit as $T \to \infty$ and let $n \to \infty$. To identify the limit as $T \to \infty$, it is necessary to show that the family of the mean empirical measures of $\hat{Q}^{n,\psi}$ is tight. This is not at all obvious as this cannot be inferred directly from the stability of the process $\hat{Q}^n$. We achieve this by first showing the tightness of the mean empirical measures of the joint processes $\left(\hat{Q}^{n,\psi^n}, h^n(\psi^n)\right)$ in $T, n$ with $h^n(\psi^n) = \left(\sqrt{n}(e-\phi^n), \sqrt{n}(e-\psi^n), \sqrt{n}(e-\psi^n)\right)$. We introduce a suitable topology to prove the tightness of the mean empirical measures of $h^n(\psi^n)$, and then use the tightness of $h^n(\psi^n)$ along with a Lyapunov function (motivated from [7]) to show that the family of the mean empirical measures of $\hat{Q}^{n,\psi^n}$ is tight (in fact, both in $n$ and $T$). Finally, from this tightness, we can show that the right hand of the above display is bounded from above by $\Lambda + \delta$. This proves (2.23).

Before proceeding to prove these bounds, we introduce the variational representations and present some preliminary results in the next section.

## 3. Variational formulations of ERSC problems

3.1. **Variational formulation for the limiting diffusion.** We develop a variational formulation of ERSC problem for the limiting diffusion. Moreover, in Theorem 3.3, we also show that the optimal value of the ERSC problem for the limiting diffusion can be represented as the optimal value of a two-player zero-sum stochastic differential game with an extended running cost. This is the main result of this subsection. The fundamental result we use is the following variational representation of exponential functionals of Brownian motion ([22, Theorem 5.1]).

**Theorem 3.1.** *For $T>0$, suppose $G : \mathfrak{C}^d_T \to \mathbb{R}$ is a non-negative Borel measurable function. Then the following holds:*

$$\frac{1}{T}\log \mathbb{E}[e^{TG(W)}] = \sup_{w\in\mathcal{A}} E\biggl[G\Bigl(W_{(\cdot)} + \int_0^{\cdot} w_s ds\Bigr) - \frac{1}{2T}\int_0^T \|w_s\|^2 ds\biggr], \tag{3.1}$$

*where $\mathcal{A}$ is the set of all $\mathcal{G}_t$–progressively measurable functions $w : \mathbb{R}_+ \to \mathbb{R}^d$ such that*

$$\frac{1}{T}\mathbb{E}\Bigl[\int_0^T \|w_s\|^2 ds\Bigr] < \infty, \ \textit{for every } T>0. \tag{3.2}$$

*Here $\mathcal{G}_t$ is the filtration generated by $\{W_s : 0\le s\le t\}$ such that $\mathcal{G}_0$ includes all the $\mathbb{P}$–null sets.*

*Remark* 3.1. We give an idea of the proof (see [22] for the details). The starting point is the following well-known entropy formula ([24, Proposition 2.2]): For some measurable space $(\mathcal{X},\mathfrak{X})$ and bounded measurable function $f : \mathcal{X}\to\mathbb{R}$ and measure $\mu$ on $(\mathcal{X},\mathfrak{X})$, we have

$$\log\int_{\mathcal{X}} e^{f(x)}\mu(dx) = \sup_{\nu\in\mathcal{P}(\mathcal{X})}\Bigl(\int_{\mathcal{X}} f(x)\nu(dx) - R(\nu||\mu)\Bigr). \tag{3.3}$$

Here, $R(\nu||\mu)$ is the relative entropy defined as $\int_{\mathcal{X}}\log\frac{d\nu}{d\mu}(x)\nu(dx)$ whenever $\nu$ is absolutely continuous (with respect to $\mu$ with $\frac{d\nu}{d\mu}$ being the associated Radon-Nikodym derivative) and as $\infty$ otherwise. We apply the above formula to the Wiener measure. Using Girsanov's theorem, we can construct a certain family of absolutely continuous measures, whose corresponding Radon-Nikodym derivatives can be easily computed. The crucial part is to show that one can replace the supremum over $\mathcal{P}(\mathcal{X})$ in (3.3) with the supremum over the family of the aforementioned absolutely continuous measures given by Girsanov's theorem. This gives us (3.1).

*Remark* 3.2. The representation in Theorem 3.1 was first used in [23] in the context of ERSC where the authors studied the problem of maximizing the escape times of diffusion under the control (which is associated with a ERSC cost function). The authors also studied the small noise asymptotics of the maximizing control (see [23, Lemma 4.5 and Theorem 4.6]).

For $v\in\mathfrak{U}_{\mathrm{SM}}$, it is easy to see that Theorem 3.1 can be applied to the case of ERSC cost for the limiting diffusion $X^v$ in (2.14), by choosing $G(W) = \frac{1}{T}\int_0^T r(X^v_t, v(X^v_t))dt$, with $X^v_t$ being regarded as a functional of the driving Brownian motion $W_t$ (noting that we write $X^v_t$ to indicate the dependence on the control $v$ explicitly). In the rest of this sub-section, we fix $X^v_0 = x$.

For $w\in\mathcal{A}$, define $X^{*,v,w}$ as the solution to

$$dX^{*,v,w}_t = b(X^{*,v,w}_t, v(X^{*,v,w}_t))dt + \Sigma w_t dt + \Sigma dW_t\,,\ \text{with } X^{*,v,w}_0 = x. \tag{3.4}$$

*Remark* 3.3. The reader will notice that all the results in the rest of the section hold if one replaces $\Sigma$ with a general function $\Sigma : \mathbb{R}^d\to\mathbb{R}^{d\times d}$ which is Lipschitz continuous and such that there exists a $\sigma>0$ with the following property: $\Sigma(x)\Sigma(x)^{\mathsf{T}} \ge \sigma\|x\|^2$, $x\in\mathbb{R}^d$.

As an immediate corollary to Theorem 3.1, we have the following result.

**Corollary 3.1.** *For $v\in\mathfrak{U}_{\mathrm{SM}}$,*

$$\Lambda_v = \limsup_{T\to\infty}\sup_{w\in\mathcal{A}}\mathbb{E}\biggl[\frac{1}{T}\int_0^T\Bigl(r(X^{*,v,w}_t, v(X^{*,v,w}_t)) - \frac{1}{2}\|w_t\|^2\Bigr)dt\biggr]. \tag{3.5}$$

Note that in (3.5), we are optimizing over $\mathcal{A}$, which is a subset of all $\mathcal{G}_t$–progressively measurable processes. In other words, $w_t = g(W_{[0,t]}, t)$, for some Borel measurable $g : \mathfrak{C}^d\times\mathbb{R}_+\to\mathbb{R}^d$.

*Remark* 3.4. The analysis that follows is applicable even when $v$ is a relaxed Markov control. It is straightforward to show that the infimum of the ERSC cost over the set of all relaxed controls is greater than or equal to the optimal ERSC cost. We reserve the same notation $X^{*,v,w}$. In this case, $b(x,v(x))$ and $r(x,v(x))$ will be replaced by $\bar{b}^v \doteq \int_{\mathbb{U}} b(x,u)v_*(du|x)$ and $\bar{r}^v \doteq \int_{\mathbb{U}} r(x,u)dv_*(du|x)$, respectively. Also, $\mathcal{L}^v$ is defined with $\bar{b}^v(x)$ instead of $b(x,v(x))$. Therefore, we restrict ourselves with only precise Markov controls in this section.

We now write $\Lambda_v$ as the optimal value of a CEC problem for an extended diffusion over an auxiliary control. To that end, let $\mathfrak{W}$ be the set of admissible $\mathbb{R}^d$–valued controls (here we use admissibility as in Definition 2.1 with $\mathbb{U}$ replaced by $\mathbb{R}^d$) $\tilde{w}$ and $Z^{v,\tilde{w}}$ be the extended diffusion process defined as the solution to

$$dZ_t^{v,\tilde{w}} = b(Z_t^{v,\tilde{w}}, v(Z_t^{v,\tilde{w}}))dt + \Sigma\tilde{w}_t dt + \Sigma dW_t \text{ with } Z_0^{v,\tilde{w}} = x.$$

Even though $Z^{v,\tilde{w}}$ is governed by the same equation as that of $X^{*,v,w}$ (which is (3.4)) with $\tilde{w}$ taking the place of $w$, we use different notation to emphasize the fact that $w$ and $\tilde{w}$ lie in possibly different spaces ($w \in \mathcal{A}$ and $\tilde{w} \in \mathfrak{W}$). Now, define the following ergodic control problem

$$\widetilde{\Lambda}_v \doteq \sup_{\tilde{w}\in\mathfrak{W}} \limsup_{T\to\infty} \mathbb{E}\left[\frac{1}{T}\int_0^T \left(r^v(Z_t^{v,\tilde{w}}) \; - \frac{1}{2}\|\tilde{w}_t\|^2\right)dt\right]. \tag{3.6}$$

First we note that for every $\bar{w} \in \mathfrak{W}$, $\bar{w}_t \doteq \bar{w}(W_{[0,t]}, t) \in \mathcal{A}$. Therefore, $\mathfrak{W} \subset \mathcal{A}$. This immediately gives the following bound.

**Lemma 3.1.** *For $v \in \mathfrak{U}_{\mathrm{SM}}$, $\Lambda_v \geq \widetilde{\Lambda}_v$.*

*Proof.* Fix $\delta > 0$ and choose $\bar{w}^* \in \mathfrak{W}$ such that

$$\widetilde{\Lambda}_v \leq \limsup_{T\to\infty} \mathbb{E}\left[\frac{1}{T}\int_0^T \left(r^v(Z_t^{v,\tilde{w}^*}) \; - \frac{1}{2}\|\tilde{w}_t^*\|^2\right)dt\right] + \delta.$$

Since $\bar{w}^* \in \mathcal{A}$, we have

$$\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T \left(r^v(X_t^{*,v,\bar{w}^*}) - \frac{1}{2}\|\bar{w}_t^*\|^2\right)dt\right] \leq \limsup_{T\to\infty} \frac{1}{T} \sup_{w\in\mathcal{A}} \mathbb{E}\left[\int_0^T \left(r^v(X_t^{*,v,w}) - \frac{1}{2}\|w_t\|^2\right)dt\right]$$
$$\implies \widetilde{\Lambda}_v - \delta \leq \Lambda_v\,.$$

This consequently gives us $\Lambda_v \geq \widetilde{\Lambda}_v$. ☐

Showing that $\Lambda_v$ can be represented as the optimal value of CEC problem of $w$ in (3.5) is equivalent to showing that $\Lambda_v = \widetilde{\Lambda}_v$ (in the light of Lemma 3.1). Therefore, it suffices for us to show the reverse inequality. In the following, we proceed to do this in an indirect way. To that end, we denote $\mathfrak{W}_{\mathrm{SM}} \subset \mathfrak{W}$ as the set of stationary Markov controls (including relaxed Markov controls) and then first study the following CEC problem: For $l > 0$, let $\mathfrak{W}_{\mathrm{SM}}(l)$ be the set of all stationary Markov controls (which is a subset of $\mathfrak{W}$) that are of the form $w = w(\cdot)$ on $B_l$ and $w = 0$ on $B_l^c$. Define

$$f_l^v(x,w) \doteq \begin{cases} r^v(x) - \frac{1}{2}\|w\|^2, \text{ whenever } x \in B_l, \\ r^v(x), \text{ otherwise,} \end{cases}$$

and

$$\Delta_l^v(x,w) \doteq \begin{cases} b(x,v(x)) + \Sigma w, \text{ whenever } x \in B_l, \\ b(x,v(x)), \text{ otherwise.} \end{cases}$$

Also, define

$$\Lambda_v(l) \doteq \inf_{w\in\mathfrak{W}_{\mathrm{SM}}(l)} \limsup_{T\to\infty} \frac{1}{T}\int_0^T \Big(r^v(Z_t^{v,w}) - \frac{1}{2}\|w_t\|^2\Big)dt.$$

We then consider the limit as $l\to\infty$. There are multiple reasons for using such an approach:

(i) For every $l>0$, the process $Z^{v,w}$ is exponentially ergodic, as outside $B_l$, the generator of $Z^{v,w}$ is the same as that of $X^v$ which is exponentially ergodic.

(ii) We can use a variant of the method of spatial truncation (introduced in [4, Section 4]) where a CEC problem can be shown to be approximated arbitrarily well, by a class of CEC problems with an appropriate truncated version of the original drifts and running costs.

(iii) This approach also tells us that we can find nearly optimal stationary Markov controls which are supported only on a large ball (this property is used in proof of Theorem 4.1).

We state and prove a modified version of [4, Theorem 4.1].

**Theorem 3.2.** *For $v\in\mathfrak{U}_{\mathrm{SM}}$ and every $l>0$, there exists a solution $\Upsilon_l^v\in W^{2,p}_{loc}(\mathbb{R}^d)$, for any $p>d$, with $\Upsilon_l^v(0)=1$, of the equation*

$$\mathcal{L}^v\Upsilon_l^v + \max_{w\in\mathbb{R}^d}\left\{f_l^v(x,w) + \Delta_l^v(x,w)\cdot\nabla\Upsilon_l^v\right\} = \rho_l^v. \tag{3.7}$$

*Moreover, $\rho_l^v = \Lambda_v(l)$ and $\rho_l$ is non-decreasing in $l$.*

*Proof.* It is evident that for every $l>0$ and $w\in\mathfrak{W}_{\mathrm{SM}}(l)$, $Z^{v,w}$ is exponentially ergodic and therefore satisfies the hypothesis of [4, Theorem 4.1] and this proves the theorem. □

The following lemma studies how $\{\Upsilon_l^v\}_{l>0}$ and $\{\rho_l^v\}_{l>0}$ behave as $l\uparrow\infty$. From [5, Lemma 2.4], for $v\in\mathfrak{U}_{\mathrm{SM}}$, there is a unique pair $(\Psi^v,\Lambda^v)$ such that $\Psi^v>0$, $\Psi^v\in W^{2,d}_{\mathrm{loc}}(\mathbb{R}^d)$ and

$$\mathcal{L}^v\Psi^v(x) + r^v(x)\Psi^v(x) = \Lambda^v\Psi^v(x). \tag{3.8}$$

In the rest of this section, we suppose that Assumption 2.1 holds.

**Lemma 3.2.** *For every $v\in\mathfrak{U}_{\mathrm{SM}}$, there exists a unique pair $(\widetilde{\Lambda}_v,\Phi_v)$ such that as $l\to\infty$, $\rho_l^v\to\widetilde{\Lambda}_v$ and $\Upsilon_l^v\to\Phi^v\doteq\log\Psi^v$ in $W^{2,p}_{loc}(\mathbb{R}^d)$, for any $p>d$. Moreover, $\widetilde{\Lambda}_v=\Lambda_v$.*

*Remark* 3.5. The content of [4, Theorem 4.1] also addresses the aforementioned convergence. The reason for giving a separate lemma is that in our case, as $l\to\infty$, the space $\mathfrak{W}_{\mathrm{SM}}$ (considered as an appropriate limit of $\mathfrak{W}_{\mathrm{SM}}(l)$ as $l\uparrow\infty$) is no longer compact and the topology of Markov controls (as done in [6, Section 2.4]) may not be compact and/or metrizable. But this will not be an issue for us as will be evident from the proof.

*Remark* 3.6. A result similar to this is showed in [5, Section 2.4], but under the following condition on the cost function $r$: $r(x,u)\to 0$ as $\|x\|\to\infty$. This is too restrictive for our setting of multiclass queueing networks.

*Proof.* From Lemma 3.1, the finiteness of $\Lambda_v$ implies the finiteness of $\widetilde{\Lambda}_v$. Since $\rho_l^v\le\widetilde{\Lambda}_v$, $\{\rho_l^v\}_{l>0}$ is convergent along a subsequence (with, say, $\rho^{*,v}\le\widetilde{\Lambda}_v$ as the limit point). Using the standard elliptic regularity theory (arguments similar to [6, Theorem 3.5.2 and Lemma 3.5.3]), we can then conclude that $\Upsilon_l^v$ converges to some function $\Upsilon^v$ in $W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$ that satisfies

$$\mathcal{L}^v\Upsilon^v + r^v(x) + \max_{w\in\mathbb{R}^d}\left\{\Sigma w.\nabla\Upsilon^v - \frac{1}{2}\|w\|^2\right\} = \rho^{*,v}. \tag{3.9}$$

It is clear that (3.9) can be rewritten as

$$\mathcal{L}^v\Upsilon^v + r^v + \frac{1}{2}\|\Sigma^{\mathsf{T}}\nabla\Upsilon^v\|^2 = \rho^{*,v}.$$

By making a substitution $\widetilde{\Upsilon}^v = e^{\Upsilon^v}$, we have

$$\mathcal{L}^v\widetilde{\Upsilon}^v + r^v\widetilde{\Upsilon}^v = \rho^{*,v}\widetilde{\Upsilon}^v.$$

Using [5, Corollary 2.1], we can conclude that $\rho^{*,v} \geq \Lambda_v$. This proves that $\Lambda_v = \widetilde{\Lambda}_v$ as we already know that $\widetilde{\Lambda}_v \geq \rho^{*,v}$ and that $\Lambda_v \geq \widetilde{\Lambda}_v$. Since [5, Lemma 2.4] gives us a unique solution to the above equation, $\widetilde{\Upsilon}^v = \Psi^v$. This completes the proof as $\Upsilon^v = \Phi^v$. □

We now give main result of this section, which is a straightforward consequence of the above analysis. In addition to proving that $\Lambda$ is equal to the optimal value of a two-player zero-sum stochastic differential game, we also show that such a game can be written as the limit of a family of two-player zero-sum stochastic differential games where the strategies of the maximizing player are only allowed to be compactly supported (see (3.12) below).

