Optimal control of Markov-modulated multiclass many-server queues

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ABSTRACT. We study multiclass many-server queues for which the arrival, service and abandonment rates are all modulated by a common finite-state Markov process. We assume that the system operates in the "averaged" Halfin–Whitt regime, which means that it is critically loaded in the average sense, although not necessarily in each state of the Markov process. We show that under any static priority policy, the Markov-modulated diffusion-scaled queueing process is exponentially ergodic. This is accomplished by employing a solution to an associated Poisson equation in order to construct a suitable Lyapunov function. We establish a functional central limit theorem for the diffusion-scaled queueing process and show that the limiting process is a controlled diffusion with piecewise linear drift and constant covariance matrix. We address the infinite-horizon discounted and long-run average (ergodic) optimal control problems and establish asymptotic optimality.

1. INTRODUCTION

Queueing networks operating in a random environment have been studied extensively. A functional central limit theorem (FCLT) for Markov-modulated infinite-server queues is established in [1], which shows that the limit process is an Ornstein–Uhlenbeck diffusion; see also [2, 3] for more recent work. Scheduling control problems for Markov-modulated multiclass single-server queueing networks have been addressed in [4–6]. In [6], the authors show that a modified $c\mu$-policy is asymptotically optimal for the infinite horizon discounted problem. For a single-server queue with only the arrival rates modulated, service rate control problems over a finite and infinite horizon have been studied in [4, 5]. For multiclass many-server queues without modulation, the infinite-horizon discounted and ergodic control problems have been studied in [7] and [8], respectively.

In this paper we address the aforementioned control problems for Markov-modulated multiclass many-server queues. We establish the weak convergence of the diffusion-scaled queueing processes, study their stability properties, characterize the optimal solutions via the associated limiting diffusion control problems, and then prove asymptotic optimality. Specifically, we assume that the arrival, service and abandonment rates are all modulated by a finite-state Markov process, and that given the state of this process, the arrivals are Poisson, and the service and patient times are exponentially distributed. The system operates in the "averaged" Halfin–Whitt (H–W) regime, namely, it is critically loaded in an average sense, but it may be underloaded or overloaded for a given state of the environment. This situation is different from the standard H–W regime for many-server queues, which requires that the system is critically loaded as the arrival rates and number of servers get large; see, e.g., [3, 7–9].

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We first establish a FCLT in Theorem 2.1 for the Markov-modulated diffusion-scaled queueing processes under any admissible scheduling policy (only considering work-conserving and preemptive policies). Proper scaling is needed in order to establish weak convergence of the queueing processes. In particular, since the arrival processes are of order n, and the switching rates of the background process are assumed to be of order n^{α} for $\alpha > 0$, the queueing processes are centered at the 'averaged' steady state, which is of order n, and are then scaled down by a factor of an n^{β}, with $\beta := \max\{1/2, 1-\alpha/2\}$, in the diffusion scale. Thus, when $\alpha \geq 1$, we have the usual diffusion scaling with $\beta = 1/2$, which is due to the fact that the very fast switching of the environment results in an 'averaging' effect for the arrival, service and abandonment processes of the queueing dynamics. The limit queueing process is a piecewise Ornstein–Uhlenbeck diffusion process with a drift and covariance given by the corresponding 'averaged' quantities under the stationary distribution of the background process. When $\alpha = 1$, both the variabilities of the queueing and background processes are captured in the covariance matrix, while when $\alpha > 1$, only the variabilities of the queueing process is captured. On the other hand, when $\alpha < 1$, the proper diffusion scaling requires $\beta = 1 - \alpha/2$, for which we obtain a similar piecewise Ornstein–Uhlenbeck diffusion process with the covariance matrix capturing the variabilities of the background process only.

The ergodic properties of this class of piecewise linear diffusions (and Lévy-driven stochastic differential equations) have been studied in [10,11], and these results can be applied directly to our model. The study of the ergodic properties of the diffusion-scaled processes, however, is challenging. Ergodicity of switching Markov processes has been an active research subject. For switching diffusions, stability has been studied in [12–14]. However, studies of ergodicity of switching Markov processes are scarce. Recently in [15,16], some kind of hypoellipticity criterion with Hörmander-like bracket conditions is provided to establish exponential convergence in the total variation distance. As pointed out in [17], this condition cannot be easily verified, even for many classes of simple Markov processes with random switching. Cloez and Hairer [17] provided a concrete criterion for exponential ergodicity in situations which do not verify any hypoellipticity assumption (as well as criterion for convergence in terms of Wasserstein distance). Their proof is based on a coupling argument and a weak form of Harris' theorem. It is worth noting that in these studies, the transition rates of the underlying Markov process are unscaled, and therefore, the Markov processes under random switching do not exhibit an 'averaging' effect. Because of the 'averaging' effect in our model, we are able to construct a suitable Lyapunov function to verify the standard Foster-Lyapunov condition in order to prove the exponential ergodicity of the diffusion-scaled queueing processes.

The technique we employ is much similar in spirit to the approach in [12] for studying p-stability of the switching diffusion processes with rapid switching. For diffusions, Khasminskii [12] observes that rapid switching results in some 'averaging' effect, and thus if the 'averaged' diffusion (modulated parameters are replaced by their averages under the invariant measure of the background process) is stable, then a Lyapunov function can be constructed by using solutions to an associated Poisson equation to verify the Foster-Lyapunov stability condition for the original diffusion process. To the best of our knowledge, this approach has not been used to study general fast switching Markov processes. We employ this technique to the Markov-modulated diffusion-scaled queueing process of the multiclass many-server model. Ergodicity properties for multiclass Markovian queues have been established in [18]; in particular, it is shown that the queueing process is ergodic under any work-conserving scheduling policy. Following the approach in [8], we show that under a static priority scheduling policy, the 'averaged' diffusion-scaled processes (with the arrival, service and abandonment parameters being replaced by the averaged quantities) are exponentially ergodic (Lemma 3.1). We then construct a Lyapunov function using a Poisson equation associated with the difference of the Markov-modulated diffusion-scaled queueing process and the 'averaged' queueing process, and thus verify the Foster-Lyapunov stability criterion for exponential ergodicity (Theorem 3.1).

To study asymptotic optimality in Theorem 2.2 for the discounted problem , we first establish a moment bound for the Markov-modulated diffusion-scaled queueing process, which is uniform under all admissible policies, that is, work-conserving and non-preemptive polices. We then adopt the approach in [7] and construct a sequence of polices which asymptotically converges to the optimal value of the discounted problem for the limiting diffusion process. To prove asymptotic optimality in Theorem 2.3 for the ergodic problem, it is critical to study the convergence of the mean empirical measures associated with the Markov-modulated diffusion-scaled queueing processes. Unlike the studies in [8, 19, 20], the Markov modulation makes this work much more challenging. For both the lower and upper bounds, we construct an auxiliary (semimartingale) process associated with a diffusion-scaled queueing process and the underlying Markov process. We then establish the convergence of the mean empirical measure of the auxiliary process, and thus prove that of the Markov-modulated diffusion-scaled queueing processes by establishing their asymptotic equivalence. In establishing the upper bound, we adopt the technique developed in [8]. Using a spatial truncation, we obtain nearly optimal controls for the ergodic problem of our controlled limiting diffusion by fixing a stable Markov control (any constant control) outside a compact set. We then map such concatenated controls for the limiting diffusion process to a family of scheduling polices for the auxiliary processes as well as the diffusion-scaled queueing processes, which also preserve the ergodic properties. With these concatenated policies, we are able to prove the upper bound for the value functions.

1.1. **Organization of the paper.** In the next subsection, we summarize the notation used in this paper. Section 2 contains a detailed description of the Markov-modulated multiclass many-server queueing model. In Section 2.1, we introduce the scheduling policies considered in this paper. In Section 2.2, we present the controlled limiting diffusions and weak convergence results. We state the main results on asymptotic optimality for the discounted and ergodic problems in Sections 2.3 and 2.4, respectively. In Section 3, we summarize the ergodic properties of the controlled limiting diffusions, and establish the exponential ergodicity of the diffusion-scaled processes. A characterization of optimal controls for the controlled limiting diffusions, and the proofs of asymptotic optimality are given in Section 4. Appendix A is devoted to the proofs of Theorem 2.1 and Lemma 3.1, while Appendix B contain the proofs of some technical results in Section 4.

1.2. **Notation.** We let $\mathbb{N}$ denote the set of positive integers. For $k \in \mathbb{N}$, $\mathbb{R}^k$ ($\mathbb{R}^k_+$) denotes the set of k-dimensional real (nonnegative) vectors, and we write $\mathbb{R}$ ($\mathbb{R}_+$) for $k = 1$. For $k \in \mathbb{N}$, $\mathbb{Z}^k_+$ stands for the set of d-dimensional nonnegative integer vectors. For $i = 1, \dots, d$, we let e_i denote the vector in $\mathbb{R}^d$ with the i^{th} element equal to 1 and all other elements equal to 0, and define $e = (1, \dots, 1)^{\mathsf{T}}$. The complement of a set $A \subset \mathbb{R}^d$ is denoted by A^c. The open ball in $\mathbb{R}^d$ with center the origin and radius R is denoted by B_R. For $a, b \in \mathbb{R}$, the minimum (maximum) of a and b is denoted by $a \wedge b$ ($a \vee b$), and we let $a^+ := a \vee 0$. For $a \in \mathbb{R}^+$, $\lfloor a \rfloor$ denotes the largest integer not greater than a. Given any vectors $a, b \in \mathbb{R}^d$, let $\langle a, b\rangle$ denote the inner product.