**Theorem 3.3.** *The following series of equalities hold:*

$$\Lambda = \inf_{v\in\mathfrak{U}_{\mathrm{SM}}} \sup_{w\in\mathfrak{W}_{\mathrm{SM}}} \limsup_{T\to\infty} \frac{1}{T}\int_0^T \Big(r^v(X_t^{*,v,w}) - \frac{1}{2}\|w_t\|^2\Big)dt \tag{3.10}$$

$$= \sup_{w\in\mathfrak{W}_{\mathrm{SM}}} \inf_{v\in\mathfrak{U}_{\mathrm{SM}}} \limsup_{T\to\infty} \frac{1}{T}\int_0^T \Big(r^v(X_t^{*,v,w}) - \frac{1}{2}\|w_t\|^2\Big)dt \tag{3.11}$$

$$= \lim_{l\to\infty} \sup_{w\in\mathfrak{W}_{\mathrm{SM}}(l)} \inf_{v\in\mathfrak{U}_{\mathrm{SM}}} \limsup_{T\to\infty} \frac{1}{T}\int_0^T \Big(r^v(X_t^{*,v,w}) - \frac{1}{2}\|w_t\|^2\Big)dt\,. \tag{3.12}$$

*Proof.* Let $v^*$ be the optimal Markov control. Then, from Theorem 2.3 (in particular, (2.21)), we know that

$$\mathcal{L}^{v^*}V(x) + r^{v^*}(x)V(x) = \Lambda V(x)$$

and from Lemma 3.2, we know that

$$\Lambda = \sup_{w\in\mathfrak{W}} \limsup_{T\to\infty} \frac{1}{T}\int_0^T \Big(r^{v^*}(X_t^{*,v^*,w}) - \frac{1}{2}\|w_t\|^2\Big)dt. \tag{3.13}$$

Letting $\widehat{V}(x) \doteq \log V(x)$ and substituting in the above equation, we get

$$\mathcal{L}^{v^*}\widehat{V}(x) + r^{v^*}(x) + \frac{1}{2}\|\Sigma^{\mathsf{T}}\nabla\widehat{V}(x)\|^2 = \Lambda. \tag{3.14}$$

This can be also written as

$$\mathcal{L}^{v^*}\widehat{V}(x) + r^{v^*}(x) + \max_{w\in\mathbb{R}^d}\Big\{\Sigma w\cdot\nabla\widehat{V}(x) - \frac{1}{2}\|w\|^2\Big\} = \Lambda.$$

From here, using the standard verification argument, we can conclude that $w^* = \Sigma^{\mathsf{T}}\nabla\widehat{V}(x)$ is the unique optimal stationary Markov control for the maximization problem in (3.13). This in particular, implies that the supremum over $\mathfrak{W}$ (in (3.13)) can be replaced by the supremum over $\mathfrak{W}_{\mathrm{SM}}$ (in (3.13)). In other words,

$$\Lambda = \inf_{v\in\mathfrak{U}_{\mathrm{SM}}} \sup_{w\in\mathfrak{W}_{\mathrm{SM}}} \limsup_{T\to\infty} \frac{1}{T}\int_0^T \Big(r^v(X_t^{*,v,w}) - \frac{1}{2}\|w_t\|^2\Big)dt.$$

To prove (3.11), we recall (2.20) with the substitution $\widehat{V} = \log V$,

$$\min_{u\in\mathbb{U}}\max_{w\in\mathbb{R}^d}\Big\{\mathcal{L}^u\widehat{V}(x) + r^u(x) + \Sigma w\cdot\nabla\widehat{V}(x) - \frac{1}{2}\|w\|^2\Big\} = \Lambda. \tag{3.15}$$

From the structure of the expression inside the parenthesis, it is clear that

$$\max_{w\in\mathbb{R}^d}\min_{u\in\mathbb{U}}\Big\{\mathcal{L}^u\widehat{V}(x) + r^u(x) + \Sigma w\cdot\nabla\widehat{V}(x) - \frac{1}{2}\|w\|^2\Big\} = \Lambda. \tag{3.16}$$

Again using the standard verification argument, we can conclude that

$$\Lambda = \sup_{w\in\mathfrak{W}_{\mathrm{SM}}} \inf_{v\in\mathfrak{U}_{\mathrm{SM}}} \limsup_{T\to\infty} \frac{1}{T}\int_0^T \big(r^v(X_t^{*,v,w}) - \frac{1}{2}\|w_t\|^2\big)dt.$$

Finally, to prove (3.12), we make an easy observation that for every $l>0$, we have

$$\Lambda \geq \sup_{w\in\mathfrak{W}_{\mathrm{SM}}(l)} \inf_{v\in\mathfrak{U}_{\mathrm{SM}}} \limsup_{T\to\infty} \frac{1}{T}\int_0^T \big(r^v(X_t^{*,v,w}) - \frac{1}{2}\|w_t\|^2\big)dt.$$

We now apply Lemma 3.2 to $v=v^*$ (which satisfies (2.21)) to conclude that for every $\delta>0$, there is a $l_0>0$ such that for $l>l_0$, we have

$$\Lambda - \delta \leq \sup_{w\in\mathfrak{W}_{\mathrm{SM}}(l)} \inf_{v\in\mathfrak{U}_{\mathrm{SM}}} \limsup_{T\to\infty} \frac{1}{T}\int_0^T \big(r^v(X_t^{*,v,w}) - \frac{1}{2}\|w_t\|^2\big)dt.$$

Combining the above two displays, we have

$$\Lambda = \lim_{l\to\infty} \sup_{w\in\mathfrak{W}_{\mathrm{SM}}(l)} \inf_{v\in\mathfrak{U}_{\mathrm{SM}}} \limsup_{T\to\infty} \frac{1}{T}\int_0^T \big(r^v(X_t^{*,v,w}) - \frac{1}{2}\|w_t\|^2\big)dt.$$

This proves the theorem. □

*Remark* 3.7. In the above analysis, observe that we choose the fixed $v$ to be a relaxed control and not a general admissible control. The reason for doing this is as follows. Suppose we choose $v$ to be some non-Markov control. Then the running cost function for the maximization problem in (3.6) is a priori dependent on the entire past. This restricts us from using the principle of dynamic programming and the analysis above is not applicable.

We next present two results that will be used in the proofs of the lower and upper bounds. They involve the mean empirical (occupation) measure and the ergodic occupation measure of the extended diffusion process. For any $\mathfrak{U} \ni u_t = u(t, W_{[0,t]})$, let $X^{*,u,w}$ be a diffusion process as a solution to

$$dX_t^{*,u,w} = b(X_t^{*,u,w}, u(t, W_{[0,t]}^w))dt + \Sigma w_t dt + \Sigma dW_t\,, \text{ with } X_0^{*,u,w} = x. \tag{3.17}$$

Here, $W_\cdot^w = W_\cdot + \int_0^\cdot w_t dt$. To keep expressions concise, we write $u_t^w = u(t, W_{[0,t]}^w)$. In case $u$ is a Markov control, we know that $u$ depends on $W$ through the diffusion, that is, we have $u_t^w = u(X_t^{*,u,w})$. In other words, for any $v\in\mathfrak{U}_{\mathrm{SM}}$ and $w\in\mathcal{A}$, $X^{*,v,w}$ is solution to (3.4).

The mean empirical (occupation) measure is defined as follows. For a given $u\in\mathfrak{U}$ and $w\in\mathcal{A}$,

$$\begin{aligned}
\mu_{u,w}^{*,T}(A\times B\times C) &\doteq \frac{1}{T}\int_0^T \mathbb{1}_{\{(X_t^{*,u,w}, u_t^w, w_t)\in A\times B\times C\}}dt\,,\\
\mu_{u,w}^{*,T,1}(A\times B) &\doteq \frac{1}{T}\int_0^T \mathbb{1}_{\{(X_t^{*,u,w}, u_t^w)\in A\times B\}}dt\,,\\
\mu_{u,w}^{*,T,2}(A\times C) &\doteq \frac{1}{T}\int_0^T \mathbb{1}_{\{(X_t^{*,u,w}, w_t)\in A\times C\}}dt\,,\\
\mu_{u,w}^{*,T,3}(A) &\doteq \frac{1}{T}\int_0^T \mathbb{1}_{\{X_t^{*,u,w}\in A\}}dt\,,
\end{aligned} \tag{3.18}$$

for any Borel sets $A, C\subset\mathbb{R}^d$ and $B\subset\mathbb{U}$. For $v\in\mathfrak{U}_{\mathrm{SM}}$ and $w\in\mathfrak{W}_{\mathrm{SM}}$, we write

$$\widetilde{\mu}_{v,w}^{*,T}(A) \doteq \frac{1}{T}\int_0^T \mathbb{1}_{\{X_t^{*,v,w}\in A\}}dt\,, \text{ for any Borel set } A\subset\mathbb{R}^d.$$

We represent their weak limits (if they exist and the associated subsequence $T_k$ is irrelevant) by $\mu_{u,w}^*, \mu_{u,w}^{*,1}, \mu_{u,w}^{*,2}, \mu_{u,w}^{*,3}, \widetilde{\mu}_{v,w}^*$, respectively.

Observe that using the above notation, we can rewrite $\Lambda_v$ in (3.5) as

$$\Lambda_v = \limsup_{T\to\infty} \sup_{w\in\mathcal{A}} \int_{\mathbb{R}^d\times\mathbb{R}^d} \bigl(r^v(x) - \frac{1}{2}\|y\|^2\bigr) d\mu^{*,T,2}_{v,w}(x,y)\,.$$

Using Theorem 3.3, we can further write $\Lambda_v$ as

$$\Lambda_v = \sup_{w\in\mathfrak{W}_{\mathrm{SM}}} \inf_{v\in\mathfrak{U}_{\mathrm{SM}}} \int_{\mathbb{R}^d} \bigl(r^v(x) - \frac{1}{2}\|w(x)\|^2\bigr) d\widetilde{\mu}^*_{v,w}(x).$$

The following corollary is important in the proof of Theorem 4.1.

**Corollary 3.2.** *For $\delta > 0$, there exist $l_0 > 0$ and $w^* \in \mathfrak{W}_{\mathrm{SM}}(l)$ (for $l > l_0$) such that for any $v \in \mathfrak{U}_{\mathrm{SM}}$,*

$$\Lambda \le \int_{\mathbb{R}^d} \bigl(r^v(x) - \frac{1}{2}\|w^*(x)\|^2\bigr) d\widetilde{\mu}^*_{v,w^*}(x) + 2\delta. \tag{3.19}$$

*Proof.* Fix $\delta > 0$. From Theorem 3.3 (in particular, (3.12)), there exists $l_0 > 0$ such that for $l > l_0$,

$$\Lambda \le \sup_{w\in\mathfrak{W}_{\mathrm{SM}}(l)} \inf_{v\in\mathfrak{U}_{\mathrm{SM}}} \limsup_{T\to\infty} \frac{1}{T}\int_0^T \bigl(r^v(X^{*,v,w}_t) - \frac{1}{2}\|w_t\|^2\bigr) dt + \delta.$$

From here, we know that there exists $w^* \in \mathfrak{W}_{\mathrm{SM}}(l)$ such that

$$\Lambda \le \inf_{v\in\mathfrak{U}_{\mathrm{SM}}} \limsup_{T\to\infty} \frac{1}{T}\int_0^T \bigl(r^v(X^{*,v,w^*}_t) - \frac{1}{2}\|w^*(X^{*,v,w^*}_t)\|^2\bigr) dt + 2\delta.$$

From the definition of $\mathfrak{W}_{\mathrm{SM}}(l)$, we know that $w^*(x) = 0$, whenever $x \in B^c_l$. Therefore, $X^{*,v,w}$ has the same infinitesimal generator as that of $X^v$ on $B^c_l$. Noting the positive recurrence (in fact, exponential ergodicity) of $X^v$ from Theorem 2.2, we can also conclude the positive recurrence (in fact, exponential ergodicity) of $X^{*,v,w}$, for every $v \in \mathfrak{U}_{\mathrm{SM}}$. From here, with any $v \in \mathfrak{U}_{\mathrm{SM}}$ and its corresponding ergodic occupation measure $\widetilde{\mu}^*_{v,w^*}$, we obtain (3.19). □

*Remark* 3.8. For every $l > 0$, the problem of existence of optimal solution to

$$\sup_{w\in\mathfrak{W}_{\mathrm{SM}}(l)} \inf_{v\in\mathfrak{U}_{\mathrm{SM}}} \int_{\mathbb{R}^d} \bigl(r^v(x) - \frac{1}{2}\|w^*(x)\|^2\bigr) d\widetilde{\mu}^*_{v,w}(x)$$

over all allowed ergodic occupation measures $\widetilde{\mu}^*_{v,w}$ can also solved using the convex analytic approach introduced by Borkar and Ghosh in [20] in the case of two-person zero-sum stochastic differential game with long-run average expected cost criterion. This approach was originally introduced in the context of CEC problem for Markov decision process by Borkar in [21]. We also referred the reader to [6, Chapter 3] for a detailed discussion in the context of diffusions.

We now give another immediate consequence of the above analysis, which will be used in the proof of Theorem 5.1.

**Lemma 3.3.** *For $v \in \mathfrak{U}_{\mathrm{SM}}$ and $w \in \mathcal{A}$, let $\{\mu^{*,T,3}_{v,w}\}_{K\in\mathbb{N}}$ be a tight family of measures in $\mathcal{P}(\mathbb{R}^d)$ (with $\mu^{*,3}_{v,w}$ being a weak limit point corresponding to a sequence $T_k$) such that*

$$\limsup_{k\to\infty} \int_{\mathbb{R}^d} r^v(x) d\mu^{*,T_k,3}_{u,w}(x) < \infty.$$

*Then,*

$$\Lambda_v \ge \int_{\mathbb{R}^d} r^v(x) d\mu^{*,3}_{v,w}(x) - \limsup_{T\to\infty} \frac{1}{2T}\int_0^T \|w_t\|^2 dt\,. \tag{3.20}$$

*Proof.* From (3.5) and the choice of $w$, we have

$$\Lambda_v \geq \limsup_{k\to\infty} \mathbb{E}\left[\frac{1}{T_k}\int_0^{T_k} \left(r^v(X_t^{*,v,w}) - \frac{1}{2}\|w_t\|^2\right) dt\right]. \tag{3.21}$$

From the hypothesis on $\mu_{u,w}^{*,T,3}$ and the definition of $\mu_{u,w}^{*,3}$, we conclude that (3.20) holds. □

To summarize, we have shown that the ERSC problem for the limiting diffusion is equivalent to a two person zero-sum game with the long-run average expected cost criterion for the extended process $X^{*,v,w}$ associated with the auxiliary control $w$.

3.2. **Variational formulation for Poisson-driven controlled queueing dynamics.** This section provides a variational formulation for the ERSC problem of the diffusion-scaled queueing processes in the context of Markovian stochastic networks. To the best of knowledge of the authors, this formulation is novel. The following variational representation of exponential functionals of Poisson process (see [24, Theorem 3.23]) is crucial in what follows. We state this result for a 1-dimensional Poisson process $\widetilde{N}$ with rate $\lambda > 0$. The function $\varkappa$ defined as

$$\varkappa(r) \doteq r\ln r - r + 1 \tag{3.22}$$

plays an important role in the analysis. We give some important lemmas regarding the function $\varkappa(\cdot)$ in Appendix B. Let $\mathcal{F}_t$ be the filtration generated by the $\widetilde{N}$ such that $\mathcal{F}_0$ contains all null sets.

**Theorem 3.4.** *For $T > 0$, suppose that $G : \mathfrak{D}_T \to \mathbb{R}$ is a bounded Borel measurable function. Then the following holds:*

$$\frac{1}{T}\log \mathbb{E}\Big[e^{TG(\widetilde{N})}\Big] = \sup_{\phi\in\widetilde{\mathcal{E}}} \mathbb{E}\left[G(\widetilde{N}^\phi) - \frac{\lambda}{T}\int_0^T \varkappa(\phi_s)ds\right]. \tag{3.23}$$

*Moreover, for every $\delta > 0$,*

$$\frac{1}{T}\log \mathbb{E}\Big[e^{TG(\widetilde{N})}\Big] \leq \sup_{\phi\in\widetilde{\mathcal{E}}_M} \mathbb{E}\left[G(\widetilde{N}^\phi) - \frac{\lambda}{T}\int_0^T \varkappa(\phi_s)ds\right] + \delta. \tag{3.24}$$

*Here, $\widetilde{\mathcal{E}}$ is the set of all the $\phi$ which is progressively measurable (with respect to $\mathcal{F}_t$) such that for every $T > 0$,*

$$\frac{\lambda}{T}\int_0^T \varkappa(\phi_s)ds < \infty,$$

*$\widetilde{\mathcal{E}}_M$ is the set of all $\phi \in \widetilde{\mathcal{E}}$ such that for $T > 0$,*

$$\frac{\lambda}{T}\int_0^T \varkappa(\phi_s)ds \leq M, \ \textit{with } M \textit{ depending only on } \delta \textit{ and } \|G\|_\infty$$

*and $\widetilde{N}^\phi$ denotes a "controlled" Poisson process which is a solution to the martingale problem below: for $f \in \mathcal{C}^2(\mathbb{R})$,*

$$f(\widetilde{N}_t^\phi) - f(0) - \lambda\int_0^t \phi_s\big[f(\widetilde{N}_s^\phi + 1) - f(\widetilde{N}_s^\phi)\big]ds \tag{3.25}$$

*is a martingale with respect to $\mathcal{F}_t$.*

*Remark* 3.9. Comments similar to those in Remark 3.1 are applicable. In this case, instead of Girsanov's theorem for Brownian motion, Girsanov's theorem for Poisson process is used. See [25, Theorem 2.1] for the proof.

*Remark* 3.10. The above theorem can be easily extended to the multi-variate independent Poisson processes. We do not state the multi-variate version in the general setting, however we state the version that is relevant to us in the context of the ERSC problem for the queueing network model (see Theorem 3.7).

Our interest in using the Theorem 3.4 lies in proving Theorem 2.4. To do this, we first understand and study how to apply Theorem 3.4 in a simple case of a diffusion-scaled Poisson process where $G(\widetilde{N})$ is replaced by $G(\widetilde{N}^n)$ with

$$\widetilde{N}^n_t \doteq \frac{\widetilde{N}_{nt} - \lambda nt}{\sqrt{n}}.$$

We revisit the weak convergence for Poisson process $\widetilde{N}_t$ with rate $\lambda$ using the associated variational representation which is given in Theorem 3.4. Such a study is also helpful for the reader to understand the intuition behind the proof of Theorem 2.4. By Theorem 3.4, we obtain

$$\frac{1}{T}\log \mathbb{E}\big[e^{TG(\widetilde{N}^n)}\big] = \sup_{\phi\in\widetilde{\mathcal{E}}} \mathbb{E}\left[G\Big(\frac{\widetilde{N}^{n,\phi} - \lambda nt}{\sqrt{n}}\Big) - \frac{\lambda n}{T}\int_0^T \varkappa(\phi_s)ds\right], \tag{3.26}$$

where $\widetilde{N}^{n,\phi}$ is the unique solution to the martingale problem below: for $f \in \mathcal{C}^2(\mathbb{R})$,

$$\begin{aligned} f\left(\frac{\widetilde{N}^{n,\phi}_t - nt}{\sqrt{n}}\right) - f(0) - \int_0^t n\lambda\phi_s\left[f\left(\frac{\widetilde{N}^{n,\phi}_s + 1 - ns}{\sqrt{n}}\right) - f\left(\frac{\widetilde{N}^{n,\phi}_s - ns}{\sqrt{n}}\right)\right]ds \\ - \int_0^t \sqrt{n}\lambda\nabla f\left(\frac{\widetilde{N}^{n,\phi}_s - ns}{\sqrt{n}}\right)ds \end{aligned} \tag{3.27}$$

is an $\mathcal{F}_t$–martingale. We prove the weak convergence of $\widetilde{N}^n_t$ using the above variational formulation.