The Euclidean norm in $\mathbb{R}^k$ is denoted by $|\cdot|$. For $x \in \mathbb{R}^k$, we let $\|x\| := \sum_{i=1}^k |x_i|$. The indicator function of a set $A \subset \mathbb{R}^k$ is denoted by $\mathbb{1}(A)$ or $\mathbb{1}_A$. We use the notations $\partial_i := \frac{\partial}{\partial x_i}$ and $\partial_{ij} := \frac{\partial^2}{\partial x_i \partial x_j}$. For a domain $D \subset \mathbb{R}^d$, the space $\mathcal{C}^k(D)$ ($\mathcal{C}^\infty(D)$) denotes the class of functions whose partial derivatives up to order k (of any order) exist and are continuous, and $\mathcal{C}^k_c(D)$ denotes the space of functions in $\mathcal{C}^k(D)$ with compact support. For $D \subset \mathbb{R}^d$, we let $\mathcal{C}^k_b(D)$ denote the set of functions in $\mathcal{C}^k(D)$, whose partial derivatives up to order k are continuous and bounded. For a nonnegative function $f \in \mathcal{C}(\mathbb{R}^d)$, we use $\mathcal{O}(f)$ to denote the space of function $g \in \mathcal{C}(\mathbb{R}^d)$ such that $\sup_{x\in\mathbb{R}^d} \frac{|g(x)|}{1+f(x)} < \infty$, and we use $\mathfrak{o}(f)$ to denote the subspace of $\mathcal{O}(f)$ consisting of functions $g \in \mathcal{C}(\mathbb{R}^d)$ such that $\limsup_{|x|\to\infty} \frac{|f(x)|}{1+g(x)} = 0$. The arrows $\to$ and $\Rightarrow$ are used to denote convergence of real numbers and convergence in distribution, respectively. For any path $X(\cdot)$,

$\Delta X(t)$ is used to denote the jump at time t. We use $\langle\cdot\rangle$ to denote the predictable quadratic variation of a square integrable martingale, and use $[\cdot]$ to denote the optional quadratic variation. We define $\mathbb{D} := \mathbb{D}(\mathbb{R}_+, \mathbb{R})$ as the real-valued function space of all cádlág functions on $\mathbb{R}_+$. We endow the space $\mathbb{D}$ with the Skorohod J_1 topology and denote this topological space as $(\mathbb{D}, \mathcal{J})$. For any complete and separable metric spaces S_1 and S_2, we use $S_1 \times S_2$ to denote their product space endowed with the maximum metric. For any complete and separable space S, and $k \in \mathbb{N}$, the k-fold product space with the maximum metric is denoted by S^k. For $k \in \mathbb{N}$, $(\mathbb{D}^k, \mathcal{J})$ denotes the k-fold product of $(\mathbb{D}, \mathcal{J})$ with the product topology. Given a Polish space E, $\mathcal{P}(E)$ denotes the space of probability measures on E, endowed with the Prokhorov metric.

2. The Model and Control Problems

We consider a sequence of d-class Markov-modulated $M/M/n+M$ queueing models indexed by n. Define the space of customer classes by $\mathcal{I} := \{1, \dots, d\}$. For $n \in \mathbb{N}$, let $J^n := \{J^n(t) \colon t \ge 0\}$ be a continuous-time Markov chain with finite state space $\mathcal{K} := \{1, \dots, K\}$, with an irreducible transition rate matrix $n^\alpha \mathcal{Q}$ for some $\alpha > 0$. Thus, J^n has a stationary distribution denoted by $\pi = (\pi_1, \cdots, \pi_K)$, for each $n \in \mathbb{N}$. We assume that J^n starts from this stationary distribution.

For each n and $i \in \mathcal{I}$, let $A^n_i := \{A^n_i(t) \colon t \ge 0\}$ denote the arrival process of class-i customers in the n^{th} system. Provided J^n is in state k, the arrival rate of class-i customers is defined by $\lambda^n_i(k) \in \mathbb{R}_+$, and the service time and the patience time are exponentially distributed with rates $\mu^n_i(k)$ and $\gamma^n_i(k)$, respectively. Let A^n denote a Markov-modulated Poisson process, that is, for $t \ge 0$, each n and $i \in \mathcal{I}$,

$$A^n_i(t) \;=\; A^n_{*,i}\left(\int_0^t \lambda^n_i(J^n(s))\,\mathrm{d}s\right)\,,$$

where $\{A^n_{*,i} \colon n \in \mathbb{N}, i \in \mathcal{I}\}$ are unit-rate Poisson processes.

Let X^n, Q^n and Z^n denote the d-dimensional processes counting the number of customers of each class in the n^{th} system, in queue and in service, respectively, and the following constraints are satisfied: for $t \ge 0$ and $i \in \mathcal{I}$,

$$\begin{aligned} X^n_i(t) \;&=\; Q^n_i(t) + Z^n_i(t)\,,\\ Q^n_i(t) \;\ge\; 0\,, \quad Z^n_i(t) \;&\ge\; 0 \quad \text{and} \quad \langle e, Z^n(t)\rangle \;\le\; n\,. \end{aligned} \tag{2.1}$$

Then, we have the following dynamic equation: for $t \ge 0$, $n \in \mathbb{N}$ and $i \in \mathcal{I}$,

$$X^n_i(t) \;=\; X^n_i(0) + A^n_i(t) - S^n_i(t) - R^n_i(t)\,, \tag{2.2}$$

where

$$S^n_i(t) \;:=\; S^n_{*,i}\left(\int_0^t \mu^n_i(J^n(s)) Z^n_i(s)\,\mathrm{d}s\right)\,, \quad R^n_i(t) \;:=\; R^n_{*,i}\left(\int_0^t \gamma^n_i(J^n(s)) Q^n_i(s)\,\mathrm{d}s\right)\,,$$

and $\{S^n_{*,i}, R^n_{*,i} \colon n \in \mathbb{N}, i \in \mathcal{I}\}$ are unit-rate Poisson processes. We assume that for each $n \in \mathbb{N}$, $\{X^n_i(0), A^n_{*,i}, S^n_{*,i}, R^n_{*,i} \colon i \in \mathcal{I}\}$ are mutually independent.

Assumption 2.1. As $n \to \infty$, for $i \in \mathcal{I}$ and $k \in \mathcal{K}$,

$$\begin{gathered} n^{-1}\lambda^n_i(k) \;\to\; \lambda_i(k) \;>\; 0\,, \quad \mu^n_i(k) \;\to\; \mu_i(k) \;>\; 0\,, \quad \gamma^n_i(k) \;\to\; \gamma_i(k) \;>\; 0\,,\\ n^{-\beta}\left(\lambda^n_i(k) - n\lambda_i(k)\right) \;\to\; \hat{\lambda}_i(k) \quad \textit{and} \quad n^{1-\beta}(\mu^n_i(k) - \mu_i(k)) \;\to\; \hat{\mu}_i(k)\,, \end{gathered}$$

where

$$\beta \;:=\; \max\{1/2, 1 - \alpha/2\}\,.$$

For $i \in \mathcal{I}$ and $n \in \mathbb{N}$, we define

$$\lambda_i^\pi \,:=\, \sum_{k\in\mathcal{K}} \pi_k \lambda_i(k) \quad \mu_i^\pi \,:=\, \sum_{k\in\mathcal{K}} \pi_k \mu_i(k)\,, \quad \gamma_i^\pi \,:=\, \sum_{k\in\mathcal{K}} \pi_k \gamma_i(k)\,,$$
$$\bar\lambda_i^n \,:=\, \sum_{k\in\mathcal{K}} \pi_k \lambda_i^n(k) \quad \bar\mu_i^n \,:=\, \sum_{k\in\mathcal{K}} \pi_k \mu_i^n(k)\,, \quad \bar\gamma_i^n \,:=\, \sum_{k\in\mathcal{K}} \pi_k \gamma_i^n(k)\,,$$

and

$$\rho_i \,:=\, \lambda_i^\pi/\mu_i^\pi, \quad \rho^n := n^{-1} \sum_{i\in\mathcal{I}} \bar\lambda_i^n/\bar\mu_i^n\,.$$

Assumption 2.2. The system is critically loaded, that is, $\sum_{i\in\mathcal{I}} \rho_i = 1$.

Under Assumptions 2.1 and 2.2, we have

$$n^{1-\beta}(1-\rho^n) \,=\, \sum_{i\in\mathcal{I}} \frac{n^{-\beta}(n\bar\mu_i^n - n\mu_i^\pi)\rho_i - n^{-\beta}(\bar\lambda_i^n - n\lambda_i^\pi)}{\bar\mu_i^n} \xrightarrow[n\to\infty]{} \sum_{i\in\mathcal{I}} \frac{\rho_i\hat\mu_i^\pi - \hat\lambda_i^\pi}{\mu_i^\pi}\,,$$

with

$$\hat\lambda_i^\pi \,:=\, \sum_{k\in\mathcal{K}} \pi_k \hat\lambda_i(k), \quad \hat\mu_i^\pi \,:=\, \sum_{k\in\mathcal{K}} \pi_k \hat\mu_i(k)\,.$$

Assumptions 2.1 and 2.2 are in effect throughout the paper, without further mention. A model satisfying these assumptions is said to be in the "averaged" H–W regime.

Let $\bar X^n$, $\bar Z^n$, $\bar Q^n$, $\hat X^n$, $\hat Z^n$ and $\hat Q^n$ denote the d-dimensional processes satisfying

$$\bar X_i^n \,=\, n^{-1} X_i^n\,, \quad \bar Z_i^n \,=\, n^{-1} Z_i^n\,, \quad \bar Q_i^n \,=\, n^{-1} Q_i^n\,,$$
$$\hat X_i^n \,=\, n^{-\beta}(X_i^n - \rho_i n)\,, \quad \hat Z_i^n \,=\, n^{-\beta}(Z_i^n - \rho_i n) \quad \text{and} \quad \hat Q_i^n \,=\, n^{-\beta} Q_i^n$$

for $i \in \mathcal{I}$. Then, for $t \geq 0$ and $i \in \mathcal{I}$, $\hat X_i^n(t)$ can be written as

$$\begin{aligned}\hat X_i^n(t) \,=\, &\hat X_i^n(0) + \hat\ell_i^n(t) + \hat L_i^n(t) + \hat A_i^n(t) - \hat S_i^n(t) - \hat R_i^n(t) \\ &- \int_0^t \mu_i^n(J^n(s))\hat Z_i^n(s)\,\mathrm{d}s - \int_0^t \gamma_i^n(J^n(s))\hat Q_i^n(s)\,\mathrm{d}s\,,\end{aligned} \tag{2.3}$$

where

$$\hat\ell_i^n(t) \,:=\, n^{-\beta} \sum_{k\in\mathcal{K}} \Big(\big(\lambda_i^n(k) - n\lambda_i(k)\big) - n\rho_i\big(\mu_i^n(k) - \mu_i(k)\big)\Big) \int_0^t \mathbb{1}(J^n(s) = k)\,\mathrm{d}s\,,$$
$$\hat L_i^n(t) \,:=\, n^{1-\beta} \int_0^t (\lambda_i(J^n(s)) - \lambda_i^\pi)\,\mathrm{d}s - n^{1-\beta}\rho_i \int_0^t (\mu_i(J^n(s)) - \mu_i^\pi)\,\mathrm{d}s\,,$$
$$\hat A_i^n(t) \,:=\, n^{-\beta}\left(A_i^n(t) - \int_0^t \lambda_i^n(J^n(s))\,\mathrm{d}s\right)\,,$$
$$\hat S_i^n(t) \,:=\, n^{-\beta}\left(S_i^n(t) - \int_0^t \mu_i^n(J^n(s))Z_i^n(s)\,\mathrm{d}s\right)\,,$$
$$\hat R_i^n(t) \,:=\, n^{-\beta}\left(R_i^n(t) - \int_0^t \gamma_i^n(J^n(s))Q_i^n(s)\,\mathrm{d}s\right)\,.$$

Define the random processes $\hat Y^n = (\hat Y_1^n, \dots, \hat Y_d^n)'$ and $\hat W^n = (\hat W_1^n, \dots, \hat W_d^n)'$ by

$$\hat Y_i^n(t) \,:=\, \hat\ell_i^n(t) - \int_0^t \mu_i^n(J^n(s))\hat Z_i^n(s)\,\mathrm{d}s - \int_0^t \gamma_i^n(J^n(s))\hat Q_i^n(s)\,\mathrm{d}s$$

for $i \in \mathcal{I}$ and $t \geq 0$, and

$$\hat W^n \,:=\, \hat L^n + \hat A^n - \hat S^n - \hat R^n\,, \tag{2.4}$$

respectively. Then, (2.3) can be written as

$$\hat{X}^n(t) \;=\; \hat{X}^n(0) + \hat{Y}^n(t) + \hat{W}^n(t)\,, \quad t \ge 0\,.$$

Throughout the paper, we assume that $\{\hat{X}^n(0)\colon n \in \mathbb{N}\}$ are deterministic and does not depend on n.

2.1. **Scheduling policies.** Let $\tau^n(t) := \inf\{r \ge t\colon J^n(r) = 1\}$ for $t \ge 0$. We define the following filtrations: for $t \ge 0, r \ge 0$,

$$\begin{aligned}
\mathcal{F}^n_t \;&:=\; \sigma\{A^n_i(s), S^n_i(s), R^n_i(s), Q^n_i(s), Z^n_i(s), X^n_i(s), J^n(s)\colon i \in \mathcal{I}, s \le t\} \vee \mathcal{N}\,,\\
\mathcal{G}^n_{t,r} \;&:=\; \sigma\{A^n_i(\tau^n(t)+r) - A^n_i(\tau^n(t)), S^n_i(\tau^n(t)+r) - S^n_i(\tau^n(t)),\\
&\qquad R^n_i(\tau^n(t)+r) - R^n_i(\tau^n(t)), \hat{L}^n_i(\tau^n(t)+r) - \hat{L}^n_i(\tau^n(t))\colon i \in \mathcal{I}\} \vee \mathcal{N}\,,
\end{aligned}$$

where $\mathcal{N}$ is a collection of $\mathbb{P}$-null sets.

Definition 2.1. We say a scheduling policy Z^n is admissible, if it satisfies following conditions.

(i) Preemptive: a server can stop serving a class of customer to serve some other class of customers at any time, and resume the original service at a later time.
(ii) Work-conserving: for each $t \ge 0$, $\langle e, Z^n(t)\rangle = \langle e, X^n(t)\rangle \wedge n$.
(iii) Non-anticipative: for $t \ge 0$ and $r \ge 0$,
  (a) $Z^n(t)$ is adapted to $\mathcal{F}^n_t$.
  (b) $\mathcal{F}^n_t$ and $\mathcal{G}^n_{t,r}$ are independent.

We only consider admissible scheduling policies. Given an admissible scheduling policy Z^n, the process X^n in (2.2) is well defined, and we say that it is governed by the scheduling policy Z^n. Abusing the terminology, we equivalently also say that $\hat{X}^n$ is governed by the scheduling policy $\hat{Z}^n$. We say that an admissible scheduling policy is stationary Markov if $Z^n(t) = z^n(X^n)$ for some $z^n\colon \mathbb{Z}^d_+ \mapsto \mathbb{Z}^d_+$.

Define the set

$$\mathbb{U} \;:=\; \{u \in \mathbb{R}^d_+\colon \langle e, u\rangle = 1\}\,,$$

It is often useful to re-parametrize and replace the scheduling policy Z^n with a new scheduling policy $\hat{U}^n$ defined as follows. Given a process X^n defined using (2.2) and an admissible scheduling policy Z^n, for $t \ge 0$, define

$$\hat{U}^n(t) \;:=\; \begin{cases} \frac{Z^n(t) - X^n(t)}{n - \langle e, X^n(t)\rangle} & \text{for } \langle e, X^n(t)\rangle > n\,,\\ e_d & \text{for } \langle e, X^n(t)\rangle \le n\,. \end{cases}$$

The process $\hat{U}^n(t)$ takes values in $\mathbb{U}$ and represents the proportion of class-i customers in the queue. Any process $\hat{U}^n$ defined as above by using some admissible scheduling policy Z^n is called an admissible proportions-scheduling policy. The set of all such admissible proportions-scheduling polices $\hat{U}^n$ is denoted by $\widehat{\mathfrak{U}}^n$. Then, given $\hat{X}^n$ and any $\hat{U}^n \in \widehat{\mathfrak{U}}^n$, the scaled processes $\hat{Z}^n$ and $\hat{Q}^n$ are determined by

$$\hat{Z}^n \;=\; \hat{X}^n - \hat{Q}^n\,, \qquad \hat{Q}^n \;=\; \langle e, \hat{X}^n\rangle^+ \hat{U}^n\,. \tag{2.5}$$

By replacing $(\hat{Z}^n, \hat{Q}^n)$ with $(\hat{X}^n, \hat{U}^n)$ in the equations, it is often easier to establish the limiting controlled diffusion as we see in the next theorem. Also, the representation in (2.5) is useful in the study of asymptotic optimality in Section 4. Abusing the terminology, we replace the term admissible proportions-scheduling policy with admissible scheduling policy.

2.2. **The limiting controlled diffusion.** By the equation (4) in [1] and Assumption 2.1, we have

$$\hat{\ell}^n(t) \;\to\; \ell t \quad \text{a.s.} \quad \text{as} \quad n \to \infty\,, \tag{2.6}$$

where $\hat{\ell}^n := (\hat{\ell}_1, \dots, \hat{\ell}_d)'$, $\ell := (\ell_1, \dots, \ell_d)'$ and $\ell_i := \hat{\lambda}^\pi_i - \rho_i \hat{\mu}^\pi_i$. Let π be the stationary distribution of J^n, that is, $\pi'\mathcal{Q} = 0$ and $\pi' e = 1$. (Note that scaling $\mathcal{Q}$ does not change the stationary distribution.) By Proposition 3.2 in [1], we obtain

$$\hat{L}^n \;\Rightarrow\; \sigma^L_\alpha \widetilde{W} \quad \text{in} \quad (\mathbb{D}^d, \mathcal{J})\,, \quad \text{as} \quad n \to \infty\,, \tag{2.7}$$

where $\hat{L}^n := (\hat{L}^n_1, \dots, \hat{L}^n_d)'$, $\widetilde{W}$ is a zero-drift standard d-dimensional Wiener process, and if $\alpha > 1$, then $\sigma^L_\alpha = 0$, while if $\alpha \le 1$, then σ^L_α satisfies $(\sigma^L_\alpha)'\sigma^L_\alpha = \Theta = [\theta_{ij}]$, with

$$\theta_{ij} \;:=\; 2 \sum_{l \in \mathcal{K}} \sum_{k \in \mathcal{K}} \bigl(\lambda_i(k) - \rho_i \mu_i(k)\bigr)\bigl(\lambda_j(l) - \rho_j \mu_j(l)\bigr) \pi_k \Upsilon_{kl} \tag{2.8}$$

for $i, j \in \mathcal{I}$, and $\Upsilon_{kl} := \int_0^{+\infty} (P_{kl}(t) - \pi_k)\,\mathrm{d}t$ with $P_{kl}(t) = [\mathrm{e}^{\mathcal{Q}t}]_{kl}$, that is, $\Upsilon = (\Pi - \mathcal{Q})^{-1} - \Pi$. Here, Π denotes the matrix whose rows are equal to the vector π.