**Theorem 3.5.** *For $T > 0$, let $G : \mathfrak{D}_T \to \mathbb{R}$ be a bounded continuous function. Then*

$$\lim_{n\to\infty}\frac{1}{T}\log\mathbb{E}[e^{TG(\widetilde{N}^n)}] = \sup_{w\in\mathcal{A}}\mathbb{E}\Big[G\Big(W + \int_0^{\cdot} w_t dt\Big) - \frac{\lambda}{2T}\int_0^T |w_t|^2 dt\Big] = \frac{1}{T}\log\mathbb{E}[e^{TG(W)}]. \tag{3.28}$$

*Here, $W$ is a 1–dimensional Brownian motion on $\mathbb{R}$ and $\mathcal{A}$ is as defined in (3.2), but for $\mathbb{R}$ instead of $\mathbb{R}^d$.*

In the following, we discuss several key elements of the proof of the theorem while deferring the proofs to the appendix. To begin with, we write

$$\begin{aligned} \frac{1}{T}\log\mathbb{E}\big[e^{TG(\widetilde{N}^n)}\big] &= \sup_{\phi\in\widetilde{\mathcal{E}}}\mathbb{E}\left[G\Big(\frac{\widetilde{N}^{n,\phi} - \lambda n\int_0^{\cdot}\phi_t dt - \lambda n\mathfrak{e}(\cdot) + \lambda n\int_0^{\cdot}\phi_t dt}{\sqrt{n}}\Big) - \frac{\lambda n}{T}\int_0^T\varkappa(\phi_t)dt\right] \\ &= \sup_{\phi\in\widetilde{\mathcal{E}}}\mathbb{E}\left[G\Big(\frac{\widetilde{N}^{n,\phi} - \lambda n\int_0^{\cdot}\phi_t dt}{\sqrt{n}} - \lambda\int_0^{\cdot}\sqrt{n}(1-\phi_t)dt\Big) - \frac{\lambda n}{T}\int_0^T\varkappa(\phi_t)dt\right], \end{aligned} \tag{3.29}$$

where $\mathfrak{e}(t) = t$ and $\widetilde{\mathcal{E}}$ is as defined in the statement of Theorem 3.4. Clearly, it suffices to show that

$$\begin{aligned} \lim_{n\to\infty}\sup_{\phi\in\widetilde{\mathcal{E}}}\mathbb{E}&\left[G\Big(\frac{\widetilde{N}^{n,\phi} - \lambda n\int_0^{\cdot}\phi_t dt}{\sqrt{n}} - \lambda\int_0^{\cdot}\sqrt{n}(1-\phi_t)dt\Big) - \frac{\lambda n}{T}\int_0^T\varkappa(\phi_t)dt\right] \\ &= \sup_{w\in\mathcal{A}}\mathbb{E}\left[G\Big(W + \int_0^{\cdot} w_s ds\Big) - \frac{\lambda}{2T}\int_0^T |w_t|^2 dt\right]. \end{aligned} \tag{3.30}$$

Here, $\mathcal{A}$ is as defined in (3.2) for some 1-dimensional Brownian motion $W$. We observe that

$$M^{n,\phi} \doteq \frac{\widetilde{N}^{n,\phi} - \lambda n \int_0^{\cdot} \phi_t dt}{\sqrt{n}}$$

is a square integrable $\mathcal{F}_t$–martingale.

Comparing (3.29) with the right hand side of (3.28), it is evident that analyzing the behavior of $\{\int_0^{\cdot} \sqrt{n}(1-\phi_t^n)dt\}_{n\in\mathbb{N}}$ is necessary. It will be shown using Lemma 3.5 that $\{\sqrt{n}(1-\phi^n)\}_{n\in\mathbb{N}}$ converges to $w \in L^2([0,T],\mathbb{R})$ in an appropriate sense, along a subsequence. This in turn, ascertains the convergence of $\{\int_0^{\cdot} \sqrt{n}(1-\phi_t^n)dt\}_{n\in\mathbb{N}}$ to $\int_0^{\cdot} w_t dt$, again in an appropriate sense, along the same subsequence. To see if the corresponding family $\{M^{n,\phi^n}\}_{n\in\mathbb{N}}$ is tight in $\mathfrak{D}_T$, we use the tightness of $\{\sqrt{n}(1-\phi^n)\}_{n\in\mathbb{N}}$ in $L^2([0,T],\mathbb{R})$ (in an appropriate sense) with $\phi^n$ being $\delta$–optimal control for every $n$. This in turn, implies that $\phi^n$ converges to 1 in $L^2([0,T],\mathbb{R})$. We can then show that $\int_0^{\cdot} \phi_s^n ds$ converges to $\mathfrak{e}(\cdot)$ in $\mathfrak{C}_T$. Using random time change lemma ([14, Pg. 151]) and martingale central limit theorem, we can conclude that $M^{n,\phi^n}$ converges to a Brownian motion.

To prove the upper bound, we choose a family of $\delta$–optimal controls (denoted by $\phi^n$), while to prove the lower bound, we let $\phi^n = 1 - \frac{w^*}{\sqrt{n}}$, which is a priori suboptimal control corresponding to the left hand side of (3.30).

Inspired by this study on the diffusion-scaled Poisson process, in the proof of lower bound (Theorem 4.1), we choose $\psi^n = (\phi^n, \psi^n, \varphi^n)$ according to (4.4) and in the proof of upper bound (Theorem 5.1), we choose $\psi^n = (\phi^n, \psi^n, \varphi^n)$ that is nearly optimal corresponding to (5.11) and then analyze the sequences $\{\sqrt{n}(e-\phi^n)\}_{n\in\mathbb{N}}$, $\{\sqrt{n}(e-\psi^n)\}_{n\in\mathbb{N}}$ and $\{\sqrt{n}(e-\varphi^n)\}_{n\in\mathbb{N}}$.

*Remark* 3.11. To avoid/clarify any confusion, we re-iterate that we have used $\widetilde{\mathcal{E}}$ ($\mathcal{A}$, respectively) to denote the set of controls in the Poisson case (Brownian case, respectively). We will use $\mathcal{E}^n$ to denote the set of controls in the case of diffusion-scaled queueing processes.

We now set up the notation to state the multi-variate version of Theorem 3.4 in the context of the ERSC problem for the queueing network.

Define

$$N^n = \Big(\{\breve{A}_i^n\}_{i=1}^d, \{\breve{S}_i^n\}_{i=1}^d, \{\breve{R}_i^n\}_{i=1}^d\Big)$$

be the $3d$–dimensional vector of independent Poisson processes with rates

$$\Big(\{\lambda_i^n\}_{i=1}^d, \{n\mu_i^n\}_{i=1}^d, \{n\gamma_i^n\}_{i=1}^d\Big).$$

The filtration of the process $N^n$ is denoted by $\bar{\mathcal{G}}_t^n$, for $t \geq 0$ (such that $\bar{\mathcal{G}}_0^n$ includes all $\mathbb{P}$–null sets). In the following, using the processes $\{\breve{A}_i^n\}_{i=1}^d$, $\{\breve{S}_i^n\}_{i=1}^d$ and $\{\breve{R}_i^n\}_{i=1}^d$, we re-define the processes $\hat{X}^n$, $\hat{Q}^n$ and $\hat{Z}^n$. Since the re-defined processes have the same laws, we reserve the original notation to denote them. In the rest of the paper, we always consider the re-defined versions of these processes. In terms of $\{\breve{A}_i^n\}_{i=1}^d$, $\{\breve{S}_i^n\}_{i=1}^d$ and $\{\breve{R}_i^n\}_{i=1}^d$, $\hat{X}_t^n$ is the diffusion-scaled queueing process, with $U_t^n$ being the corresponding control, given by

$$\begin{aligned}
\hat{X}_{i,t}^n = \hat{X}_{i,0}^n + \ell_i^n t - \mu_i^n \int_0^t \hat{Z}_{i,s}^n ds - \gamma_i^n \int_0^t \hat{Q}_{i,s}^n ds, \\
+ \frac{1}{\sqrt{n}}\Big(\breve{A}_i^n(t) - \lambda_i^n t\Big) - \frac{1}{\sqrt{n}}\left(\breve{S}_i^n\left(\int_0^t \frac{Z_{i,s}^n}{n} ds\right) - \mu_i^n \int_0^t Z_{i,s}^n ds\right) \\
- \frac{1}{\sqrt{n}}\left(\breve{R}_i^n\left(\int_0^t \frac{Q_{i,s}^n}{n} ds\right) - \gamma_i^n \int_0^t Q_{i,s}^n ds\right),
\end{aligned}$$

where as given in (2.6),

$$\hat{Q}_t^n = (e\cdot \hat{X}_t^n)^+ U_t^n, \quad \hat{Z}_t^n = \hat{X}_t^n - (e\cdot \hat{X}_t^n)^+ U_t^n\,.$$

Here, we assume that the control $U^n$ is admissible, i.e., $U^n_t = U^n(t, \breve{A}^n_{[0,t]}, \breve{S}^n_{[0,t]}, \breve{R}^n_{[0,t]})$. We enforce this assumption in the rest of the section.

For a triplet

$$\psi \doteq \Big(\{\phi_i\}_{i=1}^d, \{\psi_i\}_{i=1}^d, \{\varphi_i\}_{i=1}^d\Big)$$

such that $\phi_i, \psi_i, \varphi_i$ are positive valued functions, let $\hat{X}^{n,\psi}$ be the solution to the following equation:

$$\begin{aligned}\hat{X}^{n,\psi}_{i,t} &= \hat{X}^{n,\psi}_{i,0} + \ell^n_i t - \mu^n_i \int_0^t \hat{Z}^{n,\psi}_{i,s} ds - \gamma^n_i \int_0^t \hat{Q}^{n,\psi}_{i,s} ds \\ &\qquad + \frac{1}{\sqrt{n}}\Big(\breve{A}^n_i\Big(\int_0^t \phi_{i,s} ds\Big) - \lambda^n_i t\Big) - \frac{1}{\sqrt{n}}\Big(\breve{S}^n_i\left(\int_0^t \psi_{i,s}\frac{Z^{n,\psi}_{i,s}}{n} ds\right) - \mu^n_i \int_0^t Z^{n,\psi}_{i,s} ds\Big) \\ &\qquad - \frac{1}{\sqrt{n}}\Big(\breve{R}^n_i\left(\int_0^t \varphi_{i,s}\frac{Q^{n,\psi}_{i,s}}{n} ds\right) - \gamma^n_i \int_0^t Q^{n,\psi}_{i,s} ds\Big),\end{aligned} \tag{3.31}$$

where

$$\hat{X}^{n,\psi}_0 = \hat{X}^n_0\,, \quad \hat{X}^{n,\psi}_t = \hat{Q}^{n,\psi}_t + \hat{Z}^{n,\psi}_t\,, \quad \hat{Z}^{n,\psi}_t = \hat{X}^{n,\psi}_t - (e\cdot \hat{X}^{n,\psi}_t)^+ U^{n,\psi}_t\,,$$

with $U^{n,\psi} = U^n(t, \breve{A}^{n,\phi}_{[0,t]}, \breve{S}^{n,\psi}_{[0,t]}, \breve{R}^{n,\varphi}_{[0,t]})$. The definitions of $\breve{A}^{n,\phi}$, $\breve{S}^{n,\psi}$, $\breve{R}^{n,\varphi}$ are defined according to (3.27).

*Remark* 3.12. Observe that we choose $\{\breve{S}^n_i\}_{i=1}^d$ and $\{\breve{R}^n_i\}_{i=1}^d$ with rates $\{n\mu^n_i\}_{i=1}^d$ and $\{n\gamma^n_i\}_{i=1}^d$, respectively, instead of $\{\mu^n_i\}_{i=1}^d$ and $\{\gamma^n_i\}_{i=1}^d$. This will be convenient in proving the appropriate stability in Lemma 5.4.

Define

$$\Theta^n \doteq \limsup_{T\to\infty} \frac{1}{T}\log \mathbb{E}\Big[e^{T\left(\frac{1}{T}\int_0^T r(\hat{X}^n_t, U^n_t)dt\right)\wedge LT}\Big],$$

$$\mathfrak{K}^n(\psi, T) \doteq \frac{1}{T}\int_0^T \mathfrak{k}^n(\psi_s) ds\,, \tag{3.32}$$

where

$$\mathfrak{k}^n(\psi_s) \doteq \sum_{i=1}^d \big(\lambda^n_i \varkappa(\phi_{i,s}) + n\mu^n_i \varkappa(\psi_{i,s}) + n\gamma^n_i \varkappa(\varphi_{i,s})\big)\,.$$

Let $\mathcal{E}^n$ be the set of all $\psi$ functions that are $\bar{\mathcal{G}}^n_t$–progressively measurable such that for every $T > 0$, $\mathfrak{K}^n(\psi, T) < \infty$. Also for every $M > 0$, define $\mathcal{E}^n_M$ as the set of all $\psi \in \mathcal{E}^n$ such that $\mathfrak{K}^n(\psi, T) \le M$, for every $T > 0$. We are now in a position to state the crucial variational representation theorems. We again remark that these results are stated in the form that is convenient for us to work with.

**Theorem 3.6.** *We have*

$$\limsup_{T\to\infty} \frac{1}{T}\log \mathbb{E}[e^{\int_0^T r(\hat{X}^n_t, U^n_t)dt}] = \limsup_{T\to\infty} \sup_{\psi\in\mathcal{E}^n} \mathbb{E}\Big[\frac{1}{T}\int_0^T r(\hat{X}^{n,\psi}_t, U^{n,\psi}_t)dt - \mathfrak{K}^n(\psi, T)\Big].$$

**Theorem 3.7.**

$$\Theta^n = \limsup_{T\to\infty} \sup_{\psi\in\mathcal{E}^n} \mathbb{E}\Big[\Big(\frac{1}{T}\int_0^T r(\hat{X}^{n,\psi}_t, U^{n,\psi}_t)dt\Big)\wedge L - \mathfrak{K}^n(\psi, T)\Big]. \tag{3.33}$$

*Moreover, for every $\delta > 0$,*

$$\Theta^n \le \limsup_{T\to\infty} \sup_{\psi\in\mathcal{E}^n_M} \mathbb{E}\Big[\Big(\frac{1}{T}\int_0^T r(\hat{X}^{n,\psi}_t, U^{n,\psi}_t)dt\Big)\wedge L - \mathfrak{K}^n(\psi, T)\Big] + \delta. \tag{3.34}$$

*Here, $M$ only depends on $\delta$ and $L$.*

*Remark* 3.13. The content of the second statement of the above theorem is that for every $\delta > 0$, there are $2\delta$–optimal controls $\psi \in \mathcal{E}_M^n$. This in particular asserts that the hypothesis of Lemma 3.5 is satisfied which will in turn help us to use Lemma 3.5 in proving Lemma 5.4.

We next define the mean empirical (occupation) measures for the extended diffusion-scaled processes $\hat{X}_t^{n,\psi}$ in (3.31) as follows. For any $U^n \in \mathbb{U}^n$ and $\psi \in \mathcal{E}^n$, let

$$\begin{aligned}
\mu_{U,\psi}^{n,T}(A \times B \times C) &\doteq \frac{1}{T}\int_0^T \mathbb{1}_{\{(\hat{X}_t^{n,\psi}, U_t^{n,\psi}, \psi_t) \in A\times B\times C\}} dt, \\
\mu_{U,\psi}^{n,T,1}(A \times B) &\doteq \frac{1}{T}\int_0^T \mathbb{1}_{\{(\hat{X}_t^{n,\psi}, U_t^{n,\psi}) \in A\times B\}} dt, \qquad (3.35)\\
\mu_{U,\psi}^{n,T,2}(A \times C) &\doteq \frac{1}{T}\int_0^T \mathbb{1}_{\{(\hat{X}_t^{n,\psi}, \psi_t) \in A\times C\}} dt, \\
\mu_{U,\psi}^{n,T,3}(A) &\doteq \frac{1}{T}\int_0^T \mathbb{1}_{\{\hat{X}_t^{n,\psi} \in A\}} dt, \qquad (3.36)
\end{aligned}$$

for any Borel set $A \subset \mathbb{R}^d$, $B \subset \mathbb{U}$ and $C \subset \mathcal{Y}$ with $\mathcal{Y} \doteq \mathbb{R}_+^d \times \mathbb{R}_+^d \times \mathbb{R}_+^d$. We will use $\mu_{U,\psi}^{n,T,3}$ with only $\psi$ being Markov in the lower bound proof while with only $U^n$ being Markov in the upper bound proof below. Thus we do not define a corresponding $\widetilde{\mu}_{U,\psi}^{n,T}$ when both $U^n$ and $\psi$ being Markov as $\widetilde{\mu}_{v,w}^{*,T}$ defined in (3.18). For $n \in \mathbb{N}$, we represent their weak limits in $T$ (if they exist and the associated subsequence $T_k$ is irrelevant) by $\mu_{U,\psi}^n$, $\mu_{U,\psi}^{n,1}$, $\mu_{U,\psi}^{n,2}$, $\mu_{U,\psi}^{n,3}$, respectively.

It is evident from Theorem 3.6 that the term $\frac{1}{T}\int_0^T r(\hat{X}_t^{n,\psi}, U_t^{n,\psi})dt - \mathfrak{K}^n(\psi, T)$ can be written as the integral over the mean empirical measure of $(\hat{X}^{n,\psi}, U^{n,\psi}, \psi)$ as follows:

$$\frac{1}{T}\int_0^T \Big(r(\hat{X}_t^{n,\psi}, U_t^{n,\psi}) - \mathfrak{k}^n(\psi_t)\Big)dt = \int_{\mathbb{R}^d\times\mathbb{U}\times\mathcal{Y}} \big(r(x,u) - \mathfrak{k}^n(y)\big) d\mu_{U,\psi}^{n,T}(x,u,y)\,.$$

In the proof of the lower bound (Theorem 4.1), we work with a general admissible control $U_t^n$ (whose corresponding extended process is $U^{n,\psi}$) and a specific control $\psi_t = f_n(\hat{X}^{n,\psi})$, for some appropriate function $f_n$. Because of this, we make use of the above mean empirical measures of $(\hat{X}^{n,\psi}, U^{n,\psi})$. However, in the proof of the upper bound (Theorem 5.1), we work with a Markov control $U_t^n = g_n(\hat{X}_t^n)$ for some appropriate function $g_n$ and a general control $\psi \in \mathcal{E}_M^n$ (for some $M > 0$). In this case, the corresponding extended process has $U_t^{n,\psi} = g_n(\hat{X}_t^{n,\psi})$. Using this property, we can then write

$$\frac{1}{T}\int_0^T r(\hat{X}_t^{n,\psi}, g_n(\hat{X}_t^{n,\psi}))dt = \int_{\mathbb{R}^d} r(x, g_n(x)) d\mu_{U,\psi}^{n,T,3}(x).$$

3.3. **A suitable topology and corresponding lemmas.** In this section, we describe a suitable topology and give some important consequences (see Lemmas 3.4 and 3.5). To that end, we fix a $\delta > 0$ and choose $\delta$–optimal $\psi^n(T)$ for every $T$ (we will suppress the $\delta$-dependence throughout). From the choice of $\psi^n(T)$ and Theorem 3.7, we can assume that $\psi^n(T) \in \mathcal{E}_M^n$ and we have

$$\Theta^n \le \limsup_{T\to\infty} \mathbb{E}\Big[\Big(\frac{1}{T}\int_0^T r\big(\hat{X}_t^{n,\psi^n(T)}, U_t^{n,\psi^n(T)}\big)dt\Big) \wedge L - \mathfrak{K}^n(\psi^n(T), T)\Big] + \delta.$$

Since $\psi^n(T) \in \mathcal{E}_M^n$, we have

$$\mathfrak{K}^n(\psi^n(T), T) \le M, \text{ for every } T > 0.$$

We will use this bound extensively to prove some existence and convergence results. Therefore, it is helpful for us to work with a topology under which the above bound gives compactness. This topology is defined below. First, notice that since $\varkappa(\cdot) > 0$, $\sqrt{\varkappa(\phi_\cdot)} \in L^2_{\text{loc}}(\mathbb{R}_+, \mathbb{R})$, whenever

$$\frac{1}{T}\int_0^T \varkappa(\phi_s)ds < \infty, \text{ for every } T > 0.$$

From this observation, we can see that $\mathcal{E}^n$ can be regarded as the subset of $\mathcal{Z} \doteq L^2_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d)$. Now we equip $\mathcal{Z}$ (and thereby $\mathcal{E}^n$) with a topology that is inherited from the weak* topology of $\mathcal{Z}_T \doteq L^2([0,T], \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d)$ (which we denote by $L_{T,w}$ or simply weak convergence) which will be useful for us. $\mathcal{Z}$ is equipped with the coarsest topology for which the mapping $\mathcal{Z} \ni u \mapsto u_{|[0,T]} \in \mathcal{Z}_T$ is continuous in $L_{T,w}$ for every $T > 0$, *i.e.*, we say $u^n \to u$ for $u^n, u \in \mathcal{Z}$ under this topology if and only if $u^n_{|[0,T]}$ converges to $u_{|[0,T]}$ in $L_{T,w}$, for every $T > 0$. We denote the space $\mathcal{Z}$ by $L_{\infty,w}$ when equipped with the aforementioned topology.