The proof of the following result is in Appendix A.

Theorem 2.1. *Under Assumptions 2.1 and 2.2, and assuming that $\hat{X}^n(0)$ is uniformly bounded, the following results hold.*

(i) *As $n \to \infty$,*

$$(\bar{Z}^n, \bar{Q}^n) \;\Rightarrow\; (\rho, 0) \quad \text{in} \quad (\mathbb{D}^d, \mathcal{J})^2\,,$$

where $\rho = (\rho_1, \dots, \rho_d)$.

(ii) *As $n \to \infty$,*

$$\hat{W}^n \;\Rightarrow\; \hat{W} \quad \text{in} \quad (\mathbb{D}^d, \mathcal{J})\,,$$

where $\hat{W}$ is a d-dimensional Brownian motion with a covariance coefficient matrix $\sigma'_\alpha \sigma_\alpha$ defined by

$$\sigma'_\alpha \sigma_\alpha = \begin{cases} \Lambda^2, & \alpha > 1, \\ \Lambda^2 + \Theta, & \alpha = 1, \\ \Theta, & \alpha < 1, \end{cases}$$

and $\Lambda := \mathrm{diag}(\sqrt{2\lambda^\pi_1}, \dots, \sqrt{2\lambda^\pi_d})$.

(iii) *$(\hat{X}^n, \hat{W}^n, \hat{Y}^n)$ is tight in $(D^d, \mathcal{J})^3$.*

(iv) *Provided that $\hat{U}^n$ is tight, any limit $\hat{X}$ of $\hat{X}^n$ is a unique strong solution of*

$$\mathrm{d}\hat{X}(t) \;=\; b\bigl(\hat{X}(t), \hat{U}(t)\bigr)\mathrm{d}t + \mathrm{d}\hat{W}(t)\,, \tag{2.9}$$

where $\hat{X}(0) = x$, $x \in \mathbb{R}^d$, is a limit of $\hat{X}^n(0)$, $\hat{U}$ is a limit of $\hat{U}^n$, and $b\colon \mathbb{R}^d \times \mathbb{U} \mapsto \mathbb{R}^d$ satisfies

$$b(x, u) \;=\; \ell - M(x - \langle e, x\rangle^+ u) - \Gamma \langle e, x\rangle^+ u\,,$$

with $M = \mathrm{diag}(\mu^\pi_1, \dots, \mu^\pi_d)$ and $\Gamma = \mathrm{diag}(\gamma^\pi_1, \dots, \gamma^\pi_d)$. Furthermore, $\hat{U}$ is non-anticipative, that is, for $s \le t$, $\hat{W}(t) - \hat{W}(s)$ is independent of

$$\mathcal{F}_s \;:=\; \sigma(\hat{U}(r), \hat{W}(r)\colon r \le s) \vee \mathcal{N}\,,$$

where $\mathcal{N}$ is the collection of $\mathbb{P}$-null sets.

2.3. **The discounted cost problem.** Let $\widetilde{\mathcal{R}}\colon \mathbb{R}^d_+ \mapsto \mathbb{R}_+$ take the form

$$\widetilde{\mathcal{R}}(x) = c|x|^m, \tag{2.10}$$

for some $c > 0$ and $m \geq 1$. Define the running cost function $\mathcal{R}\colon \mathbb{R}^d \times \mathbb{U} \mapsto \mathbb{R}_+$ by

$$\mathcal{R}(x,u) \;\coloneqq\; \widetilde{\mathcal{R}}(\langle e, x\rangle^+ u)\,.$$

Note that the running cost function is penalizing the size of the queues, and depends on the scheduling policy.

Remark 2.1. In place of (2.10) one may merely stipulate that $\widetilde{\mathcal{R}}(x)$ is a locally Hölder continuous function such that

$$c_1|x|^{\underline{m}} \;\leq\; \widetilde{\mathcal{R}}(x) \;\leq\; c_2(1+|x|^{\overline{m}}) \tag{2.11}$$

for some constants $1 \leq \underline{m} \leq \overline{m}$. See, e.g., Remark 3.1 in [20]. For the discounted problem, the lower bound in (2.11) is not required.

For each n and $\vartheta > 0$, given $\hat{X}^n(0)$, the ϑ-discounted problem can be written as

$$\hat{V}^n_\vartheta(\hat{X}^n(0)) \;\coloneqq\; \inf_{\hat{U}^n\in\widehat{\mathfrak{U}}^n} \mathfrak{J}^n_\vartheta(\hat{X}^n(0),\hat{U}^n)\,,$$

and

$$\mathfrak{J}^n_\vartheta(\hat{X}^n(0),\hat{U}^n) \;\coloneqq\; \mathbb{E}\left[\int_0^\infty \mathrm{e}^{-\vartheta s}\mathcal{R}\bigl(\hat{X}^n(s),\hat{U}^n(s)\bigr)\,\mathrm{d}s\right]\,.$$

Let $\mathfrak{U}$ denote the set of all admissible controls for the limiting diffusion in (2.9). The ϑ-discounted cost criterion for the limiting controlled diffusion is defined by

$$\mathfrak{J}_\vartheta(x,\hat{U}) \;\coloneqq\; \mathbb{E}^{\hat{U}}_x\left[\int_0^\infty \mathrm{e}^{-\vartheta s}\mathcal{R}\bigl(\hat{X}(s),\hat{U}(s)\bigr)\,\mathrm{d}s\right]\,,$$

for $\hat{U} \in \mathfrak{U}$, and the ϑ-discounted problem is

$$\hat{V}_\vartheta(x) \;\coloneqq\; \inf_{\hat{U}\in\mathfrak{U}} \mathfrak{J}_\vartheta(x,\hat{U})\,.$$

The main result concerning the discounted cost problem is stated in the next theorem, the proof of which is given in Section 4.2.

Theorem 2.2. *Grant Assumptions 2.1 and 2.2. If $\hat{X}^n(0) \to x \in \mathbb{R}^d$ as $n \to \infty$, then it holds that*

$$\lim_{n\to\infty} \hat{V}^n_\vartheta(\hat{X}^n(0)) \;=\; \hat{V}_\vartheta(x) \qquad \forall\,\vartheta > 0\,.$$

2.4. **The ergodic control problem.** Given $\hat{X}^n(0)$, define the ergodic cost associated with $\hat{X}^n$ and $\hat{U}^n$ by

$$\mathfrak{J}^n(\hat{X}^n(0),\hat{U}^n) \;\coloneqq\; \limsup_{T\to\infty} \frac{1}{T}\,\mathbb{E}\left[\int_0^T \mathcal{R}\bigl(\hat{X}^n(s),\hat{U}^n(s)\bigr)\,\mathrm{d}s\right]\,,$$

and the associated ergodic control problem by

$$\hat{V}^n(\hat{X}^n(0)) \;\coloneqq\; \inf_{\hat{U}^n\in\widehat{\mathfrak{U}}^n} \mathfrak{J}^n(\hat{X}^n(0),\hat{U}^n)\,.$$

Analogously, we define the ergodic cost associated with the limiting controlled process $\hat{X}$ in (2.9) by

$$\mathfrak{J}(x,\hat{U}) \;\coloneqq\; \limsup_{T\to\infty} \frac{1}{T}\,\mathbb{E}\left[\int_0^T \mathcal{R}\bigl(\hat{X}(s),\hat{U}(s)\bigr)\,\mathrm{d}s\right]\,,$$

and the ergodic control problem by

$$\varrho_*(x) \;\coloneqq\; \inf_{U\in\mathfrak{U}} \mathfrak{J}(x,U)\,. \tag{2.12}$$

The value $\varrho_*(x)$ is independent of x. As we show in Theorem 4.2, the infimum is realized with a stationary Markov control and $\varrho_*(x) = \varrho_*$.

The asymptotic optimality of the value functions is stated below (proof in Section 4.3).

Theorem 2.3. *Grant Assumptions 2.1 and 2.2. If $\hat{X}^n(0) \to x \in \mathbb{R}^d$ as $n \to \infty$, then it holds that*

$$\lim_{n\to\infty} \hat{V}^n(\hat{X}^n(0)) \;=\; \varrho_*(x)\,.$$

3. Ergodic Properties

We first review the ergodic properties of the limiting diffusion. Then, we establish some ergodic results for the diffusion-scaled process. This second task forms the central part of this section, and we explain why the results established here are needed for the study of ergodic control problems. It is worth pointing out that equivalent results exist for non-modulated diffusion-scaled processes (see [8]). However, the presence of modulation requires a fresh approach.

3.1. **The limiting controlled diffusion.** The limiting diffusion belongs to the class of piecewise linear diffusions studied in [10]. Applying Theorem 3 in [10], we deduce that the limiting process $\hat{X}$ with abandonment in (2.9) is exponentially ergodic (see, e.g., [21, Section 6], for definition) under a constant control $\bar{u} = e_d = (0, \dots, 0, 1)'$. By Theorem 3.5 in [11], the limiting process $\hat{X}$ in (2.9) is exponentially ergodic under any constant control. We summarize the ergodic properties of the limiting controlled process $\hat{X}$ in the following proposition.

Proposition 3.1. *The controlled diffusion $\hat{X}$ in (2.9) is exponentially ergodic under any constant control $u \in \mathbb{U}$.*

Remark 3.1. As a consequence of the proposition, if $\tilde{v}$ is a stationary Markov control which is constant on the complement of some compact set, then the controlled diffusion $\hat{X}$ in (2.9) is exponentially ergodic under this control. For the diffusion-scaled process $\hat{X}^n$, we first prove exponential ergodicity under a static priority scheduling policy in Theorem 3.1. It then follows that any stationary Markov scheduling policy, which agrees with this static priority policy outside a compact set, is exponentially ergodic. We remark here that exponential ergodicity of the diffusion-scaled process under any stationary Markov scheduling policy is an open problem (compare with the study of ergodicity for the standard 'V' network in [18]).

3.2. **Diffusion-scaled processes.** Theorem 2.1 asserts that the scaled process $\hat{X}^n$ converges in a weak sense to the diffusion $\hat{X}$, under suitable controls. This does not mean that the optimal ergodic control problem for the limiting diffusion is a good approximation for the optimal ergodic control problem for the diffusion-scaled processes in general. Loosely speaking, in order to establish this, we need to show that under some "near optimal controls", the ergodic occupation measures of the diffusion-scaled processes converge to the corresponding measures of the limiting diffusion process. We make this formal in Section 4. Crucially, to establish Theorem 2.3, we need to establish that the diffusion-scaled process under the "near optimal controls" are "exponentially ergodic uniformly in n". We make this last notion precise through the following definitions.

Definition 3.1. For each $n \in \mathbb{N}$, let $\tilde{z}^n = \tilde{z}^n(x)$, for $x \in \mathbb{Z}^d_+$, denote the scheduling policy defined by

$$\tilde{z}^n_i(x) \;:=\; x_i \wedge \left(n - \sum_{i'=1}^{i-1} x_{i'}\right)^+ \quad \text{for} \quad i \in \mathcal{I}\,.$$

By using the balance equation $x_i = \tilde{z}_i^n(x) + \tilde{q}_i^n(x)$ and Definition 3.1, we obtain for $x \in \mathbb{Z}_+^d$ and $i \in \mathcal{I}$, that

$$\tilde{q}_i^n(x) \;=\; \left[x_i - \left(n - \sum_{i'=1}^{i-1} x_{i'}\right)^+\right]^+ .$$

Definition 3.2. For $x \in \mathbb{R}^d$, define

$$\tilde{x}^n(x) \;\coloneqq\; (x_1 - \rho_1 n, \dots, x_d - \rho_d n)' \,, \quad \hat{x}^n(x) \;\coloneqq\; n^{-\beta} \tilde{x}^n(x) \,,$$

$\mathfrak{X}^n \coloneqq \{\hat{x}^n(x) \colon x \in \mathbb{Z}_+^d\}$, and $\tilde{\mathfrak{X}}^n \coloneqq \{\hat{x}^n(x) \colon x \in \mathfrak{A}^n\}$, with

$$\mathfrak{A}^n \;\coloneqq\; \bigl\{x \in \mathbb{R}_+^d \colon \|x - \rho n\| \;\le\; c_0 n^\beta\bigr\}$$

for some positive constant c_0.

Definition 3.3. We let $\widehat{\mathcal{L}}_n^{z^n}$ denote the generator of the process $(\hat{X}^n, J^n)$, governed by a stationary Markov scheduling policy z^n. We say that $\bigl\{(\hat{X}^n, J^n)\bigr\}_{n \in \mathbb{N}}$ governed by a sequence of stationary Markov policies $\{z^n\}_{n \in \mathbb{N}}$ is *uniformly exponentially ergodic of order* m if for each $n \ge N_0$, where $N_0 \in \mathbb{N}$ is fixed, there exists a nonnegative function $\widehat{\mathcal{V}}^n \colon \mathfrak{X}^n \times \mathcal{K} \to \mathbb{R}$, which is continuous in its first argument, and positive constants c, C_1 and C_2, independent of n, such that

$$\widehat{\mathcal{V}}^n(x, k) \;\ge\; c \sum_{i=1}^d |x_i|^m \,,$$

and

$$\widehat{\mathcal{L}}_n^{z^n} \widehat{\mathcal{V}}^n(\hat{x}, k) \;\le\; C_1 - C_2 \widehat{\mathcal{V}}^n(\hat{x}, k) \,, \quad \forall (\hat{x}, k) \in \mathfrak{X}^n \times \mathcal{K} \,.$$

Since $(\hat{X}^n, J^n)$ is irreducible and aperiodic, it is well known that uniform exponential ergodicity of order m implies that the transition probabilities of the process converge to the invariant distribution with an exponential rate which is independent of $n \ge N_0$. It also implies that

$$\sup_{n \ge N_0} \; \limsup_{T \to \infty} \frac{1}{T} \, \mathbb{E}^{z^n} \left[\int_0^T |\hat{X}^n(s)|^m \, \mathrm{d}s\right] \;<\; \infty \,.$$

We then say that the sequence of controls z^n are stabilizing (of order m). We begin by showing in Theorem 3.1 that "static" scheduling policies (Definition 3.1) are stabilizing. Then, roughly speaking, we proceed to show in Lemma 3.3 that scheduling policies which agree with a static policy outside a ball are also stabilizing. Finally in Section 4 we choose near optimal policies inside the ball and static policy outside the ball, by Lemma 3.3 these are stabilizing, and hence the ergodic occupation measures of the diffusion-scaled process are well approximated by the corresponding ones of the limiting diffusion.

Definition 3.4. Let z^n be a stationary Markov policy. We denote the infinitesimal generator of the "average" process by

$$\begin{aligned}\bar{\mathcal{L}}_n^{z^n} f(x) \;\coloneqq\; \sum_{i \in \mathcal{I}} \bar{\lambda}_i^n \bigl(f(x + e_i) - f(x)\bigr) + \sum_{i \in \mathcal{I}} \bar{\mu}_i^n z_i^n(x) \bigl(f(x - e_i) - f(x)\bigr) \\ + \sum_{i \in \mathcal{I}} \bar{\gamma}_i^n q_i^n(x, z) \bigl(f(x - e_i) - f(x)\bigr)\end{aligned}$$

for $f \in \mathcal{C}_b(\mathbb{R}^d)$ and all $x \in \mathbb{Z}_+^d$, where $q_i^n(x, z) \coloneqq x_i - z_i^n(x)$, $i \in \mathcal{I}$.

Lemma 3.1. *Let $\tilde{z}^n$ be the scheduling policy in Definition 3.1. Then for any even integer $m \geq 2$, there exist a positive vector ξ, positive constants C_1 and C_2, and $n_0 \in \mathbb{N}$, such that the functions f_n, $n \in \mathbb{N}$, defined by*

$$f_n(x) := \sum_{i \in \mathcal{I}} \xi_i |x_i - \rho_i n|^m \,, \quad \forall\, x \in \mathbb{Z}^d_+ \,, \tag{3.1}$$

satisfy

$$\bar{\mathcal{L}}^{\tilde{z}^n}_n f_n(x) \leq C_1 n^{m\beta} - C_2 f_n(x) \,, \quad \forall x \in \mathbb{Z}^n_+ \,, \quad \forall n \geq n_0 \,.$$

For a proof of Lemma 3.1, see Appendix A. This lemma shows that, under the static priority policy $\tilde{z}^n$, the "average" process is exponentially ergodic.