**Lemma 3.4.** *For every $M > 0$, the set of all $w \in \mathcal{Z}$ such that*

$$\sup_{T>0} \frac{1}{T}\int_0^T \|w_t\|^2 dt \le M$$

*is compact in $L_{\infty,w}$.*

*Proof.* For a fixed $M > 0$, let the set in the hypothesis be denoted by $\mathcal{K}$. Now assume the contrary, that is, there is a sequence $\{w^n\}_{n\in\mathbb{N}} \subset \mathcal{K}$ which is not convergent. In other words, there is a $T_0 > 0$ for which $\{w^n_{|[0,T_0]}\}_{n\in\mathbb{N}} \subset L_{T_0,w}$ is not convergent in $L_{T_0,w}$. Since $w^n \in \mathcal{K}$, we have $\sup_{T>0} \frac{1}{T}\int_0^T \|w^n_t\|^2 dt \le M$. In particular,

$$\int_0^{T_0} \|w^n_t\|^2 dt = \int_0^{T_0} \|w^n_{|[0,T_0],t}\|^2 dt < MT_0$$

which implies that $\{w^n_{|[0,T_0]}\}_{n\in\mathbb{N}}$ is convergent in $L_{T_0,w}$. This is a contradiction and also proves the result. □

Even though we have defined the topology above for $\mathcal{Z}$, the crucial compactness result below is only proved for a simpler case of $L^2_{\text{loc}}(\mathbb{R}^+, \mathbb{R})$ to keep the expressions in the proof concise. To this end, we equip $L^2_{\text{loc}}(\mathbb{R}^+, \mathbb{R})$ with the coarsest topology for which the mapping $L^2_{\text{loc}}(\mathbb{R}^+, \mathbb{R}) \ni u \mapsto u_{|[0,T]} \in L^2([0,T], \mathbb{R})$ is continuous in weak topology of $L^2([0,T], \mathbb{R})$ for every $T > 0$. From now on, the space $L^2_{\text{loc}}(\mathbb{R}^+, \mathbb{R})$ is always assumed to be equipped with such a topology.

**Lemma 3.5.** *Suppose $\phi^n > 0$ for every $n$, and*

$$\sup_{n\in\mathbb{N}} \sup_{T>0} \frac{n}{T}\int_0^T \varkappa(\phi^n_t)dt \le M, \text{ for some } M > 0. \tag{3.37}$$

*Then,*

$$\limsup_{n\to\infty} \sup_{T>0} \frac{1}{T}\int_0^T |\sqrt{n}(1-\phi^n_t)|^2 dt \le 2M.$$

*In particular, $\{\sqrt{n}(1-\phi^n)\}_{n\in\mathbb{N}}$ is compact in $L^2_{loc}(\mathbb{R}^+, \mathbb{R})$.*

The proof of this result is given in the Appendix. The following is an easy corollary (a multi-dimensional version) to the above lemma. For $\psi \in \mathcal{E}^n$, define $h^n(\psi) = h^n(\phi, \psi, \varphi) \doteq \big(\sqrt{n}(e-\phi), \sqrt{n}(e-\psi), \sqrt{n}(e-\varphi)\big)$. Recall $e = (1,1,\dots,1)^{\mathsf{T}} \in \mathbb{R}^d$.

**Corollary 3.3.** *Suppose $\{\psi^n\}_{n\in\mathbb{N}} \subset \mathcal{E}^n_M$ and*

$$\limsup_{n\to\infty} \sup_{T>0} \sum_{i=1}^d \frac{1}{T}\int_0^T \Big(\lambda^n_i|\sqrt{n}(1-\phi^n_{i,s})|^2 + \mu^n_i|\sqrt{n}(1-\psi^n_{i,s})|^2 + \gamma^n_i|\sqrt{n}(1-\varphi^n_{i,s})|^2\Big)ds \le M\,.$$

*Then, $\{h^n(\psi^n)\}_{n\in\mathbb{N}}$ is a tight family of $L_{\infty,w}$–valued random variables.*

An immediate consequence of the above corollary is that for a fixed $n$, $\{\psi^n(T)\}_{T>0}$ is compact in $L_{\infty,w}$ and therefore, has a weak limit point $\psi^n \in L_{\infty,w}$ along a subsequence (say $T_k$). Moreover,

$$\mathbb{E}\Big[\limsup_{T\to\infty} \mathfrak{K}^n(\psi^n(T),T)\Big] \geq \mathbb{E}\Big[\limsup_{T\to\infty} \mathfrak{K}^n(\psi^n,T)\Big] \tag{3.38}$$

which follows from the weak lower-semicontinuity of the norm.

3.4. **An auxiliary diffusion limit arising from the variational formulation.** Let $\mathfrak{D}_T^{\mathbb{U}}$ be the set of $\mathbb{U}$–valued càdlàg functions on $[0,T]$ equipped with the Skorohod topology.

**Theorem 3.8.** *Suppose $\psi^n = (\phi^n, \psi^n, \varphi^n) \in \mathcal{E}^n$ is such that*

$$\sup_{n\in\mathbb{N}} \sup_{T>0} \mathfrak{K}^n(\psi^n,T) < \infty.$$

*Then the family of processes $\{(\hat{X}^{n,\psi^n}, U^{n,\psi^n}, h^n(\psi^n))\}_{n\in\mathbb{N}}$ (defined by (3.31) with $\psi = \psi^n$) is tight in $\mathfrak{D}_T^d \times \mathfrak{D}_T^{\mathbb{U}} \times L_{\infty,w}$, for every $T>0$. Moreover, every limit point $(X,u,w) \doteq (X^{*,u,w},u,w)$ with $w=(w^1,w^2,w^3)$ satisfies*

$$dX_t^{*,u,w} = b(X_t^{*,u,w},u_t))dt + \Sigma\widetilde{w}_t dt + \Sigma dW_t, \tag{3.39}$$

*for some $d$–dimensional Brownian motion $W$ and $\widetilde{w}$ is defined as*

$$\widetilde{w}_{i,t} \doteq \frac{\lambda_i w_{i,t}^1 + \mu_i\rho_i w_{i,t}^2}{\sqrt{2}} = \frac{\lambda_i}{\sqrt{2}}(w_{i,t}^1 + w_{i,t}^2).$$

We only give the sketch of the proof below. To begin with, fix $T>0$ and re-write (3.31) for $\psi = \psi^n$ as follows:

$$\begin{aligned}
\hat{X}_{i,t}^{n,\psi^n} &= \hat{X}_{i,0}^{n,\psi^n} + \ell_i^n t - \mu_i^n \int_0^t \hat{Z}_{i,s}^{n,\psi^n} ds - \gamma_i^n \int_0^t \hat{Q}_{i,s}^{n,\psi^n} ds \\
&\quad + \frac{1}{\sqrt{n}}\Big(\breve{A}_i^n\big(\int_0^t \phi_{i,s}^n ds\big) - \lambda_i^n \int_0^t \phi_{i,s}^n ds\Big) \\
&\quad - \frac{1}{\sqrt{n}}\left(\breve{S}_i^n\left(\int_0^t \frac{\psi_{i,s}^n Z_{i,s}^{n,\psi^n}}{n} ds\right) - n\mu_i^n \int_0^t \frac{\psi_{i,s}^n Z_{i,s}^{n,\psi^n}}{n} ds\right) \\
&\quad - \frac{1}{\sqrt{n}}\left(\breve{R}_i^n\left(\int_0^t \frac{\varphi_{i,s}^n Q_{i,s}^{n,\psi^n}}{n} ds\right) - n\gamma_i^n \int_0^t \frac{\varphi_{i,s}^n Q_{i,s}^{n,\psi^n}}{n} ds\right) \\
&\quad - \frac{\lambda_i^n}{\sqrt{n}} \int_0^t (1-\phi_{i,s}^n) ds - \mu_i^n \sqrt{n} \int_0^t (1-\psi_{i,s}^n) \frac{Z_{i,s}^{n,\psi^n}}{n} ds \\
&\quad - \gamma_i^n \sqrt{n} \int_0^t (1-\varphi_{i,s}^n) \frac{Q_{i,s}^{n,\psi^n}}{n} ds.
\end{aligned} \tag{3.40}$$

Here,

$$\hat{X}_0^{n,\psi^n} = \hat{X}_0^n, \quad \hat{X}_t^{n,\psi^n} = \hat{Q}_t^{n,\psi^n} + \hat{Z}_t^{n,\psi^n}, \quad \hat{Z}_t^{n,\psi^n} = \hat{X}_t^{n,\psi^n} - (e\cdot \hat{X}_t^{n,\psi^n})^+ U_t^{n,\psi^n}.$$

We now observe that

$$\begin{aligned}
\hat{M}_{i,t}^{n,A,\psi^n} &\doteq \frac{1}{\sqrt{n}}\Big(\breve{A}_i^n\big(\int_0^t \phi_{i,s}^n ds\big) - \lambda_i^n \int_0^t \phi_{i,s}^n ds\Big), \\
\hat{M}_{i,t}^{n,S,\psi^n} &\doteq \frac{1}{\sqrt{n}}\left(\breve{S}_i^n\left(\int_0^t \frac{\psi_{i,s}^n Z_{i,s}^{n,\psi^n}}{n} ds\right) - n\mu_i^n \int_0^t \frac{\psi_{i,s}^n Z_{i,s}^{n,\psi^n}}{n} ds\right),
\end{aligned}$$

$$\hat{M}^{n,R,\psi^n}_{i,t} \doteq \frac{1}{\sqrt{n}}\Bigg(\breve{R}^n_i\left(\int_0^t \frac{\varphi^n_{i,s} Q^{n,\psi^n}_{i,s}}{n} ds\right) - n\gamma^n_i \int_0^t \frac{\varphi^n_{i,s} Q^{n,\psi^n}_{i,s}}{n} ds\Bigg),$$

are square integrable $\bar{\mathcal{G}}^n_t$–martingales. Let

$$\hat{M}^{n,\psi^n} \doteq \hat{M}^{n,A,\psi^n} + \hat{M}^{n,S,\psi^n} + \hat{M}^{n,R,\psi^n}$$

and

$$\hat{\Xi}^n \doteq \frac{\lambda^n_i}{\sqrt{n}} \int_0^t (1-\phi^n_{i,s}) ds + \mu^n_i \sqrt{n} \int_0^t (1-\psi^n_{i,s}) \frac{Z^{n,\psi^n}_{i,s}}{n} ds + \gamma^n_i \sqrt{n} \int_0^t (1-\varphi^n_{i,s}) \frac{Q^{n,\psi^n}_{i,s}}{n} ds.$$

Below we state the tightness of the following families of random variables which are the key aspects of the proof.

(i) $\{U^{n,\psi^n}\}_{n\in\mathbb{N}}$: Since $U^{n,\psi^n}$ is a $\mathbb{U}$–valued process which is compact, the family is trivially tight in $\mathfrak{D}^{\mathbb{U}}_T$ with a limit point $u$.

(ii) $\{h^n(\psi^n)\}_{n\in\mathbb{N}}$: Using Corollary 3.3 and the hypothesis of the theorem, we can conclude that this family of random variables is tight in $L_{\infty,w}$.

(iii) $\{\psi^n\}_{n\in\mathbb{N}}$: From the definition of $h^n$, it is clear that $\phi^n_i, \psi^n_i, \varphi^n_i \Rightarrow e$, for every $1 \le i \le d$, as $n \to \infty$. This implies that $\psi^n \Rightarrow (e,e,e)$, as $n \to \infty$. Recall that $e = (1,\dots,1)^{\mathsf{T}} \in \mathbb{R}^d$.

(iv) $\big\{((n^{-1}Z^{n,\psi^n}_i, n^{-1}Q^{n,\psi^n}_i)\big\}_{n\in\mathbb{N}}$: Following the arguments of the proof of [11, Lemma 4(ii)], we can conclude that

$$(n^{-1}Z^{n,\psi^n}_i, n^{-1}Q^{n,\psi^n}_i) \Rightarrow (\rho_i, 0) \quad \text{in } \mathfrak{D}_T \quad \text{as } n \to \infty.$$

(v) $\{\hat{M}^{n,\psi^n}\}_{n\in\mathbb{N}}$: From the martingale central limit theorem, the random time change lemma, the fact that $\psi^n \Rightarrow (e,e,e)$ as $n \to \infty$ and the above display, we can conclude that

$$\hat{M}^{n,\psi^n} \Rightarrow \Sigma W \quad \text{in } \mathfrak{D}^d_T \quad \text{as } n \to \infty.$$

Here, $W$ is a $d$–dimensional Brownian motion and $\Sigma = \sqrt{2}\,\mathrm{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \dots, \sqrt{\lambda_d})$.

(vi) $\{\hat{\Xi}^n\}_{n\in\mathbb{N}}$: From the above tightness in (ii) and the following arguments used in showing tightness of $\{m^k\}_{k\in\mathbb{N}}$ (in the proof of Theorem 3.5), we can conclude the tightness of $\{\int_0^\cdot h^n_t dt\}_{n\in\mathbb{N}}$ in $\mathfrak{C}^d_T \times \mathfrak{C}^d_T \times \mathfrak{C}^d_T$. Combining this with tightness in (iv), gives us the following. If $h^n \Rightarrow w = (w^1, w^2, w^3) \in L_{\infty,w}$, then

$$\int_0^\cdot h^n_t dt \Rightarrow \Big(\int_0^\cdot w^1_t dt, \int_0^\cdot w^2_t dt, \int_0^\cdot w^3_t dt\Big) \text{ in } \mathfrak{C}^d_T \times \mathfrak{C}^d_T \times \mathfrak{C}^d_T \text{ as } n \to \infty$$

and

$$\hat{\Xi}^n_i \Rightarrow \lambda_i w^1 + \mu_i \rho_i w^2.$$

Following the arguments of the proof of [11, Lemma 4(iii)], we can show that $\{\hat{X}^{n,\psi^n}\}_{n\in\mathbb{N}}$ is tight in $\mathfrak{D}^d_T$. Therefore, along a subsequence (again denoted by $n$),

$$\big(\hat{X}^{n,\psi^n}, U^{n,\psi^n}, h^n(\psi^n)\big) \Rightarrow (X^{*,u,w}, u, w) \text{ in } \mathfrak{D}^d_T \times L_{\infty,w}, \text{ as } n \to \infty.$$

Here, $X^{*,u,w}$ and $w$ are related according to (3.39).

*Remark* 3.14. If in the above theorem, $U^{n,\psi^n} = v(\hat{X}^{n,\psi^n})$, for some $v : \mathbb{R}^d \to \mathbb{U}$ that is continuous, then the limit point $\hat{X}^{n,\psi^n}$ is given as the solution to

$$dX^{*,v,w}_t = b(X^{*,v,w}_t, v(X^{*,v,w}_t))dt + \Sigma \widetilde{w}_t dt + \Sigma dW_t.$$

This diffusion resembles very much a well studied process in relation to the ERSC cost known as the *ground diffusion process* which is defined as the solution to the following equation

$$dX^{*,v}_t = b(X^{*,v}_t, v(X^{*,v}_t))dt + \Sigma\Sigma^{\mathsf{T}} \nabla \Phi^v(X^{*,v}_t)dt + \Sigma dW_t. \tag{3.41}$$

It turns out that $w^* = \Sigma^{\mathsf{T}} \nabla \Phi^v(\cdot)$ is the optimal stationary Markov control for (3.6). We do not go into further details and interested reader can refer [1, 5] and the reference therein. The control in (3.39) corresponds to a sub-optimal control for (3.6).

## 4. Proof of the Lower bound for Theorem 2.4

In this section we prove the following lower bound result.

**Theorem 4.1.** *Under Assumption 2.1,*

$$\liminf_{n\to\infty} \hat{\Lambda}^n(\hat{X}_0^n) \geq \Lambda.$$

*Proof.* Let $n_k$ be a sequence along which $\liminf_{n\to\infty} \hat{\Lambda}^n(\hat{X}_0^n)$ is attained. We simply denote this sequence as $n$. For a fixed $\delta > 0$, choose an admissible control $U^n = U^n(t, \breve{A}^n_{[0,t]}, \breve{S}^n_{[0,t]}, \breve{R}^n_{[0,t]})$ (with the associated SCP $\hat{Z}^n = \hat{Z}^n(t, \breve{A}^n_{[0,t]}, \breve{S}^n_{[0,t]}, \breve{R}^n_{[0,t]})$ defined according to (2.6)) such that

$$\hat{\Lambda}^n(\hat{X}_0^n) + \delta \geq J(\hat{X}_0^n, U^n). \tag{4.1}$$

In the following, we will show that

$$\liminf_{n\to\infty} J(\hat{X}_0^n, U^n) \geq \Lambda - \delta. \tag{4.2}$$

We will use the notation from Section 3.2. Applying Theorem 3.6, we have

$$\begin{aligned} J(\hat{X}_0^n, U^n) &= \limsup_{T\to\infty} \sup_{\psi\in\mathcal{E}^n} \mathbb{E}\left[\frac{1}{T}\int_0^T r(\hat{X}_t^{n,\psi}, U_t^{n,\psi})dt - \mathfrak{K}^n(\psi, T)\right] \\ &\geq \limsup_{T\to\infty} \mathbb{E}\left[\frac{1}{T}\int_0^T r(\hat{X}_t^{n,\psi}, U_t^{n,\psi})dt - \mathfrak{K}^n(\psi, T)\right] \end{aligned}$$

where the inequality holds for any $\psi = (\phi, \psi, \varphi) \in \mathcal{E}^n$ and $U_t^{n,\psi} \doteq U^n(t, \breve{A}^{n,\phi}_{[0,t]}, \breve{S}^{n,\psi}_{[0,t]}, \breve{R}^{n,\varphi}_{[0,t]})$ which is a $\mathbb{U}$–valued process, and $\mathfrak{K}^n(\psi, T)$ is defined as in (3.32). The definitions of $\breve{A}^{n,\phi}$, $\breve{S}^{n,\psi}$, $\breve{R}^{n,\varphi}$ are defined according to (3.27).