Definition 3.5. Under a stationary Markov policy $z^n = z^n(x)$, the infinitesimal generator of $(X^n(t), J^n(t))$ is defined by

$$\widetilde{\mathcal{L}}^{z^n}_n f(x,k) := \mathcal{L}^{z^n}_{n,k} f(x,k) + \sum_{k' \in \mathcal{K}} n^\alpha q_{kk'} \big(f(x,k') - f(x,k)\big) \,,$$

for $f \in \mathcal{C}_b(\mathbb{R}^d \times \mathcal{K})$, where

$$\begin{aligned}\mathcal{L}^{z^n}_{n,k} f(x,k) := \sum_{i \in \mathcal{I}} \lambda^n_i(k)\big(f(x+e_i,k) - f(x,k)\big) + \sum_{i \in \mathcal{I}} \mu^n_i(k) z^n_i(x)\big(f(x-e_i,k) - f(x,k)\big) \\ + \sum_{i \in \mathcal{I}} \gamma^n_i(k) q^n_i(x,z)\big(f(x-e_i,k) - f(x,k)\big) \,.\end{aligned}$$

Let $\Delta\lambda^n_i(k) := \bar{\lambda}^n_i - \lambda^n_i(k)$ for $i \in \mathcal{I}$ and $k \in \mathcal{K}$, and define $\Delta\mu^n_i$ and $\Delta\gamma^n_i$, analogously. Let $\Delta\mathcal{L}^{z^n}_{n,k} \colon \mathcal{C}_b(\mathbb{R}^d \times \mathcal{K}) \mapsto \mathcal{C}_b(\mathbb{R}^d \times \mathcal{K})$ be the operator defined by

$$\begin{aligned}\Delta\mathcal{L}^{z^n}_{n,k} f(x,k) := \sum_{i \in \mathcal{I}} \Delta\lambda^n_i(k)\big(f(x+e_i,k) - f(x,k)\big) + \sum_{i \in \mathcal{I}} \Delta\mu^n_i(k) z^n_i(x)\big(f(x-e_i,k) - f(x,k)\big) \\ + \sum_{i \in \mathcal{I}} \Delta\gamma^n_i(k) q^n_i(x,z)\big(f(x-e_i,k) - f(x,k)\big) \,.\end{aligned}$$

Define the embedding $\mathfrak{M} \colon \mathcal{C}_b(\mathbb{R}^d) \hookrightarrow \mathcal{C}_b(\mathbb{R}^d \times \mathcal{K})$ by $\mathfrak{M}(f) = \tilde{f}$, where $\tilde{f}(\cdot,k) = f(\cdot)$ for all $k \in \mathcal{K}$. It is easy to see, by Definitions 3.4 and 3.5, that for all $f \in \mathcal{C}_b(\mathbb{R}^d)$, $\tilde{f} = \mathfrak{M}(f)$, and $k \in \mathcal{K}$, we have

$$\bar{\mathcal{L}}^{z^n}_n f(x) = \mathcal{L}^{z^n}_{n,k} \tilde{f}(x,k) + \Delta\mathcal{L}^{z^n}_{n,k} \tilde{f}(x,k). \tag{3.2}$$

Abusing the notation, we can identify $\tilde{f} = \mathfrak{M}(f)$ with f, and thus (3.2) can be written as

$$\bar{\mathcal{L}}^{z^n}_n f(x) = \mathcal{L}^{z^n}_{n,k} f(x) + \Delta\mathcal{L}^{z^n}_{n,k} f(x) \,.$$

Lemma 3.2. *Let $f_n(x)$ be the function defined in (3.1), and z^n be any stationary Markov policy. There exists a function $g_n[f_n] \in \mathcal{C}(\mathbb{R}^d \times \mathcal{K})$, satisfying*

$$g_n[f_n](x,k) = \frac{1}{n^\alpha} \sum_{k' \in \mathcal{K}} c_{kk'} \Delta\mathcal{L}^{z^n}_{n,k'} f_n(x) \,, \quad \forall\, (x,k) \in \mathbb{R}^d \times \mathcal{K} \tag{3.3}$$

with some constants $c_{kk'}$ independent of n, and

$$\sum_{k' \in \mathcal{K}} n^\alpha q_{kk'} \big(g_n[f_n](x,k') - g_n[f_n](x,k)\big) = \Delta\mathcal{L}^{z^n}_{n,k} f_n(x) \,, \quad \forall\, (x,k) \in \mathbb{R}^d \times \mathcal{K} \,.$$

As a consequence, we have, for fixed $\alpha > 0$ and each $n \in \mathbb{N}$,

$$\big|g_n[f_n](x,k)\big| \leq C_3(1 + n^{m(1-\alpha)}) + \epsilon_n f_n(x) \,, \quad \forall (x,k) \in \mathbb{R}^d \times \mathcal{K} \,, \tag{3.4}$$

where C_3 is some positive constant, and $\epsilon_n > 0$ can be chosen arbitrarily small for large enough n.

Proof. The existence of $g_n[f_n](x,k)$ directly follows from the Fredholm alternative. The version applicable here may be found in [12].

For $k \in \mathcal{K}$, we observe that

$$
\begin{aligned}
\frac{|\Delta\mathcal{L}^{z^n}_{n,k} f_n(x)|}{n^\alpha} &\le \frac{1}{n^\alpha}\sum_{i\in\mathcal{I}} \xi_i \big|\Delta\lambda^n_i(k) - \Delta\mu^n_i(k) z^n_i - \Delta\gamma^n_i(k) q^n_i\big|\big|m(\tilde{x}^n_i)^{m-1} + \mathcal{O}\big(|\tilde{x}^n_i|^{m-2}\big)\big| \\
&\le \frac{1}{n^\alpha}\sum_{i\in\mathcal{I}} \xi_i \Big(|\Delta\lambda^n_i(k)| + |\Delta\mu^n_i(k)| x_i + |\Delta\gamma^n_i(k)| x_i\Big)\big|m(\tilde{x}^n_i)^{m-1} + \mathcal{O}\big(|\tilde{x}^n_i|^{m-2}\big)\big| \\
&\le C_4\big(1 + n^{m(1-\alpha)}\big) + \epsilon_n f_n(x)\,,
\end{aligned}
$$

where C_4 is some positive constant, and the last inequality follows by using $x_i = \tilde{x}^n_i + n\rho_i$, Assumption 2.1, and following inequalities with sufficiently small $\epsilon > 0$:

$$
\begin{aligned}
n^{-\alpha}|\Delta\lambda^n_i(k)||\tilde{x}^n_i|^{m-1} &\le \epsilon^{1-m} n^{m(1-\alpha)} + \epsilon|\tilde{x}^n_i|^m\,, \\
n^{1-\alpha}|\tilde{x}^n_i|^{m-1} &\le \epsilon^{1-m} n^{m(1-\alpha)} + \epsilon|\tilde{x}^n_i|^m\,, \\
\mathcal{O}(n^{1-\alpha})\mathcal{O}(|\tilde{x}^n_i|^{m-2}) &\le \epsilon^{1-m/2} n^{m(1-\alpha)/2} + \epsilon\big(\mathcal{O}(|\tilde{x}^n_i|^{m-2})\big)^{m/m-2}\,.
\end{aligned} \tag{3.5}
$$

Note that when $\alpha > 1$, $n^{m(1-\alpha)} \le 1$. Thus, by the expression of $g_n[f_n]$ in (3.3), we obtain (3.4). This completes the proof. □

For each n, define the function $\hat{f}_n \in \mathcal{C}(\mathbb{R}^d \times \mathcal{K})$ by

$$\hat{f}_n(x,k) \;\coloneqq\; f_n(x) + g_n[f_n](x,k)\,.$$

The norm-like function $\mathcal{V}_{m,\xi}$ is defined by $\mathcal{V}_{m,\xi}(x) \coloneqq \sum_{i\in\mathcal{I}} \xi_i |x_i|^m$ for $x \in \mathbb{R}^d$, with $m > 0$ and a positive vector ξ defined in (3.1). Recall from Definition 3.3 that $\widehat{\mathcal{L}}^{z^n}_n$, denote thes generator of $(\hat{X}^n, J^n)$ governed by a stationary Markov policy z^n. Using $\hat{x}^n$ in Definition 3.2, we can write $\widehat{\mathcal{L}}^{z^n}_n$ as

$$\Big[\widehat{\mathcal{L}}^{z^n}_n f(\cdot,\cdot)\Big]\,(\hat{x}^n(x), k) \;=\; \Big[\widetilde{\mathcal{L}}^{z^n}_n f(\hat{x}^n(\cdot),\cdot)\Big]\,(x,k)$$

for $f \in \mathcal{C}_b(\mathbb{R}^d \times \mathcal{K})$. We also define the operator $\Delta\widehat{\mathcal{L}}^{z^n}_{n,k}\colon \mathcal{C}_b(\mathbb{R}^d\times\mathcal{K}) \mapsto \mathcal{C}_b(\mathbb{R}^d\times\mathcal{K})$ satisfying

$$\Big[\Delta\widehat{\mathcal{L}}^{z^n}_{n,k} f(\cdot,\cdot)\Big]\,(\hat{x}^n(x), k) \;=\; \big[\Delta\mathcal{L}^{z^n}_{n,k} f(\hat{x}^n(\cdot),\cdot)\big]\,(x,k)$$

for $f \in \mathcal{C}_b(\mathbb{R}^d \times \mathcal{K})$. Let $\widehat{\mathcal{V}}_{m,\xi}$ be the function defined by

$$\widehat{\mathcal{V}}^n_{m,\xi}(x,k) \;\coloneqq\; \mathcal{V}_{m,\xi}(x) + \frac{1}{n^\alpha}\sum_{k'\in\mathcal{K}} c_{kk'}\Delta\widehat{\mathcal{L}}^{z^n}_{n,k'}\mathcal{V}_{m,\xi}(x)$$

for $x \in \mathbb{R}^d$ with the constants $c_{kk'}$ defined in (3.3).