For every $n$, we now make a particular choice for $\psi^n = (\phi^n, \psi^n, \varphi^n)$. Fix a $\delta > 0$. Then using Corollary 3.2, there exist a $l > 0$ large enough and $w^* \in \mathfrak{W}_{\mathrm{SM}}(l)$ such that

$$\Lambda \leq \int_{\mathbb{R}^d\times\mathbb{U}} \left(r^v(x) - \frac{1}{2}\|w^*(x)\|^2\right)\widetilde{\mu}^*_{v,w^*}(dx) + \delta, \tag{4.3}$$

for every $v \in \mathfrak{U}_{\mathrm{SM}}$. Now we define

$$\widetilde{\phi}_i^n(x) \doteq 1 - \frac{\big(w^*(x)\big)_i}{\sqrt{n}}, \text{ and } \widetilde{\psi}_i^n(x) \doteq 1 - \frac{\big(w^*(x)\big)_i}{\sqrt{n}}, \text{ for } x \in \mathbb{R}^d. \tag{4.4}$$

Finally, we set

$$\phi_{i,t}^n = \widetilde{\phi}_i^n(\hat{X}_t^{n,\psi^n}), \;\; \psi_{i,t}^n = \widetilde{\psi}_i^n(\hat{X}_t^{n,\psi^n}) \text{ and } \varphi_{i,t}^n = 1, \text{ for } t \geq 0.$$

This particular choice is motivated from the arguments used in Theorem 3.5. From the above choice, we can ensure that $\phi_i^n > 0$ and $\psi_i^n > 0$, for large enough $n$. From now on, we simply write $\hat{X}^{n,\psi^n}$ as $\hat{X}^{n,\psi}$. With the above choice, we have

$$J(\hat{X}_0^n, U^n) \geq \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T \Big(r(\hat{X}_t^{n,\psi}, U_t^{n,\psi}) - \sum_{i=1}^d \lambda_i^n \varkappa(\widetilde{\phi}_i^n(\hat{X}_t^{n,\psi})) - \sum_{i=1}^d n\mu_i^n \varkappa(\widetilde{\psi}_i^n(\hat{X}_t^{n,\psi}))\Big)dt\right].$$

From the definitions of $\hat{X}^{n,\psi}$, $\widetilde{\phi}^n$, $\widetilde{\psi}^n$ and the fact that $w^*$ is compactly supported, the infinitesimal generators of $\hat{X}^n$ and $\hat{X}^{n,\psi}$ coincide outside the compact set $\overline{B}_l$, $\hat{X}^{n,\psi}$ will inherit the Lyapunov function corresponding to $\hat{X}^n$ (albeit with different coefficients and compact sets in the Foster-Lyapunov equation).This implies that the family of mean empirical (occupation) measures $\{\mu^{n,T,1}_{U,\psi}\}_{T>0,n\in\mathbb{N}}$ defined in (3.35) is tight in $\mathcal{P}(\mathbb{R}^d\times\mathbb{U})$. Therefore, along a subsequence $T_k$, $\{\mu^{n,T_k,1}_{U,\psi}\}_{k\in\mathbb{N}}$ converges weakly to some $\mu^{n,1}_{U,\psi}\in\mathcal{P}(\mathbb{R}^d\times\mathbb{U})$ and we have

$$\begin{aligned}
J(\hat{X}^n_0,U^n) &\geq \limsup_{k\to\infty}\frac{1}{T_k}\mathbb{E}\Bigg[\int_0^{T_k} r(\hat{X}^{n,\psi}_t,U^{n,\psi}_t)dt-\sum_{i=1}^d\lambda^n_i\int_0^{T_k}\varkappa(\widetilde{\phi}^n_i(\hat{X}^{n,\psi}_t))dt \\
&\qquad -\sum_{i=1}^d n\mu^n_i\int_0^{T_k}\varkappa(\widetilde{\psi}^n_i(\hat{X}^{n,\psi}_t))dt\Bigg] \\
&\geq \int_{\mathbb{R}^d}\Big(r(x,u)-\sum_{i=1}^d\lambda^n_i\varkappa(\widetilde{\phi}^n_i(x))-\sum_{i=1}^d n\mu^n_i\varkappa(\widetilde{\psi}^n_i(x))\Big)d\mu^{n,1}_{U,\psi}(x,u).
\end{aligned}$$

From the choice of $\widetilde{\phi}^n$ and $\widetilde{\psi}^n$, it is clear that $\widetilde{\phi}^n(\cdot)$ and $\widetilde{\psi}^n(\cdot)$ converge to $e=(1,\ldots,1)^{\mathsf{T}}$ uniformly on $\mathbb{R}^d$ and using the fact that $n^{-1}\lambda^n_i\to\lambda_i$ and $\mu^n_i\to\mu_i$, we obtain

$$\Big|\lambda^n_i\varkappa(\widetilde{\phi}^n_i(x))-\frac{1}{2}\lambda_i|\big(w^*(x)\big)_i|^2\Big|\leq C\frac{\sup_{y\in B_l}\|w^*(y)\|^3}{\sqrt{n}}\to 0$$

and

$$\Big|n\mu^n_i\varkappa(\widetilde{\psi}^n_i(x))-\frac{1}{2}\mu_i|\big(w^*(x)\big)_i|^2\Big|\leq C\frac{\sup_{y\in B_l}\|w^*(y)\|^3}{\sqrt{n}}\to 0,$$

uniformly in $x\in\mathbb{R}^d$, for some $C>0$.

Following the computation similar to the one in [4, Pg. 3559-3560], we can conclude that $\mu^{n,1}_{U,\psi}$ converges weakly along a subsequence (still denoted by $n$) to an ergodic occupation measure $\widetilde{\mu}^*_{v,w^*}$, for some $v\in\mathfrak{U}_{\mathrm{SM}}$. From here and using (4.3), we have

$$\liminf_{n\to\infty}\hat{\Lambda}^n(\hat{X}^n_0)\geq\liminf_{n\to\infty}J(\hat{X}^n_0,U^n)-\delta\geq\int_{\mathbb{R}^d}\Big(r^v(x)-\frac{1}{2}\|w^*(x)\|^2\Big)d\widetilde{\mu}^*_{v,w^*}(x)\geq\Lambda-2\delta.$$

From the arbitrariness of $\delta$, we have the result. □

## 5. Proof of the upper bound for Theorem 2.4

In this section we prove the upper bound for asymptotic optimality. This is much more involved than that of the proof of Theorem 4.1 which involved the already existing techniques (in the context of CEC). In contrast, as will be seen below, results similar to Lemma 3.2 and Theorem 3.3 (in the case where the driving noise is a Poisson process) can simplify the proof. Unfortunately, such results cannot be proved in a straightforward way. Therefore, we give a novel approach to overcome the difficulties due to the lack of results similar to Lemma 3.2 and Theorem 3.3.

**Theorem 5.1.** *Under Assumption 2.1,*

$$\limsup_{n\to\infty}\hat{\Lambda}^n(\hat{X}^n_0)\leq\Lambda.$$

*Proof.* Let $v^*\in\mathfrak{U}_{\mathrm{SM}}$ be an optimal control corresponding to $\Lambda$. This control, as we know from Theorem 2.3, satisfies (2.21). This creates an issue in invoking weak convergence of measures later on because of the following: $v^*=v^*(\cdot)$ is in general, merely a Borel measurable function from $\mathbb{R}^d$ to $\mathbb{U}$ and the family of integrals of a Borel measurable function with respect to a weakly convergent family of measures does not necessarily converge to the corresponding integral with respect to the limiting measure. To overcome this, we construct a $\delta$–optimal control $v^\delta$ that is a continuous map from $\mathbb{R}^d\to\mathbb{U}$ and is $e_d\doteq(0,0,\ldots,1)^{\mathsf{T}}$ outside a sufficiently large ball (say, $K$ is the radius of the ball).

The construction of the aforementioned $\delta$–optimal is carried out in Lemma A.1. Therefore, we just invoke this lemma and ascertain the existence of such a control which we from now on denote by $v^\delta = v^\delta(\cdot)$.

From the construction of $v^\delta$ in the proof of Lemma A.1, we know that $v^\delta : \mathbb{R}^d \to \mathbb{U}$ is a continuous function such that

$$v^\delta(x) = \begin{cases} v^\delta, \text{ whenever } \|x\| \le K - \frac{1}{l}, \\ e_d, \text{ whenever } \|x\| > K, \end{cases}$$

for sufficiently large $l$. We focus our attention on the case when $\|x\| \le K$ and $\|x\| > K$ as there is a distinct change in behavior of $v^\delta$ at $\|x\| = K$. To that end, we will define the set $R_n$ below which captures this behavior. In the following, using $v^\delta$, we define a SCP $Z^n$ and the corresponding pre-limit process $X^n$, for large enough $n$. Subsequently, we will define diffusion-scaled versions $\hat{Z}^n$ and $\hat{X}^n$. We follow the construction in [4, Pg. 3561] and [11, Section 2.6]. We first note that $v_i^\delta$, for $i = 1, \ldots, d$ plays the role of the fraction of class-$i$ customers in queue, when total queue size is positive. Since $\hat{X}^n$ is the argument of $v^\delta$, we write $\hat{X}_t^n = \hat{x}^n(X_t^n)$ (recall $\hat{x}^n$ in (2.9)). It is important to note that we cannot simply define

$$Z^n = X^n - (e \cdot X^n - n)^+ v^\delta(\hat{X}^n),$$

because $X^n - Z^n$ will not necessarily lie in $\mathbb{Z}_+^d$ if defined as above. To overcome this, define a measurable map $\vartheta : \{z \in \mathbb{R}_+^d : (e \cdot z) \in \mathbb{Z}\} \to \mathbb{Z}_+^d$ as

$$\vartheta(z) \doteq \Big(\lfloor z_1 \rfloor, \lfloor z_2 \rfloor, \ldots, \lfloor z_d \rfloor + \sum_{i=1}^{d} (z_i - \lfloor z_i \rfloor)\Big). \tag{5.1}$$

Observe that $\|\vartheta(z) - z\| \le 2d$ and $e \cdot \vartheta(z) = 1$. Define the set

$$R_n \doteq \Big\{x \in \mathbb{R}_+^d : \max_{1 \le i \le d} |x_i - \rho_i n| \le K\sqrt{n}\Big\}.$$

For $1 \le i \le d$,

$$Z_i^n = Z_i^n[X^n] = \begin{cases} X_i^n - q_i^n(X^n), \text{ whenever } X^n \in R_n, \\ X_i^n \wedge \Big(n - \sum_{j=1}^{i-1} X_j^n\Big)^+, \text{ otherwise}, \end{cases} \tag{5.2}$$

with $q^n(x) \doteq \vartheta\big((e \cdot x - n)^+ v^\delta(\hat{x}^n)\big)$.

From the discussion in [4, Pg. 3561], this is a well-defined work conserving SCP, for large enough $n$. From now on we restrict ourselves to such large enough $n$. To proceed further, we need to understand the conditions under which we have

$$Q_t^n = X_t^n - Z_t^n = q^n(X_t^n).$$

It is easy to check that the sufficiency condition is $\sum_{i=1}^{d-1} X_{i,t}^n \le n$, which can also be re-written as

$$\sum_{i=1}^{d-1} \hat{X}_{i,t}^n \le \rho_d \sqrt{n}.$$

Define $S_n \doteq \{x : \sum_{i=1}^{d-1} \hat{x}_i^n(x) \le \rho_d \sqrt{n}\}$ with $\hat{x}^n(x)$ as defined in (2.9). We will suppress $x$ and just write $\hat{x}^n$ to keep the expression concise.

Now that we have defined the processes $Z^n$ and $X^n$ completely (see [11, Proposition 1]), we move on to define the diffusion-scaled versions $\hat{X}^n$, $\hat{Z}^n$ and $\hat{Q}^n = \hat{X}^n - \hat{Z}^n$. To do that, we define

$$\hat{q}^n(\hat{x}^n) \doteq \vartheta\big(\sqrt{n}(e \cdot \hat{x}^n) v^\delta(\hat{x}^n)\big).$$

From the above discussion,

$$\hat{Q}^n_t = \hat{X}^n_t - \hat{Z}^n_t = \frac{1}{\sqrt{n}}\hat{q}^n(\hat{X}^n_t) \quad \text{whenever } X^n \in S_n.$$

Therefore, recalling that $\widetilde{r}(q) = \kappa \cdot q$, we have

$$\begin{aligned}
\int_0^T \widetilde{r}(\hat{Q}^n_t)dt &= \int_0^T \widetilde{r}\big(\frac{1}{\sqrt{n}}\hat{q}^n(\hat{X}^n_t)\big)\mathbb{1}_{\{\hat{X}^n_t \in S_n\}}dt + \int_0^T \widetilde{r}\big(\hat{X}^n_t - \hat{Z}^n_t\big)\mathbb{1}_{\{\hat{X}^n_t \notin S_n\}}dt \\
&= \int_0^T \widetilde{r}\big(\frac{1}{\sqrt{n}}\hat{q}^n(\hat{X}^n_t)\big)\mathbb{1}_{\{\hat{X}^n_t \in S_n\}}dt + \int_0^T \widetilde{r}\big(\hat{X}^n_t - \hat{Z}^n_t\big)\mathbb{1}_{\{\hat{X}^n_t \notin S_n\}}dt \\
&\qquad - \int_0^T \widetilde{r}\big(\frac{1}{\sqrt{n}}\hat{q}^n(\hat{X}^n_t)\big)\mathbb{1}_{\{\hat{X}^n_t \notin S_n\}}dt + \int_0^T \widetilde{r}\big(\frac{1}{\sqrt{n}}\hat{q}^n(\hat{X}^n_t)\big)\mathbb{1}_{\{\hat{X}^n_t \notin S_n\}}dt \\
&= \int_0^T \widetilde{r}\big(\frac{1}{\sqrt{n}}\hat{q}^n(\hat{X}^n_t)\big)dt + \int_0^T \widetilde{r}\big(\hat{X}^n_t - \hat{Z}^n_t\big)\mathbb{1}_{\{\hat{X}^n_t \notin S_n\}}dt \\
&\qquad - \int_0^T \widetilde{r}\big(\frac{1}{\sqrt{n}}\hat{q}^n(\hat{X}^n_t)\big)\mathbb{1}_{\{\hat{X}^n_t \notin S_n\}}dt \\
&= \int_0^T \widetilde{r}\big(\frac{1}{\sqrt{n}}\hat{q}^n(\hat{X}^n_t)\big)dt + \Delta^n(T).
\end{aligned}$$

Here,

$$\Delta^n(T) = \Delta^n_1(T) - \Delta^n_2(T)$$

where

$$\Delta^n_1(T) \doteq \int_0^T \widetilde{r}\big(\hat{X}^n_t - \hat{Z}^n_t\big)\mathbb{1}_{\{\hat{X}^n_t \notin S_n\}}dt, \tag{5.3}$$

$$\Delta^n_2(T) \doteq \int_0^T \widetilde{r}\big(\frac{1}{\sqrt{n}}\hat{q}^n(\hat{X}^n_t)\big)\mathbb{1}_{\{\hat{X}^n_t \notin S_n\}}dt. \tag{5.4}$$

The rest of the proof is a consequence of several lemmas that follow. □

To keep the expressions concise, we define $\mathring{Q}^n_t \doteq \frac{1}{\sqrt{n}}\hat{q}^n(\hat{X}^n_t)$. Observe that it suffices for us to show that

$$\limsup_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T}\log \mathbb{E}\Big[e^{\int_0^T \widetilde{r}(\mathring{Q}^n_t)dt + \Delta^n(T)}\Big] \le \Lambda + \delta$$

as this will then imply

$$\begin{aligned}
\limsup_{n\to\infty} \hat{\Lambda}^n &\le \limsup_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T}\log \mathbb{E}\Big[e^{\int_0^T \widetilde{r}(\hat{Q}^n_t)dt}\Big] \\
&= \limsup_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T}\log \mathbb{E}\Big[e^{\int_0^T \widetilde{r}(\mathring{Q}^n_t)dt + \Delta^n(T)}\Big] \le \Lambda.
\end{aligned}$$

For $i = 1, 2$, we first note that since $\hat{X}^n_t - \hat{Z}^n_t = (e \cdot \hat{X}^n_t)^+ \tilde{U}^n_t$, for some process $0 \le \tilde{U}^n \le 1$, we have

$$\begin{aligned}
\Delta^n_i(T) &\le \sqrt{d}\max_{1\le j\le d}\kappa_j \int_0^T \|\hat{X}^n_t\|\mathbb{1}_{\{\hat{X}^n_t \notin S_n\}}dt \\
&\le \sqrt{d}\max_{1\le j\le d}\kappa_j \int_0^T \|\hat{X}^n_t\|\mathbb{1}_{\{\|\hat{X}^n_t\| > \bar{\rho}_d\sqrt{n}\}}dt,
\end{aligned} \tag{5.5}$$

where $\bar{\rho}_d \doteq \frac{\rho_d}{\sqrt{d}}$.

**Lemma 5.1.** *The following holds:*

$$\limsup_{L\to\infty}\limsup_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\log\mathbb{E}\Big[e^{\int_0^T\widetilde{r}(\mathring{Q}^n_t)dt}\mathbb{1}_{\{\int_0^T\widetilde{r}(\mathring{Q}^n_t)dt\geq LT\}}\Big]=-\infty. \tag{5.6}$$

*Proof.* Define a random variable

$$Z^n_T\doteq e^{\int_0^T\widetilde{r}(\mathring{Q}^n_t)dt-LT}.$$

Using this, we clearly have

$$\begin{aligned}e^{-LT}\mathbb{E}\Big[e^{\int_0^T\widetilde{r}(\mathring{Q}^n_t)dt}\mathbb{1}_{\{\int_0^T\widetilde{r}(\mathring{Q}^n_t)dt\geq LT\}}\Big]&=\mathbb{E}\Big[Z^n_T\mathbb{1}_{\{Z^n_T\geq 1\}}\Big]\\&\leq\mathbb{E}\Big[(Z^n_T)^\rho\Big]=e^{-\rho LT}\mathbb{E}\Big[e^{\rho\int_0^T\widetilde{r}(\mathring{Q}^n_t)dt}\Big],\end{aligned}$$

for $\rho>1$. This gives us

$$\begin{aligned}&\limsup_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\log\mathbb{E}\Big[e^{\int_0^T\widetilde{r}(\mathring{Q}^n_t)dt}\mathbb{1}_{\{\int_0^T\widetilde{r}(\mathring{Q}^n_t)dt\geq LT\}}\Big]\\&\qquad\leq-(\rho-1)L+\limsup_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\log\mathbb{E}\Big[e^{\rho\int_0^T\hat{C}_1\|\hat{X}^n_t\|dt}\Big].\end{aligned}$$

Under Assumption 2.1, using Corollary 2.1, it is clear that for $\rho=1+\epsilon$ (for small enough $\epsilon>0$), we can ensure that the second term on the right hand side is finite. Now we take $L\to\infty$ to get the desired result. □

**Lemma 5.2.** *The following holds: for $i=1,2$,*

$$\limsup_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\log\mathbb{E}\Big[e^{\Delta^n_i(T)}\Big]=-\infty, \tag{5.7}$$

*where $\Delta^n_1(T)$ and $\Delta^n_2(T)$ are defined in (5.3) and (5.4), respectively.*

*Proof.* The proof follows exactly along the same lines as the proof of Lemma 5.1. Here, we define random variable

$$\bar{Z}^n_T\doteq\exp\left(\sqrt{d}\max_{1\leq j\leq d}\kappa_j\int_0^T\Big(\|\hat{X}^n_t\|\mathbb{1}_{\{\|\hat{X}^n_t\|>\bar{\rho}_d\sqrt{n}\}}-\bar{\rho}_d\sqrt{n}\Big)dt\right).$$

Using the fact that $\bar{Z}^n_T\geq 1$, we have

$$\mathbb{E}\Big[\bar{Z}^n_T\Big]\leq\mathbb{E}\Big[(\bar{Z}^n_T)^\eta\Big]=\mathbb{E}\left[\exp\left(\eta\sqrt{d}\max_{1\leq j\leq d}\kappa_j\int_0^T\Big(\|\hat{X}^n_t\|\mathbb{1}_{\{\|\hat{X}^n_t\|>\bar{\rho}_d\sqrt{n}\}}-\bar{\rho}_d\sqrt{n}\Big)dt\right)\right].$$

for $\eta>1$. This gives us

$$\begin{aligned}&\limsup_{T\to\infty}\frac{1}{T}\log\mathbb{E}\left[\exp\left(\sqrt{d}\max_{1\leq j\leq d}\kappa_j\int_0^T\Big(\|\hat{X}^n_t\|\mathbb{1}_{\{\|\hat{X}^n_t\|>\bar{\rho}_d\sqrt{n}\}}\Big)dt\right)\right]\\&\leq-(\eta-1)\bar{\rho}_d\sqrt{n}+\limsup_{T\to\infty}\frac{1}{T}\log\mathbb{E}\left[\exp\left(\eta\sqrt{d}\max_{1\leq j\leq d}\kappa_j\int_0^T\|\hat{X}^n_t\|dt\right)\right].\end{aligned}$$

Under Assumption 2.1, using Corollary 2.1, it is clear that for $\eta=1+\epsilon$ (for small enough $\epsilon>0$), we can again ensure that the second term on the right hand side is uniformly bounded in $n$. Now taking $n\to\infty$ and using (5.5), we get the desired result. □

A couple of immediate consequences of the previous two lemmas are the following:

$$\limsup_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\log\mathbb{E}\Big[e^{\int_0^T\widetilde{r}(\mathring{Q}^n_t)dt+\Delta^n(T)}\Big]\leq\limsup_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\log\mathbb{E}\Big[e^{\int_0^T\widetilde{r}(\mathring{Q}^n_t)dt}\Big]\doteq\mathring{\Lambda}$$

and the corollary below.

**Corollary 5.1.** *The following holds:*

$$\mathring{\Lambda} = \lim_{L\to\infty} \limsup_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \log \mathbb{E}\Big[e^{\int_0^T \widetilde{r}(\mathring{Q}_t^n)dt} \mathbb{1}_{\{\int_0^T \widetilde{r}(\mathring{Q}_t^n)dt \le LT\}}\Big].$$

*Proof.* It is clear that for $L > 0$,

$$\limsup_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \log \mathbb{E}\Big[e^{\int_0^T \widetilde{r}(\mathring{Q}_t^n)dt} \mathbb{1}_{\{\int_0^T \widetilde{r}(\mathring{Q}_t^n)dt \le LT\}}\Big] \le \mathring{\Lambda}. \tag{5.8}$$

It is also easy to see that

$$\mathring{\Lambda} \le \max\Bigg\{ \limsup_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \log \mathbb{E}\Big[e^{\int_0^T \widetilde{r}(\mathring{Q}_t^n)dt} \mathbb{1}_{\{\int_0^T \widetilde{r}(\mathring{Q}_t^n)dt \le LT\}}\Big],$$
$$\limsup_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \log \mathbb{E}\Big[e^{\int_0^T \widetilde{r}(\mathring{Q}_t^n)dt} \mathbb{1}_{\{\int_0^T \widetilde{r}(\mathring{Q}_t^n)dt \ge LT\}}\Big]\Bigg\}. \tag{5.9}$$

Taking $L \to \infty$ and using Lemma 5.1, we have the desired result by combining (5.8) and (5.9). □

*Remark* 5.1. Corollary 5.1 implies that Theorem 5.1 follows once we prove that

$$\lim_{L\to\infty} \limsup_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \log \mathbb{E}\Big[e^{\int_0^T \widetilde{r}(\mathring{Q}_t^n)dt} \mathbb{1}_{\{\int_0^T \widetilde{r}(\mathring{Q}_t^n)dt \le LT\}}\Big] \le \Lambda + \delta. \tag{5.10}$$

This is what we do next.

For $L > 0$, define

$$\widetilde{R}_L(\mathring{Q}^n, T) \doteq \Big(\frac{1}{T}\int_0^T \widetilde{r}(\mathring{Q}_t^n)dt\Big) \wedge L \quad \text{and} \quad \Pi_L^n \doteq \limsup_{T\to\infty} \frac{1}{T} \log \mathbb{E}\Big[e^{T\widetilde{R}_L(\mathring{Q}^n,T)}\Big].$$

In what follows, we adopt the notation of Section 3.

**Lemma 5.3.**

$$\limsup_{L\to\infty} \limsup_{n\to\infty} \Pi_L^n \le \Lambda.$$

*Proof.* Applying Theorem 3.7, we write

$$\Pi_L^n = \limsup_{T\to\infty} \sup_{\uppsi\in\mathcal{E}^n} \mathbb{E}\Big[\widetilde{R}_L(\mathring{Q}^{n,\uppsi}, T) - \mathfrak{K}^n(\uppsi, T)\Big],$$

where $\mathring{Q}_t^{n,\Psi} \doteq \frac{1}{\sqrt{n}}\hat{q}^n(\hat{X}_t^{n,\uppsi})$ and $\uppsi = (\phi, \psi, \varphi)$. Recall that $\mathfrak{K}^n(\uppsi, T)$ is defined in (3.32). Now fix $\delta > 0$. From Theorem 3.7, it is also clear that

$$\Pi_L^n \le \limsup_{T\to\infty} \sup_{\uppsi\in\mathcal{E}_M^n} \mathbb{E}\Big[\widetilde{R}_L(\mathring{Q}^{n,\uppsi}, T) - \mathfrak{K}^n(\uppsi, T)\Big] + \delta, \tag{5.11}$$

for $M > 0$ depending only on $L$ and $\delta$. Let $\uppsi^n(T) = (\phi^n(T), \psi^n(T), \varphi^n(T))$ be the $\delta$–optimal control in the above supremum (it is clear that this depends on $n$ and $T$). From this choice (we simply write $\uppsi^n(T)$ as $\uppsi^n = (\phi^n, \psi^n, \varphi^n)$) and the definition of $\mathcal{E}_M^n$, we have

$$\Pi_L^n \le \limsup_{T\to\infty} \mathbb{E}\Big[\widetilde{R}_L(\mathring{Q}^{n,\uppsi^n}, T) - \mathfrak{K}^n(\uppsi^n, T)\Big] + 2\delta, \tag{5.12}$$

and

$$\mathfrak{K}^n(\uppsi^n, T) \le M.$$

Since $\mathfrak{K}^n(\uppsi^n, T) \le M$ implies

$$\limsup_{n\to\infty} \sup_{T>0} \breve{\mathfrak{K}}^n(\uppsi^n, T) \doteq \limsup_{n\to\infty} \sup_{T>0} \frac{1}{T}\int_0^T \|h_t^n(\uppsi^n)\|^2 dt \le 2M,$$

from Corollary 3.3, we can conclude that that $\{h^n(\psi^n)\}_{\{n,T\}} \subset L_{\infty,w}$ is tight in $n$ and $T$. Recall that $h^n(\psi^n) = \big(\sqrt{n}(e-\phi^n), \sqrt{n}(e-\psi^n), \sqrt{n}(e-\varphi^n)\big)$. Therefore, we can choose a compact set $\mathcal{K} = \mathcal{K}_\delta \subset \mathbb{R}^d_+ \times \mathbb{R}^d_+ \times \mathbb{R}^d_+$ (or even bigger compact set if necessary, which will be independent of $n$ and $T$) such that

$$\left|\mathbb{E}\Big[\mathfrak{K}^n(\psi^n, T)\Big] - \mathbb{E}\Big[\mathfrak{K}^n(\psi^n \mathbb{1}_{\{h^n(\psi^n)\in\mathcal{K}\}}, T)\Big]\right| < \delta, \text{ for every } T>0 \text{ and large enough } n$$

and

$$\mathbb{P}\Big(h^n(\psi^n) \in \mathcal{K}^c\Big) \le \frac{\delta}{L}.$$

Therefore, using the above two equations, (5.12) becomes

$$\begin{aligned}
\Pi^n_L &\le \limsup_{T\to\infty}\left(\mathbb{E}\Big[\widetilde{R}_L(\mathring{Q}^{n,\psi^n}, T)\Big(\mathbb{1}_{\{h^n(\psi^n)\in\mathcal{K}\}} + \mathbb{1}_{\{h^n(\psi^n)\in\mathcal{K}^c\}}\Big)\Big] - \mathbb{E}\Big[\mathfrak{K}^n(\psi^n \mathbb{1}_{\{h^n(\psi^n)\in\mathcal{K}\}}, T)\Big]\right) + 3\delta \\
&\le \limsup_{T\to\infty}\left(\mathbb{E}\Big[\widetilde{R}_L(\mathring{Q}^{n,\psi^n}, T)\mathbb{1}_{\{h^n(\psi^n)\in\mathcal{K}\}}\Big] - \mathbb{E}\Big[\mathfrak{K}^n(\psi^n \mathbb{1}_{\{h^n(\psi^n)\in\mathcal{K}\}}, T)\Big]\right) \\
&\qquad + L\mathbb{P}\Big(h^n(\psi^n) \in \mathcal{K}^c\Big) + 3\delta \\
&\le \limsup_{T\to\infty}\left(\mathbb{E}\Big[\widetilde{R}_L(\mathring{Q}^{n,\psi^n}, T)\mathbb{1}_{\{h^n(\psi^n)\in\mathcal{K}\}}\Big] - \mathbb{E}\Big[\mathfrak{K}^n(\psi^n \mathbb{1}_{\{h^n(\psi^n)\in\mathcal{K}\}}, T)\Big]\right) + 4\delta.
\end{aligned}$$

Our next goal is to analyze

$$\limsup_{T\to\infty}\left(\mathbb{E}\Big[\widetilde{R}_L(\mathring{Q}^{n,\psi^n}, T)\mathbb{1}_{\{h^n(\psi^n)\in\mathcal{K}\}}\Big] - \mathbb{E}\Big[\mathfrak{K}^n(\psi^n \mathbb{1}_{\{h^n(\psi^n)\in\mathcal{K}\}}, T)\Big]\right).$$

To do this, we assume without loss of generality that $h^n(\psi^n) \in \mathcal{K}$. Since we already know that the mean empirical measures of $\{h^n(\psi^n)\}_{n\in\mathbb{N}}$ are tight, we will focus on showing the tightness of the mean empirical measures of $\hat{X}^{n,\psi^n}$. Moreover, we will show that

$$\limsup_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\Big[\int_0^T \|\hat{X}^{n,\psi^n}_t\| dt\Big] \le R, \tag{5.13}$$

for some $R > 0$. We assume this for now. It is rigorously stated and proved in Lemma 5.4. Using (5.13), we now proceed to analyze

$$\begin{aligned}
&\limsup_{T\to\infty}\left(\mathbb{E}\Big[\widetilde{R}_L(\mathring{Q}^{n,\psi^n}, T)\mathbb{1}_{\{h^n(\psi^n)\in\mathcal{K}\}}\Big] - \mathbb{E}\Big[\mathfrak{K}^n(\psi^n \mathbb{1}_{\{h^n(\psi^n)\in\mathcal{K}\}}, T)\Big]\right) \\
&= \limsup_{T\to\infty}\left(\mathbb{E}\Big[\Big\{\Big(\frac{1}{T}\int_0^T \widetilde{r}(\mathring{Q}^{n,\psi^n}_t)dt\Big)\wedge L\Big\}\mathbb{1}_{\{h^n(\psi^n)\in\mathcal{K}\}}\Big] - \mathbb{E}\Big[\mathfrak{K}^n(\psi^n \mathbb{1}_{\{h^n(\psi^n)\in\mathcal{K}\}}, T)\Big]\right).
\end{aligned}$$

Since the first term above on the right hand side can be expressed as an integral over the mean empirical measures of the processes $\hat{X}^{n,\psi^n}$ (denoted by $\mu^{n,T,3}_{U,\psi}$; see (3.36) for its definition), it is clear that $\{\mu^{n,T,3}_{U,\psi}\}_{n,T} \subset \mathcal{P}(\mathbb{R}^d)$ is tight in $n$ and $T$, from (5.13). Therefore, along a subsequence $T_k$, there are measures $\mu^{n,3}_{U,\psi} \in \mathcal{P}(\mathbb{R}^d)$ which are the weak limits of $\{\mu^{n,T_k,3}_{U,\psi}\}_{n,k}$. This gives us

$$\begin{aligned}
&\limsup_{k\to\infty}\left(\mathbb{E}\Big[\Big\{\Big(\frac{1}{T_k}\int_0^{T_k} \widetilde{r}(\mathring{Q}^{n,\psi^n}_t)dt\Big)\wedge L\Big\}\mathbb{1}_{\{h^n(\psi^n)\in\mathcal{K}\}}\Big] - \mathbb{E}\Big[\mathfrak{K}^n(\psi^n \mathbb{1}_{\{h^n(\psi^n)\in\mathcal{K}\}}, T_k)\Big]\right) \\
&= \Big\{\int_{\mathbb{R}^d} \widetilde{r}\Big(\vartheta\big(\sqrt{n}(e\cdot\hat{x}^n)v^\delta(\hat{x}^n)\big)\Big)d\mu^{n,3}_{U,\psi}(x)\Big\}\wedge L - \limsup_{k\to\infty}\mathbb{E}\Big[\mathfrak{K}^n(\psi^n \mathbb{1}_{\{h^n(\psi^n)\in\mathcal{K}\}}, T_k)\Big].
\end{aligned} \tag{5.14}$$

It now only remains to take the limit as $n \to \infty$. From Theorem 3.8, we know that for $T > 0$, $\hat{X}^{n,\psi^n}$ converges weakly to $X^{*,v}$ on $\mathfrak{D}^d_T$, for some $L^2([0,T],\mathbb{R}^d)$–valued random variable $v$. Here, $X^{*,v}$ is given as the solution to (3.39).

Again, from (5.13), we can ensure that there exist a subsequence $n_k$ of $n$ and a measure $\widetilde{\mu}^*_{U,\psi}$ such that

$$\begin{aligned}\lim_{k\to\infty}&\Bigg\{\bigg\{\int_{\mathbb{R}^d}\widetilde{r}\Big(\vartheta\big(\sqrt{n}_k(e\cdot\hat{x}^{n_k})v^\delta(\hat{x}^{n_k})\big)\Big)d\mu^{n_k,3}_{U,\psi}(x)\bigg\}\wedge L-\limsup_{m\to\infty}\mathbb{E}\Big[\mathfrak{K}^{n_k}(\psi^{n_k}\mathbb{1}_{\{h^{n_k}(\psi^{n_k})\in\mathcal{K}\}},T_m)\Big]\Bigg\}\\ &=\bigg\{\int_{\mathbb{R}^d}\widetilde{r}\big((e\cdot x)v^\delta(x)\big)d\widetilde{\mu}^*_{U,\psi}(x)\bigg\}\wedge L-\limsup_{k\to\infty}\limsup_{m\to\infty}\mathbb{E}\Big[\mathfrak{K}^{n_k}(\psi^{n_k}\mathbb{1}_{\{h^{n_k}(\psi^{n_k})\in\mathcal{K}\}},T_m)\Big].\end{aligned}$$

In the above, we have used the fact that

$$\Big\|\frac{1}{\sqrt{n}}\vartheta\big(\sqrt{n}(e\cdot\hat{x}^n)v^\delta(\hat{x}^n)\big)-(e\cdot x)^+v^\delta(x)\Big\|\le\frac{2d}{\sqrt{n}}\,.$$

Thus, from (5.14) and the above, we obtain

$$\begin{aligned}&\limsup_{n\to\infty}\limsup_{T\to\infty}\Bigg(\mathbb{E}\Big[\Big\{\Big(\frac{1}{T}\int_0^T\widetilde{r}(\mathring{Q}^{n,\psi^n}_t)dt\Big)\wedge L\Big\}\mathbb{1}_{\{h^n(\psi^n)\in\mathcal{K}\}}\Big]-\mathbb{E}\Big[\mathfrak{K}^n(\psi^n\mathbb{1}_{\{h^n(\psi^n)\in\mathcal{K}\}},T)\Big]\Bigg)\\ &\le\bigg\{\int_{\mathbb{R}^d}\widetilde{r}\big((e\cdot x)v^\delta(x)\big)d\widetilde{\mu}^*_{U,\psi}(x)\bigg\}\wedge L-\limsup_{k\to\infty}\limsup_{m\to\infty}\mathbb{E}\Big[\mathfrak{K}^{n_k}(\psi^{n_k}\mathbb{1}_{\{h^{n_k}(\psi^{n_k})\in\mathcal{K}\}},T_m)\Big].\end{aligned}\tag{5.15}$$

Now let us compute the second term on the right hand side of the above display. From Lemma 3.5 and Theorem 3.8, we have

$$\limsup_{k\to\infty}\limsup_{m\to\infty}\mathbb{E}\Big[\mathfrak{K}^{n_k}(\psi^{n_k}\mathbb{1}_{\{h^{n_k}(\psi^{n_k})\in\mathcal{K}\}},T_m)\Big]\ge\limsup_{T\to\infty}\frac{1}{2T}\int_0^T\|v^*_t\|^2dt\,.\tag{5.16}$$

Taking $L\to\infty$ and using Corollary 5.1 and (5.16), we obtain that equation (5.15) becomes

$$\limsup_{n\to\infty}\hat{\Lambda}^n\le\mathring{\Lambda}\le\int_{\mathbb{R}^d}\widetilde{r}\big((e\cdot x)v^\delta(x)\big)d\widetilde{\mu}^*_{U,\psi}(x)-\limsup_{T\to\infty}\frac{1}{2T}\int_0^T\|v^*_t\|^2dt.\tag{5.17}$$

Now using Lemma 3.3, we have

$$\limsup_{n\to\infty}\hat{\Lambda}^n\le\Lambda_{v^\delta}\le\Lambda+\delta.$$

Arbitrariness of $\delta>0$ gives us the result. □

Now all that remains to be shown is (5.13). To that end, we recall and set up a few useful definitions. Recall that with SCP $Z^n$ as defined in (5.2), $\hat{X}^{n,\psi}$ is given as the solution to

$$\begin{aligned}\hat{X}^{n,\psi}_{i,t}&=\hat{X}^{n,\psi}_{i,0}+\ell^n_it-\mu^n_i\int_0^t\hat{Z}^{n,\psi}_{i,s}ds-\gamma^n_i\int_0^t\hat{Q}^{n,\psi}_{i,s}ds\\ &\quad+\frac{1}{\sqrt{n}}\Big(\breve{A}^n_i(\int_0^t\phi_{i,s}ds)-\lambda^n_it\Big)-\frac{1}{\sqrt{n}}\bigg(\breve{S}^n_i\bigg(\int_0^t\psi_{i,s}\frac{Z^{n,\psi}_{i,s}}{n}ds\bigg)-\mu^n_i\int_0^tZ^{n,\psi}_{i,s}ds\bigg)\\ &\quad-\frac{1}{\sqrt{n}}\bigg(\breve{R}^n_i\bigg(\int_0^t\varphi_{i,s}\frac{Q^{n,\psi}_{i,s}}{n}ds\bigg)-\gamma^n_i\int_0^tQ^{n,\psi}_{i,s}ds\bigg),\end{aligned}$$

where $\hat{X}^{n,\psi}_t=\hat{Q}^{n,\psi}_t+\hat{Z}^{n,\psi}_t$.