Theorem 3.1. *Let $\widehat{\mathcal{L}}^{\tilde{z}^n}_n$ denote the generator of the $(\hat{X}^n, J^n)$ under the scheduling policy defined in Definition 3.1. For any even integer $m \ge 2$, there exists $n_2 \in \mathbb{N}$ such that*

$$\widehat{\mathcal{L}}^{\tilde{z}^n}_n \widehat{\mathcal{V}}^n_{m,\xi}(\hat{x},k) \;\le\; \widetilde{C}_1 - \widetilde{C}_2\widehat{\mathcal{V}}^n_{m,\xi}(\hat{x},k)\,, \quad \forall(\hat{x},k) \in \mathfrak{X}^n \times \mathcal{K}\,, \quad \forall n > n_2\,, \tag{3.6}$$

for some positive constants $\widetilde{C}_1$, $\widetilde{C}_2$ and $n_2 \ge n_0$ depending on ξ and m. As a consequence, $(\hat{X}^n, J^n)$ under the scheduling policy $\tilde{z}^n$ is exponentially ergodic, and for any $m > 0$,

$$\sup_{n\ge n_2}\,\limsup_{T\to\infty}\,\frac{1}{T}\,\mathbb{E}^{\tilde{z}^n}\left[\int_0^T |\hat{X}^n(s)|^m\,\mathrm{d}s\right] \;<\; \infty\,. \tag{3.7}$$

Proof. Since operators defined in Definitions 3.4 and 3.5 are linear, we have

$$\widehat{\mathcal{V}}^n_{m,\xi}(\hat{x}^n(x),k) \;=\; n^{-m\beta}\hat{f}_n(x,k)$$

for $x \in \mathbb{Z}^d_+$ and $k \in \mathcal{K}$. Thus, it suffices to show that

$$\widetilde{\mathcal{L}}^{\tilde{z}^n}_n \hat{f}_n(x,k) \;\le\; \widetilde{C}_1 n^{m\beta} - \widetilde{C}_2\hat{f}_n(x,k)\,, \quad \forall (x,k)\in\mathbb{Z}^d_+\times\mathcal{K}\,, \quad \forall n \ge n_2\,. \tag{3.8}$$

Let ξ be the vector in (3.1). It is easy to see that

$$\begin{aligned}\widetilde{\mathcal{L}}^{\tilde{z}^n}_n \hat{f}_n(x,k) \;&=\; \bar{\mathcal{L}}^{\tilde{z}_n}_{n,k} f_n(x) + \mathcal{L}^{\tilde{z}_n}_{n,k} g_n[f_n](x,k)\\ &\le\; C_1 n^{m\beta} - C_2 f_n(x) + \mathcal{L}^{\tilde{z}_n}_{n,k} g_n[f_n](x,k)\,, \quad \forall n\ge n_0\,,\end{aligned} \tag{3.9}$$

where the inequality follows from Lemma 3.1. Applying Lemma 3.2, we see that there exist positive constants C_6, C_7, and $\tilde{n}_1$, such that

$$C_7 n^{m\beta} - C_6\hat{f}_n(x,k) \;\ge\; C_1 n^{m\beta} - C_2 f_n(x)\,, \quad \forall\, n > \tilde{n}_1\,, \quad \forall (x,k)\in\mathbb{R}^d\times\mathcal{K}\,. \tag{3.10}$$

Thus, to prove (3.8), by using (3.9) and (3.10), it suffices to show that, for large enough n,

$$\mathcal{L}^{\tilde{z}_n}_{n,k} g_n[f_n](x,k) \;\le\; C_8 n^{m\beta} + \epsilon f_n(x)\,, \tag{3.11}$$

where C_8 is some positive constant, and $\epsilon > 0$ can be chosen arbitrarily small for large enough n. Recall the definition of $g_n[f_n]$ in (3.3), and observe that

$$\frac{\Delta\mathcal{L}^{\tilde{z}_n}_{n,k} f_n(x)}{n^\alpha} \;=\; \sum_{i\in\mathcal{I}} \frac{\xi_i}{n^\alpha}\Big(\Delta\lambda^n_i(k) - \Delta\mu^n_i(k)\tilde{z}^n_i(x) - \Delta\gamma^n_i(k)\tilde{q}^n_i(x)\Big)\Big(m(\tilde{x}^n_i)^{m-1} + \mathcal{O}(|\tilde{x}^n_i|^{m-2})\Big)\,.$$

Let

$$h_n(x) \;\coloneqq\; \frac{1}{n^\alpha}\sum_{i\in\mathcal{I}} \xi_i \tilde{q}^n_i(x)(\tilde{x}^n_i)^{m-1}\,.$$

Note that in order to prove (3.11), by using (3.5) and the balance equation $\tilde{z}^n_i(x) = \tilde{x}^n_i(x) - \tilde{q}^n_i(x) + n\rho_i$, we only need to show that

$$\mathcal{L}^{\tilde{z}_n}_{n,k} h_n(x) \;\le\; C_9 n^{m\beta} + \epsilon f_n(x)\,, \tag{3.12}$$

where C_9 is some positive constant, and $\epsilon > 0$ can be chosen arbitrarily small for all large enough n; the other terms in $\mathcal{L}^{\tilde{z}_n}_{n,k} g_n[f_n]$ can be treated similarly. We obtain

$$\mathcal{L}^{\tilde{z}_n}_{n,k} h_n(x) \;=\; \sum_{i\in\mathcal{I}}\Big(F^{(1)}_{n,i}(x) + F^{(2)}_{n,i}(x)\Big)\,,$$

where

$$\begin{aligned}F^{(1)}_{n,i}(x) \;\coloneqq\; & n^{-\alpha}\xi_i\lambda^n_i(k)\big(\tilde{q}^n_i(x+e_i)(\tilde{x}^n_i+1)^{m-1} - \tilde{q}^n_i(x)(\tilde{x}^n_i)^{m-1}\big)\\ &+ n^{-\alpha}\xi_i\big(\mu^n_i(k)\tilde{z}^n_i(x) + \gamma^n_i(k)\tilde{q}^n_i(x)\big)\big(\tilde{q}^n_i(x-e_i)(\tilde{x}^n_i-1)^{m-1} - \tilde{q}^n_i(x)(\tilde{x}^n_i)^{m-1}\big)\,,\end{aligned}$$

and

$$\begin{aligned}F^{(2)}_{n,i}(x) \;\coloneqq\; n^{-\alpha}\sum_{j\ne i}\big[&\lambda^n_i(k)\xi_j\big(\tilde{q}^n_j(x+e_i) - \tilde{q}^n_j(x)\big)(\tilde{x}^n_j)^{m-1}\\ &+ \big(\mu^n_i(k)\tilde{z}^n_i(x) + \gamma^n_i(k)\tilde{q}^n_i(x)\big)\xi_j\big(\tilde{q}^n_j(x-e_i) - \tilde{q}^n_j(x)\big)(\tilde{x}^n_j)^{m-1}\big]\,.\end{aligned}$$

Note that for $i' \in \mathcal{I}$,

$$|\tilde{q}^n_{i'}(x\pm e_i) - \tilde{q}^n_{i'}(x)| \;\le\; 1\,,$$

and $\tilde{q}^n_{i'}(x)$ is the unscaled queueing process. We first consider $F^{(1)}_{n,i}(x)$. We have

$$\sum_{i\in\mathcal{I}} F^{(1)}_{n,i}(x) \le n^{-\alpha}\sum_{i\in\mathcal{I}}\Big[\xi_i\lambda^n_i(k)\Big(\tilde{q}^n_i(x+e_i)\big((\tilde{x}^n_i+1)^{m-1}-(\tilde{x}^n_i)^{m-1}\big)+|\tilde{x}^n_i|^{m-1}\Big)$$
$$+\xi_i\Big(\mu^n_i(k)(\tilde{x}^n_i+n\rho_i-\tilde{q}^n_i(x))+\gamma^n_i(k)\tilde{q}^n_i(x)\Big)\Big(\tilde{q}^n_i(x-e_i)\big((\tilde{x}^n_i-1)^{m-1}-(\tilde{x}^n_i)^{m-1}\big)+|\tilde{x}^n_i|^{m-1}\Big)\Big].$$

Note that

$$\left[n\rho_i-\left(n-\sum_{j=1}^{i-1}n\rho_j\right)^+\right]^+ = 0.$$

By using the fact that $a^+-b^+=\eta(a-b)$, for $a,b\in\mathbb{R}^d$ and $\eta\in[0,1]$, we have

$$\begin{aligned}\tilde{q}^n_i(x) &= -\left[n\rho_i-\left(n-\sum_{j=1}^{i-1}n\rho_i\right)^+\right]^+ + \left[x_i-\left(n-\sum_{i'=1}^{i-1}x_{i'}\right)^+\right]^+\\ &= -\eta_i(x)(n\rho_i-x_i)+\bar{\eta}_i(x)\sum_{j=1}^{i-1}(x_j-n\rho_j)\\ &= \eta_i(x)\tilde{x}^n_i+\bar{\eta}_i(x)\sum_{j=1}^{i-1}\tilde{x}^n_j \quad \forall x\in\mathbb{R}^d,\end{aligned} \tag{3.13}$$

for the mappings $\eta_i,\bar{\eta}_i\colon\mathbb{R}^d\mapsto[0,1]^d$. By using (3.13) and Young's inequality, we have

$$\tilde{q}^n_i(x\pm e_i)\big((\tilde{x}^n_i\pm1)^{m-1}-(\tilde{x}^n_i)^{m-1}\big) \le \mathcal{O}\big(|\tilde{x}^n_i|^{m-1}\big)+\sum_{i'=1}^{i-1}\mathcal{O}\big(|\tilde{x}^n_{i'}|^{m-1}\big),$$
$$\tilde{q}^n_i(x)\tilde{q}^n_i(x-e_i)\big((\tilde{x}^n_i-1)^{m-1}-(\tilde{x}^n_i)^{m-1}\big) \le \mathcal{O}\big(|\tilde{x}^n_i|^{m}\big)+\sum_{i'=1}^{i-1}\mathcal{O}\big(|\tilde{x}^n_{i'}|^{m}\big).$$