To understand the tightness of the empirical measures of $\hat{X}^{n,\psi}_t$, we consider the operator $\mathfrak{L}^{n,w}$ with

$$w=\Big(\{w^1_i\}^d_{i=1},\{w^2_i\}^d_{i=1},\{w^3_i\}^d_{i=1}\Big)$$

defined by

$$\begin{aligned}\mathfrak{L}^{n,w} f(x) &\doteq \sum_{i=1}^{d} \Big(\lambda_i^n w_i^1 \mathfrak{d} f(x; e_i) + \big(\mu_i^n z_i w_i^2 + \gamma_i^n q_i(x,z) w_i^3\big)\mathfrak{d} f(x, -e_i)\Big)\\ &= \mathcal{L}^{n,u} f(x) + \sum_{i=1}^{d} \Big(\lambda_i^n (1-w_i^1)\mathfrak{d} f(x; e_i) + \big(\mu_i^n z_i (1-w_i^2) + \gamma_i^n q_i(x,z)(1-w_i^3)\big)\mathfrak{d} f(x, -e_i)\Big).\end{aligned}$$

Note that $\mathcal{L}^{n,u}$ is the infinitesimal generator of the process $X^n$ (under a constant SCP $u$) given in (2.10). We suppressed the dependence of $\mathfrak{L}^{n,w}$ on $u$. Here, $q_i(x,z) = x_i - z_i$ and $e_i$ is a $\mathbb{R}^d$ vector with 1 as $i$th coordinate and 0, everywhere else. Motivated from [7, Theorem 3.4], let $\eta : \mathbb{R} \to \mathbb{R}$ be a convex function in $\mathcal{C}^2(\mathbb{R})$ defined as follows:

$$\eta(t) \doteq \begin{cases} -\frac{1}{2}, \text{ for } t \le -1,\\ (t+1)^3 - \frac{1}{2}(t+1)^4 - \frac{1}{2}, \text{ for } -1 \le t \le 0,\\ t, \text{ for } t \ge 0\,, \end{cases} \tag{5.18}$$

and let $\xi : \mathbb{R}^d \to \mathbb{R}$ be defined as

$$\xi(x) \doteq \sum_{i=1}^{d} \frac{\eta(x_i)}{\mu_i}\,,$$

and finally let

$$\mathcal{Z}(x) \doteq \epsilon_0\epsilon_1\xi(-x) + \epsilon_0\xi(x).$$

Here, $\epsilon_0$ and $\epsilon_1$ are some positive constants whose values not relevant for us.

**Lemma 5.4.** *Suppose that $\psi^n = (\phi^n, \psi^n, \varphi^n)$ is as in Lemma 5.3. Then,*

$$\limsup_{n\to\infty}\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\Big[\int_0^T \|\hat{X}_t^{n,\psi^n}\| dt\Big] \le R,$$

*for some $R > 0$.*

*Proof.* Following the similar calculations as done in the proof of [7, Theorem 3.4], we have

$$\begin{aligned}\mathfrak{L}^{n,w}\mathcal{Z}(\hat{x}^n(x)) \le& \hat{C}_0 - \hat{C}_1\|\hat{x}^n(x)\| + \sum_{i=1}^{d}\Big(\lambda_i^n(1-w_i^1)\mathfrak{d}\mathcal{Z}(\hat{x}^n(x); n^{-\frac{1}{2}}e_i)\\ &+ \big(\mu_i^n z_i(1-w_i^2) + \gamma_i^n q_i(x,z)(1-w_i^3)\big)\mathfrak{d}\mathcal{Z}(\hat{x}^n(x), -n^{-\frac{1}{2}}e_i)\Big)\,.\end{aligned} \tag{5.19}$$

Note that $|\mathfrak{d}\mathcal{Z}(\hat{x}^n(x), \pm n^{-\frac{1}{2}}e_i)| \le \hat{C}_4 n^{-\frac{1}{2}}$, for some $\hat{C}_4 > 0$ independent of $n$. We now apply the Itô's formula to $\mathcal{Z}(\hat{X}_t^{n,\psi^n})$, to get

$$\begin{aligned}\mathbb{E}[\mathcal{Z}(\hat{X}_T^{n,\psi^n})] &= \mathcal{Z}(\hat{X}_0^{n,\psi^n}) + \mathbb{E}\Big[\int_0^T \mathfrak{L}^{n,w}\mathcal{Z}(\hat{X}_t^{n,\psi^n})dt\Big]\\ &\le \mathcal{Z}(\hat{X}_0^{n,\psi^n}) + \hat{C}_0 T - \hat{C}_1\mathbb{E}\Big[\int_0^T \|\hat{X}_t^{n,\psi^n}\|dt\Big]\\ &+ \sum_{i=1}^{d}\mathbb{E}\Bigg[\int_0^T \Big(\hat{C}_4\lambda_i^n|1-\phi_{i,t}^n|n^{-\frac{1}{2}} + \hat{C}_4\hat{C}_5\mu_i^n|1-\psi_{i,t}^n|\|X_t^{n,\psi^n}\|n^{-\frac{1}{2}}\\ &\qquad + \hat{C}_4\hat{C}_5\gamma_i^n|1-\varphi_{i,t}^n|\|X_t^{n,\psi^n}\|n^{-\frac{1}{2}}\Big)dt\Bigg].\end{aligned}$$

In the above, to get the second term in the integral, we have used the fact that $\max\{\|Z_t^{n,\psi^n}\|, \|Q_t^{n,\psi^n}\|\} \le \hat{C}_5\|X_t^{n,\psi^n}\|$ for some $\hat{C}_5 > 0$ which is independent of $t$ and $n$.

From the above equation, it is clear that

$$\begin{aligned}&\frac{\hat{C}_1}{T}\mathbb{E}\Big[\int_0^T \|\hat{X}_t^{n,\Psi^n}\|dt\Big]\\&\leq \hat{C}_0+\sum_{i=1}^d \frac{1}{T}\mathbb{E}\Bigg[\int_0^T \Big(\hat{C}_4\lambda_i^n|1-\phi_{i,t}^n|n^{-\frac{1}{2}}+\hat{C}_4\hat{C}_5\mu_i^n|1-\psi_{i,t}^n|\|X_t^{n,\Psi^n}\|n^{-\frac{1}{2}}\\&\qquad\qquad+\hat{C}_4\hat{C}_5\gamma_i^n|1-\varphi_{i,t}^n|\|X_t^{n,\Psi^n}\|n^{-\frac{1}{2}}\Big)dt\Bigg].\end{aligned}\tag{5.20}$$

We will show that the following hold:

$$\begin{aligned}\max\Big\{&\frac{1}{T}\mathbb{E}\Big[\int_0^T \mu_i^n|1-\psi_{i,t}^n|\|X_t^{n,\Psi^n}\|n^{-\frac{1}{2}}dt\Big],\\&\frac{1}{T}\mathbb{E}\Big[\int_0^T \gamma_i^n|1-\varphi_{i,t}^n|\|X_t^{n,\Psi^n}\|n^{-\frac{1}{2}}dt\Big]\Big\}=\hat{C}_6+O(n^{-\frac{1}{2}})\frac{1}{T}\mathbb{E}\Big[\int_0^T \|\hat{X}_t^{n,\Psi^n}\|dt\Big],\end{aligned}\tag{5.21}$$

for some $\hat{C}_6>0$ and

$$\limsup_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \lambda_i^n|1-\phi_{i,t}^n|n^{-\frac{1}{2}}dt<\infty.\tag{5.22}$$

We will first show (5.22). Since $n^{-1}\lambda_i^n\to\lambda_i$, we have

$$\limsup_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \lambda_i^n|1-\phi_{i,t}^n|n^{-\frac{1}{2}}dt\leq\lambda_i\limsup_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \sqrt{n}|1-\phi_{i,t}^n|dt.$$

From Corollary 3.3 and Cauchy-Schwartz inequality, we have

$$\limsup_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \sqrt{n}|1-\phi_{i,t}^n|dt\leq\limsup_{n\to\infty}\limsup_{T\to\infty}\Big(\frac{1}{T}\int_0^T n|1-\phi_{i,t}^n|^2dt\Big)^{\frac{1}{2}}<\infty.$$

Combining the previous two displays gives us (5.22). We now proceed to show (5.21). We only show for one of the terms of (5.21) *viz.,*

$$\frac{1}{T}\mathbb{E}\Big[\int_0^T \mu_i^n|1-\psi_{i,t}^n|\|X_t^{n,\Psi^n}\|n^{-\frac{1}{2}}dt\Big]=O(n^{-\frac{1}{2}})\frac{1}{T}\mathbb{E}\Big[\int_0^T \|\hat{X}_t^{n,\Psi^n}\|dt\Big]$$

as the proof for the other term in (5.21) follows along the similar lines. To that end,

$$\begin{aligned}&\frac{1}{T}\mathbb{E}\Big[\int_0^T \mu_i^n|1-\psi_{i,t}^n|\|X_t^{n,\Psi^n}\|n^{-\frac{1}{2}}dt\Big]\\&=\frac{1}{T}\mathbb{E}\Big[\int_0^T \mu_i^n|1-\psi_{i,t}^n|\|X_t^{n,\Psi^n}-\rho n+\rho n\|n^{-\frac{1}{2}}dt\Big]\\&\leq\frac{1}{T}\mathbb{E}\Big[\int_0^T \mu_i^n|1-\psi_{i,t}^n|\Big(\|X_t^{n,\Psi^n}-\rho n\|+\rho n\Big)n^{-\frac{1}{2}}dt\Big]\\&\leq\frac{1}{T}\mathbb{E}\Big[\int_0^T \mu_i^n|\sqrt{n}(1-\psi_{i,t}^n)|\|n^{-\frac{1}{2}}(X_t^{n,\Psi^n}-\rho n)\|n^{-\frac{1}{2}}dt\Big]+\frac{1}{T}\mathbb{E}\Big[\int_0^T \mu_i^n|1-\psi_{i,t}^n|\rho n^{\frac{1}{2}}dt\Big]\\&\leq\frac{n^{-\frac{1}{2}}}{T}\mathbb{E}\Big[\int_0^T \mu_i^n|\sqrt{n}(1-\psi_{i,t}^n)|\|\hat{X}_t^{n,\Psi^n}\|dt\Big]+\frac{1}{T}\mathbb{E}\Big[\int_0^T \mu_i^n|1-\psi_{i,t}^n|\rho n^{\frac{1}{2}}dt\Big].\end{aligned}$$

Using the similar argument in proving (5.22), we can show that the second term above is bounded uniformly in $T$ and $n$. Recall that we have assumed that $h^n(\Psi^n)\in\mathcal{K}$. This means that

$$\frac{n^{-\frac{1}{2}}}{T}\mathbb{E}\Big[\int_0^T \mu_i^n|\sqrt{n}(1-\psi_{i,t}^n)|\|\hat{X}_t^{n,\Psi^n}\|dt\Big]\leq\frac{\mathrm{diam}(\mathcal{K})\mu_i^n n^{-\frac{1}{2}}}{T}\mathbb{E}\Big[\int_0^T \|\hat{X}_t^{n,\Psi^n}\|dt\Big].$$

This proves (5.21) as $\mu_i^n \to \mu_i$. Now substituting (5.21) and (5.22) in (5.20) for sufficiently large $n$ gives us

$$\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\Big[\int_0^T \|\hat{X}_t^{n,\Psi^n}\|dt\Big] \le R,$$

for some $R > 0$. Now taking $n \to \infty$ gives us the desired result. □

## Appendix: Auxiliary results

### Appendix A. A result on ERSC problem for limiting diffusion

**Lemma A.1.** *For every $\delta > 0$ and $u_0 \in \mathbb{U}$, there exists a control $v \in \mathfrak{U}_{\mathrm{SM}}$ such that $\Lambda_v \le \Lambda + 2\delta$ such that $v(\cdot) = u_0 \in \mathbb{U}$ outside a compact set $K = K(\delta)$ and $v(\cdot)$ is continuous.*

*Remark* A.1. This result is the risk-sensitive analog to [4, Theorem 4.1] and we adapt the proof of [1, Proposition 1.3]. The main difference in the proof of [1, Proposition 1.3] and the proof below is that, instead of being a nearly optimal control in [1, Theorem 1.3], $u_0$ (in (A.1)) is replaced by any control in $\mathbb{U}$. This helps us in constructing nearly optimal Markov controls that are continuous. This fact is used in invoking the continuity of the integrand in (5.17) and weak convergence of measures to conclude the equality in (5.17).

*Proof.* For $l > 0$, define

$$b_l(x,u) = \begin{cases} b(x,u), \ \text{if } x \in B_l\,, \\ b(x,u_0), \ \text{if } x \notin B_l\,. \end{cases} \tag{A.1}$$

$r_l(x,u)$ is similarly defined. The corresponding generator is denoted by $\mathcal{L}_l^u$. Following the similar argument as that of [1, Proposition 1.3], we can conclude that there exists a pair $(V_l, \Lambda_{l,u_0}) \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d) \times \mathbb{R}$ such that

$$\min_{u\in\mathbb{U}} \Big[\mathcal{L}_l^u V_l + r_l(x,u)V_l\Big] = \Lambda_{l,u_0} V_l, \tag{A.2}$$

where $V_l$ restricted to $B_l$ is in $\mathcal{C}^2(\mathbb{R}^d)$ and $\Lambda_{u_0} \ge \Lambda_{l,u_0} \ge \Lambda$. Moreover,

$$\inf_{l>0} \inf_{x\in\mathbb{R}^d} V_l(x) > 0.$$

Let $v_l^*$ be the measurable selector of (A.2). Since $v_l^* = u_0$ on $B_l^c$, we can infer that $v_l^* \in \mathfrak{U}_{\mathrm{SSM}}$.

Again following along the same lines as in [1, Proposition 1.3], we can conclude that there exists a subsequence $l_n$ along which $(V_{l_n}, \Lambda_{l_n,u_0})$ converge to $(\breve{V}, \breve{\Lambda})$ in $W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d) \times \mathbb{R}$ such that

$$\min_{u\in\mathbb{U}} \Big[\mathcal{L}^u \breve{V} + r(x,u)\breve{V}\Big] = \breve{\Lambda}\breve{V} \tag{A.3}$$

with $\breve{V} \in \mathcal{C}^2(\mathbb{R}^d)$ and $\Lambda \le \breve{\Lambda} \le \Lambda_{u_0}$. Fix $u \in \mathfrak{U}$. From here, the usual application of Itô's formula to $e^{\int_0^T r(X_t,u_t)dt}V(X_t)$ (then followed by Fatou's lemma) gives us the following:

$$\breve{\Lambda} \le \limsup_{T\to\infty} \frac{1}{T} \log \mathbb{E}\Big[e^{\int_0^T r(X_t,u_t)dt}\Big].$$

This together with the fact that $\breve{\Lambda} \ge \Lambda$ implies that $\breve{\Lambda} = \Lambda$.

A trivial consequence of this fact is that for every $\delta > 0$, we can choose a large enough $l$ such at that $v_l^*$ (which is constant $u_0$ outside $B_l$) is the $\delta$–optimal control for our ERSC problem.

To deduce the continuity of the control $v_l^*$ on $B_l$, we invoke [11, Proposition 3]. From here, we cannot yet conclude the result because the $v_l^*$ can still be discontinuous on $\partial B_l$. To resolve this, we follow the approach of [4, Pg. 3566], *viz.,* consider a family $\{\rho^k\}_{k\in\mathbb{N}}$ of cut-off functions such that $\rho^k = 0$ on $B^c_{l-1/k}$ and $\rho^k = 1$ on $B_{l-2/k}$. Now consider a new control $v_l^k(x) \doteq \rho^k(x)v_l^*(x) +$

$(1-\rho^k(x))u_0$. Clearly, $v_l^k \to v_l^*$, as $k \to \infty$, uniformly on compliment of any neighborhood of $\partial B_l$. Using [5, Theorem 4.3], we can conclude that $\Lambda_{v_l^k} \to \Lambda_{v_l^*}$ as $k \to \infty$. Therefore, we can choose $k$ large enough such that $\Lambda_{v_l^k} \le \Lambda_{v_l^*} + \delta \le \Lambda + 2\delta$. This finishes the proof. □

## Appendix B. Some properties on the function $\varkappa$ in (3.22)

**Lemma B.1.** *For $-1 < r \le 0$, we have the following:*

$$\varkappa(1+r) \ge \frac{1}{2}|r|^2.$$

*Proof.* Using Taylor's theorem around $r=0$, we have

$$\varkappa(1+r) = \frac{1}{2}|r|^2 - \frac{r^3}{6r_0^2}, \text{ for some } r_0 \in (r,0)\,.$$

In the above, we have used the fact that $\varkappa(1)=0$, $\frac{d}{dr}\varkappa(1+r)_{|r=0} = 0$ and $\frac{d^2}{dr^2}\varkappa(1+r)_{|r=0} = \frac{1}{2}$ and $\frac{d^3}{dr^3}\varkappa(1+r)_{|r=0} = -1$. It is now clear that the desired result holds. □

**Lemma B.2.** *For $l > 0$ and $\alpha(l) = \frac{1}{2(1+l)}$,*

$$\varkappa(1+r) \ge \alpha(l)|r|^2, \quad \textit{whenever } 0 \le r < l.$$

*Proof.* From the definition of $\varkappa$,

$$\varkappa(1+r) = (1+r)\ln(1+r) - (1+r) + 1 = (1+r)\ln(1+r) - r.$$

Clearly at $r=0$, $\varkappa(1+r) - \alpha|r|^2 = 0$. Therefore, to prove the lemma, it suffices to show that $\beta(r) \doteq \varkappa(1+r) - \alpha|r|^2$ is non-decreasing in $(0,l)$. Indeed, if $\beta(r)$ is non-decreasing, then for any $r \in (0,l)$, we will have $\beta(r) \ge 0$.

We will next show that

$$\gamma(r) \doteq \frac{d}{dr}\beta(r) = \ln(1+r) - 2\alpha r \ge 0, \quad \text{whenever } 0 < r < l.$$

First note that $\gamma(r) = 0$ at $r=0$. Consider $\frac{d}{dr}\gamma(r) = \frac{1}{1+r} - 2\alpha \ge 0$ in $(0,l)$. We have

$$\frac{d}{dr}\gamma(r) \ge 0 \implies r < \frac{1}{2\alpha} - 1.$$

Choosing $\alpha \doteq \alpha(l) = \frac{1}{2(1+l)}$, we have the result. □

*Proof of Lemmas 3.5.* Fix a subsequence of $n$ which we again denote by $n$. By Lemmas B.1 and B.2, we obtain

$$\begin{aligned}\frac{1}{T}\int_0^T \varkappa(\phi_t^n)dt &= \frac{1}{T}\int_0^T \varkappa(\phi_t^n)\mathbb{1}_{\{0<\phi_t^n\le 1\}}dt + \frac{1}{T}\int_0^T \sum_{L=0}^{\infty}\varkappa(\phi_t^n)\mathbb{1}_{\{1+\frac{L}{\sqrt{n}}<\phi_t^n\le 1+\frac{L+1}{\sqrt{n}}\}}dt \\ &\ge \frac{1}{2T}\int_0^T |1-\phi_t^n|^2\mathbb{1}_{\{0<\phi_t^n<1\}}dt \\ &\qquad + \sum_{L=0}^{\infty}\alpha\big(1+\frac{L+1}{\sqrt{n}}\big)\frac{1}{T}\int_0^T |1-\phi_t^n|^2\mathbb{1}_{\{1+\frac{L}{\sqrt{n}}<\phi_t^n\le 1+\frac{L+1}{\sqrt{n}}\}}dt. \end{aligned} \tag{B.1}$$

Using monotone convergence theorem, we can justify the interchange of the integral and summation in the above the equation. In the last equation, we have used Lemmas B.1 and B.2 for the first and second terms, respectively. Define $\mathcal{N} \doteq \{-1,0,1,2,\dots\}$ and a family of functions $\{f^{n,L}\}_{n,L}$ for $n \in \mathbb{N}$ and $L \in \mathcal{N}$ as

$$f_t^{n,L} = \begin{cases} (1-\phi_t^n)\mathbb{1}_{\{0<\phi_t^n\le 1\}}, & \text{for } L=-1, \\ (1-\phi_t^n)\mathbb{1}_{\{1+\frac{L}{\sqrt{n}}<\phi_t^n\le 1+\frac{L+1}{\sqrt{n}}\}}, & \text{for } L \ne -1. \end{cases}$$

It is clear that from (3.37), we have

$$\frac{1}{2T}\int_0^T |\sqrt{n}f_t^{n,-1}|^2 dt + \sum_{L=0}^{\infty}\frac{\alpha\left(1+\frac{L+1}{\sqrt{n}}\right)}{T}\int_0^T |\sqrt{n}f_t^{n,L}|^2 dt \le \frac{n}{T}\int_0^T \varkappa(\phi_t^n)dt \le M < \infty.$$

From here, it is clear that

$$\left\{g^{n,L} \doteq \sqrt{n}f^{n,L}\right\}_{n\in\mathbb{N}} \text{ is compact in } L^2_{\text{loc}}(\mathbb{R}^+,\mathbb{R}), \text{ for every } L\in\mathbb{N}.$$

Therefore, there exists $v^L \in L^2_{\text{loc}}(\mathbb{R}^+,\mathbb{R})$, for every $L\in\mathbb{N}$ such that

$$g^{n_k^L,L} \to v^L \quad \text{in } L^2_{\text{loc}}(\mathbb{R}^+,\mathbb{R}) \text{ as } k\to\infty.$$

Clearly, the subsequence $n_k^L$ depends on $L$. Since $L$ varies over a countable set, we can choose a single subsequence along which above convergence occurs for every $L\in\mathbb{N}$ which from now on is still denoted by $n$ for simplicity.

We will now study $\sqrt{n}(1-\phi_t^n)$. First note that for any $L\in\mathbb{N}$,

$$\left\{h_t^{n,L} \doteq g_t^{n,-1} + \sum_{l=0}^{L} g_t^{n,l}\right\}_{n\in\mathbb{N}} \text{ is compact in } L^2_{\text{loc}}(\mathbb{R}^+,\mathbb{R}).$$

Indeed, to see this observe that

$$\frac{1}{T}\int_0^T |h_t^{n,L}|^2 dt = \frac{1}{T}\int_0^T |g_t^{n,-1}|^2 dt + \sum_{l=0}^{L}\frac{1}{T}\int_0^T |g_t^{n,l}|^2 dt \le \frac{M}{c_n}.$$

Here, $c_n \doteq \min\{\frac{1}{2}, \alpha(1+\frac{L}{\sqrt{n}}\}$. To arrive at the inequality above, we used (B.1) and the fact that $\alpha(\cdot)$ is a strictly decreasing function on $\mathbb{R}^+$. This concludes the compactness of $\{h^{n,L}\}_{n\in\mathbb{N}}$ for every $L\in\mathbb{N}$ as $\inf_n c_n = \frac{1}{2}$. Moreover, we have

$$\sup_{L\in\mathbb{N}}\limsup_{n\to\infty}\sup_{T>0}\frac{1}{T}\int_0^T |h_t^{n,L}|^2 dt \le 2M$$

and $h^{n,L}$ converges in $L^2_{\text{loc}}(\mathbb{R}^+,\mathbb{R})$ as $n\to\infty$, to $v^{-1}+\sum_{l=0}^{L} v^l$, for every $L\in\mathbb{N}$. From the weak lower semi-continuity of the norm, we have

$$\sup_{L\in\mathbb{N}}\sup_{T>0}\left\{\frac{1}{T}\int_0^T |v_t^{-1}|^2 dt + \sum_{l=0}^{L}\frac{1}{T}\int_0^T |v_t^l|^2 dt\right\} \le \sup_{L\in\mathbb{N}}\liminf_{n\to\infty}\sup_{T>0}\frac{1}{T}\int_0^T |h_t^{n,L}|^2 dt \le 2M.$$

From here, it is clear that

$$\sup_{T>0}\frac{1}{T}\int_0^T |v_t^*|^2 dt \le \liminf_{n\to\infty}\sup_{T>0}\left\{\frac{1}{T}\int_0^T |g_t^{n,-1}|^2 dt + \sum_{l=0}^{\infty}\frac{1}{T}\int_0^T |g_t^{n,l}|^2 dt\right\} \le 2M.$$

Here, $v^* \doteq v^{-1}+\sum_{L=0}^{\infty} v^L$. From the above, observe that

$$\left\{\sqrt{n}(1-\phi_t^n) = g_t^{n,-1} + \sum_{l=0}^{\infty} g_t^{n,l}\right\}_{n\in\mathbb{N}} \quad \text{lies in } L^2_{\text{loc}}(\mathbb{R}^+,\mathbb{R}), \text{ for large } n.$$

To be concise, we have shown that

$$\sup_{T>0}\frac{1}{T}\int_0^T |v_t^*|^2 dt \le \liminf_{n\to\infty}\sup_{T>0}\frac{1}{T}\int_0^T |\sqrt{n}(1-\phi_t^n)|^2 dt \le 2M. \tag{B.2}$$

Recalling that we had started with an arbitrary subsequence of $n$, we can replace $\liminf$ with $\limsup$ in the above display to get

$$\limsup_{n\to\infty}\sup_{T>0}\frac{1}{T}\int_0^T|\sqrt{n}(1-\phi_t^n)|^2dt\le 2M.$$

This finishes the proof. □

## Appendix C. Proof of Theorem 3.5

In the following, we show that

$$\limsup_{n\to\infty}\frac{1}{T}\log\mathbb{E}[e^{TG(\widetilde{N}^n)}]\le\sup_{w\in\mathcal{A}}\mathbb{E}\Big[G\Big(W+\int_0^\cdot w_tdt\Big)-\frac{\lambda}{2T}\int_0^T|w_t|^2dt\Big]=\frac{1}{T}\log\mathbb{E}[e^{TG(W)}],\quad\text{(C.1)}$$

and

$$\liminf_{n\to\infty}\frac{1}{T}\log\mathbb{E}[e^{TG(\widetilde{N}^n)}]\ge\sup_{w\in\mathcal{A}}\mathbb{E}\Big[G\Big(W+\int_0^\cdot w_tdt\Big)-\frac{\lambda}{2T}\int_0^T|w_t|^2dt\Big]=\frac{1}{T}\log\mathbb{E}[e^{TG(W)}].\quad\text{(C.2)}$$

Recall that $W$ is a 1–dimensional Brownian motion and the associated set $\mathcal{A}$ is as defined in Theorem 3.1, but for 1–dimensional case. In the rest of the proof, whenever a Brownian motion is involved, we denote it by $W$.

**Proof of** (C.1): Fix $\delta>0$. Then from (3.24), we have

$$\frac{1}{T}\log\mathbb{E}\big[e^{TG(\widetilde{N}^n)}\big]\le\sup_{\phi\in\widetilde{\mathcal{E}}_M}\mathbb{E}\Big[G\Big(M^{n,\phi}-\lambda\int_0^\cdot\sqrt{n}(1-\phi_t)dt\Big)-\frac{\lambda n}{T}\int_0^T\varkappa(\phi_t)dt\Big]+\delta\,.\quad\text{(C.3)}$$

We remark that $\phi$ in the above equation is a $\widetilde{\mathcal{E}}_M$–valued random variable. Since $\phi\in\widetilde{\mathcal{E}}_M$, it satisfies

$$\sup_{n\in\mathbb{N}}\frac{\lambda n}{T}\int_0^T\varkappa(\phi_t)dt\le M.$$

Now choose a $\delta$–optimal (corresponding to (C.3)) $\phi^n\in\widetilde{\mathcal{E}}_M$, for every $n$. Then using Lemma 3.5, we can conclude that

$$\limsup_{n\to\infty}\frac{1}{T}\int_0^T|\sqrt{n}(1-\phi_t^n)|^2dt\le 2M.$$

This clearly implies that there exists a subsequence (denoted by $n_k$) such that the family of random variables $\{\sqrt{n_k}(1-\phi^{n_k})\}_{k\in\mathbb{N}}$ is tight in $L^2([0,T],\mathbb{R})$ when equipped with weak$^*$ topology (denoted by $L_T^{2,*}$ from now on). From the tightness of $\{\sqrt{n_k}(1-\phi^{n_k})\}_{k\in\mathbb{N}}$ in $L_T^{2,*}$, we know that there exists a $L_T^{2,*}$–valued random variable $w$ such that $\{\sqrt{n_k}(1-\phi^{n_k})\}_{k\in\mathbb{N}}$ converges weakly to $w$, along that subsequence.

Define the following family of $\mathfrak{C}_T$–valued random variables: $m_t^{n_k}\doteq\int_0^t\sqrt{n_k}(1-\phi_s^{n_k})ds$. We show that $\{m^{n_k}\}_{k\in\mathbb{N}}$ is a pre-compact set of $\mathfrak{C}_T$. To that end, first observe that

$$\begin{aligned}\sup_{k\in\mathbb{N}}\sup_{t\in[0,T]}|m_t^{n_k}|&\le\sup_{k\in\mathbb{N}}\sup_{t\in[0,T]}t\sqrt{\frac{T}{tT}\int_0^t|\sqrt{n_k}(1-\phi_s^{n_k})|^2ds}\\&\le\sup_{k\in\mathbb{N}}\sup_{t\in[0,T]}T\sqrt{\frac{1}{T}\int_0^t|\sqrt{n_k}(1-\phi_s^{n_k})|^2ds}\le T\sqrt{M}.\end{aligned}$$

To show equicontinuity, for $0\le s<t\le T$, we consider

$$m_t^{n_k}-m_s^{n_k}=\int_s^t\sqrt{n_k}(1-\phi_u^{n_k})du.$$

Using Cauchy-Schwartz inequality, we have

$$|m_t^{n_k} - m_s^{n_k}| \leq \sqrt{T}\sqrt{t-s}\sqrt{\frac{1}{T}\int_0^T |\sqrt{n_k}(1-\phi_u^{n_k})|^2 du} \leq \sqrt{MT}(t-s)^{\frac{1}{2}}.$$

This proves that $\{m^{n_k}\}_{k\in\mathbb{N}}$ is a pre-compact set of $\mathfrak{C}_T$ and hence, is tight in $\mathfrak{C}_T$.

We now analyze $\{M^{n,\phi^n}\}_{n\in\mathbb{N}}$. To do this, we first show that $\int_0^\cdot \phi_t^n dt \to \mathfrak{e}(\cdot)$ in $\mathfrak{C}_T$. We begin by observing that $\phi^n \in L_T^{2,*}$ and $\{\phi^n\}_{n\in\mathbb{N}}$ is convergent in $L_T^{2,*}$ with $\phi^* \equiv 1$ being the limit. Therefore arguing similarly as above will imply boundedness (uniformly in $n$) and equicontinuity of $\{j_n(\cdot) \doteq \int_0^\cdot \phi_t^n dt\}_{n\in\mathbb{N}}$ (hence, tightness of $\{j_n\}_{n\in\mathbb{N}}$ in $\mathfrak{C}_T$). It is clear that the limit point of $\{j_n\}_{n\in\mathbb{N}}$ is $\mathfrak{e}(\cdot)$. Therefore, using martingale central limit theorem and random change of time lemma ([14, Pg. 151]), we can conclude that $M^{n,\phi^n}$ converges weakly to a Brownian motion $W$ in $\mathfrak{D}_T$, along a subsequence.

$$\begin{aligned}\limsup_{k\to\infty}\frac{1}{T}\log\mathbb{E}\big[e^{TG(\widetilde{N}^{n_k})}\big] &\leq \limsup_{k\to\infty}\mathbb{E}\Big[G\Big(M^{n_k,\phi^{n_k}} + \lambda\int_0^\cdot \sqrt{n_k}(\phi_t^{n_k}-1)dt\Big) - \frac{\lambda n_k}{T}\int_0^T \varkappa(\phi_t^{n_k})dt\Big] + 2\delta \\ &\leq \mathbb{E}\Big[G\Big(W + \int_0^\cdot w dt\Big)\Big] - \liminf_{k\to\infty}\mathbb{E}\Big[\frac{\lambda n_k}{T}\int_0^T \varkappa(\phi_t^{n_k})dt\Big] + 2\delta\,.\end{aligned}$$

In the above, we have used the fact that $\{\big(M^{n_k,\phi^{n_k}}, \int_0^\cdot \sqrt{n}_k(1-\phi_t^{n_k})dt\big)\}_{k\in\mathbb{N}}$ converges weakly in $\mathfrak{D}_T\times\mathfrak{C}_T$ to $(W, \int_0^\cdot w_t dt)$ and the continuous mapping theorem. To simplify

$$\liminf_{k\to\infty}\mathbb{E}\Big[\frac{\lambda n_k}{T}\int_0^T \varkappa(\phi_t^{n_k})dt\Big],$$

we use (B.2) to obtain

$$\begin{aligned}\liminf_{k\to\infty}\mathbb{E}\Big[\frac{\lambda n_k}{T}\int_0^T \varkappa(\phi_t^{n_k})dt\Big] &\geq \liminf_{k\to\infty}\frac{1}{2T}\int_0^T |\sqrt{n_k}(1-\phi_t^{n_k})|^2 dt, \\ &\geq \frac{1}{2T}\int_0^T |w_t|^2 dt.\end{aligned}$$

Lower semicontinuity of the functional $f(w) \doteq \frac{1}{T}\int_0^T |w_t|^2 dt$ in $L_T^{2,*}$ is used to get the last inequality above. Therefore, from arbitrariness of $\delta > 0$, we have shown that

$$\limsup_{k\to\infty}\frac{1}{T}\log\mathbb{E}\big[e^{TG(\widetilde{N}^{n_k})}\big] \leq \mathbb{E}\Big[G\Big(W+\int_0^\cdot w_t dt\Big) - \frac{1}{2T}\int_0^T |w_t|^2 dt\Big] \leq \frac{1}{T}\mathbb{E}\big[e^{TG(W)}\big]\,.$$

**Proof of** (C.2): Fix $\delta > 0$ and a 1–dimensional Brownian motion $W$. Then choose a $w^* \in \mathcal{A}$ such that

$$\sup_{w\in\mathcal{A}}\mathbb{E}\Big[G\Big(W+\int_0^\cdot w_t dt\Big) - \frac{\lambda}{2T}\int_0^T |w_t|^2 dt\Big] \leq \mathbb{E}\Big[G\Big(W+\int_0^\cdot w_t^* dt\Big) - \frac{\lambda}{2T}\int_0^T |w_t^*|^2 dt\Big] + \delta$$

and define $\phi_t^n \doteq 1 - \frac{w_t^*}{\sqrt{n}}$. Clearly, $\{\sqrt{n}(1-\phi^n)\}_{n\in\mathbb{N}}$ is weakly convergent family of random variables in $L_T^{2,*}$ with limit being $w^*$. Moreover, it is also clear that $M^{n,\phi^n}$ converges weakly to $W$ in $\mathfrak{D}_T$. Now we obtain

$$\begin{aligned}\liminf_{n\to\infty}\frac{1}{T}\log\mathbb{E}\big[e^{TG(\widetilde{N}^n)}\big] &\geq \liminf_{n\to\infty}\mathbb{E}\Bigg[G\Big(M^{n,\phi^n} + \frac{\lambda n\int_0^\cdot \phi_t^n dt - \lambda n\mathfrak{e}(\cdot)}{\sqrt{n}}\Big) - \frac{\lambda n}{T}\int_0^T \varkappa(\phi_t^n)dt\Bigg], \\ &\geq \mathbb{E}\Big[G\Big(W+\int_0^\cdot w_t^* dt\Big) - \frac{\lambda}{2T}\int_0^T |w_t^*|^2 dt\Big], \\ &\geq \frac{1}{T}\log\mathbb{E}[e^{TG(W)}] - \delta\,.\end{aligned}$$

From arbitrariness of $\delta > 0$, we have the result.

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