Therefore, applying inequalities in (3.5), we obtain

$$\sum_{i\in\mathcal{I}} F^{(1)}_{n,i}(x) \le C_{10}n^{m(1-\alpha)}+\epsilon f_n(x). \tag{3.14}$$

where C_{10} is some positive constant, and ϵ can be chosen arbitrarily small for large enough n. On the other hand, since $\tilde{z}^n_{i'}(x)\le x_{i'}$, and $\tilde{q}^n_{i'}(x)\le x_{i'}$ for $i'\in\mathcal{I}$, applying Young's inequality, we obtain

$$\begin{aligned}F^{(2)}_{n,i}(x) &\le n^{-\alpha}\sum_{j\ne i}\xi_j\big(|\lambda^n_i(k)|+(|\mu^n_i(k)|+|\gamma^n_i(k)|)(\tilde{x}^n_i+n\rho_i)\big)|\tilde{x}^n_j|^{m-1}\\ &\le C_{11}n^{m(1-\alpha)}+\epsilon f_n(x),\end{aligned} \tag{3.15}$$

where C_{11} is some positive constant, and ϵ can be chosen arbitrarily small for large enough n, and the second inequality follows from (3.5). By Lemma 3.2, there exists $n_1>0$ such that for all $\hat{x}\in\mathfrak{X}^n$, $k\in\mathcal{K}$ and $n>n_1$, we have

$$\widehat{\mathcal{V}}^n_{m,\xi}(\hat{x},k) \ge \frac{1}{2}\mathcal{V}_{m,\xi}(\hat{x})+\mathfrak{o}(1). \tag{3.16}$$

We choose $n_2\in\mathbb{N}$ satisfying $n_2\ge\max\{n_0,\tilde{n}_1,n_1\}$. Thus, since $1-\alpha<\beta$, by (3.14) and (3.15) we have shown (3.12). As a result, we have proved (3.8), which implies (3.6).

Let $\mathbb{E}^{\tilde{z}_n}=\mathbb{E}$. By Itô's formula, we obtain

$$\mathbb{E}\Big[\widehat{\mathcal{V}}^n_{m,\xi}(\hat{X}^n(T),J^n(T))\Big]-\mathbb{E}\Big[\widehat{\mathcal{V}}^n_{m,k}(\hat{X}^n(0),J^n(0))\Big] = \mathbb{E}\left[\int_0^T\widehat{\mathcal{L}}^{\tilde{z}^n}_n\widehat{\mathcal{V}}^n_{m,\xi}(\hat{X}^n(s),J^n(s))\,\mathrm{d}s\right].$$

Then, using (3.6), we have, for $\forall n \ge n_2$,

$$- \mathbb{E}\left[\widehat{\mathcal{V}}^n_{m,k}(\hat{X}^n(0), J^n(0))\right] \;\le\; \widetilde{C}_1 T - \widetilde{C}_2\, \mathbb{E}\left[\int_0^T \widehat{\mathcal{V}}^n_{m,\xi}(\hat{X}^n(s), J^n(s))\, \mathrm{d}s\right]. \tag{3.17}$$

Applying (3.16) and (3.17), we obtain that, for some positive constant C_{12}, and for all $n \ge n_2$,

$$\frac{1}{T}\,\mathbb{E}\left[\int_0^T \sum_{i\in\mathcal{I}} |\hat{X}^n_i(s)|^m \,\mathrm{d}s\right] \le C_{12}\,.$$

This proves (3.7). □

Definition 3.6. Let $\omega\colon \mathbb{R}^d_+ \mapsto \mathbb{Z}^d_+$ be a measurable map defined by

$$\omega(x) \;\coloneqq\; \left(\lfloor x_1\rfloor, \dots, \lfloor x_{d-1}\rfloor, \lfloor x_d\rfloor + \sum_{i=1}^d (x_i - \lfloor x_i\rfloor)\right).$$

Definition 3.7. Let $v^n\colon \mathbb{R}^d \mapsto \mathbb{U}$ be any sequence of functions satisfying $v^n(\hat{x}^n(x)) = e_d$, for all $x \notin \mathfrak{A}^n$, and such that $x \mapsto v^n(\hat{x}^n(x))$ is continuous. Define

$$q^n[v^n](x) \;\coloneqq\; \begin{cases} \omega\big((\langle e, x\rangle - n)^+ v^n(\hat{x}^n(x))\big) & \text{for } \sup_{i\in\mathcal{I}} |x_i - n\rho_i| \le \kappa n\,,\\ \tilde{q}^n(x) & \text{for } \sup_{i\in\mathcal{I}} |x_i - n\rho_i| > \kappa n\,, \end{cases}$$

where $\tilde{q}^n(x)$ is as in Definition 3.1, and $\kappa < \inf_{i\in\mathcal{I}}\{\rho_i\}$. Define the admissible scheduling policy

$$z^n[v^n](x) \;\coloneqq\; x - q^n[v^n](x)\,.$$

We have the following lemma on stabilization of the diffusion-scaled queueing processes.

Lemma 3.3. *The process $(\hat{X}^n, J^n)$ governed by the scheduling policy z^n in Definition 3.7 is uniformly exponentially ergodic (of any order m).*

Proof. Observe that for all $n \in \mathbb{N}$, we have

(i) For $i \in \mathcal{I}$, there exists a constant C such that $|q^n[v^n](x \pm e_i) - q^n[v^n](x)| \le C$;
(ii) For $i \in \mathcal{I}$, there exists functions $\epsilon^n_i,\ \tilde{\epsilon}^n_i\colon \mathbb{R}^d \mapsto [0,1]$ such that

$$q^n_i[v^n](x) \;=\; \epsilon^n_i(x)(x_i - n\rho_i) + \tilde{\epsilon}^n_i(x)\sum_{i'=1}^{i-1}(x_{i'} - n\rho_{i'}) + \mathcal{O}(n^\beta)\,.$$

Hence the same proof as that of Theorem 3.1 may be employed to obtain the result. □

Remark 3.2. Lemma 3.3 shows that any sequence of scheduling policies which satisfies (i) and (ii) in the proof of Lemma 3.3 is "stabilizing".

4. Asymptotic Optimality

4.1. **Optimal solutions to the limiting diffusion control problems.** The characterization of optimal control for the limiting diffusion follow from the known results: the discounted problem in [22, Section 3.5.2] and the ergodic problem in [8, Sections 3 and 4]. We summarize these for our model.

We first introduce some notation for the limiting diffusion. For $u \in \mathbb{U}$, let $\mathcal{L}_u\colon \mathcal{C}^2(\mathbb{R}^d) \mapsto \mathcal{C}^2(\mathbb{R}^d \times \mathbb{U})$ be the controlled generator of $\hat{X}$ in (2.9), defined by

$$\mathcal{L}_u f(x) \;=\; \sum_{i\in\mathcal{I}} b_i(x,u)\partial_i f(x) + \sum_{i,j\in\mathcal{I}} a_{ij}\partial_{ij} f(x)\,, \tag{4.1}$$

where $a_{ii} \coloneqq \mathbb{1}(\alpha \ge 1)\lambda^\pi_i + \frac{1}{2}\mathbb{1}(\alpha \le 1)\theta_{ii}$, and $a_{ij} \coloneqq \frac{1}{2}\mathbb{1}(\alpha \le 1)\theta_{ij}$ for $i \ne j$. Recall that a control is Markov if $\hat{U}(t) = v(t, \hat{X}(t))$ for a Borel map v on $\mathbb{R}_+ \times \mathbb{R}^d$, and we say that it is stationary Markov if $v\colon \mathbb{R}^d \mapsto \mathbb{U}$. The set of stationary Markov controls is denoted by $\mathfrak{U}_{\mathrm{SM}}$. We extend the definition of b and $\mathcal{R}$ by using the relaxed control framework (see, for example, Section 2.3 in [22]). Without changing the notation, for $v \in \mathfrak{U}_{\mathrm{SM}}$, we replace b_i by

$$b_i(x, v(x)) \;=\; \int_{\mathbb{U}} b_i(x,u)\, v(du\,|\,x)\,, \quad \text{for } i \in \mathcal{I}\,,$$

where $v(du\,|\,x)$ denotes a Borel measurable kernel on $\mathbb{U}$ given x, and replace $\mathcal{R}$ analogously. A control which is a measurable map from $\mathbb{R}^d$ to $\mathbb{U}$ is called a precise control. We say that a control $v \in \mathfrak{U}_{\mathrm{SM}}$ is stable if the controlled diffusion is positive recurrent, and the set of such controls is denoted by $\mathfrak{U}_{\mathrm{SSM}}$. Let $\nu_v \in \mathcal{P}(\mathbb{R}^d)$ denote the unique invariant probability measure of (2.9) under the control $v \in \mathfrak{U}_{\mathrm{SSM}}$. Here, $\mathcal{P}(\mathbb{R}^d)$ denotes the space of Borel probability measures on $\mathbb{R}^d$ under the Prokhorov topology. We define the corresponding ergodic occupation measure $\pi_v \in \mathcal{P}(\mathbb{R}^d \times \mathbb{U})$ by $\pi_v(\mathrm{d}x, \mathrm{d}u) := \nu_v(\mathrm{d}x)v(\mathrm{d}u\,|\,x)$. The set of ergodic occupation measures corresponding to all controls in $\mathfrak{U}_{\mathrm{SSM}}$ is denoted by $\mathcal{G}$, and satisfies

$$\mathcal{G} \;=\; \left\{\pi \in \mathcal{P}(\mathbb{R}^d \times \mathbb{U})\colon \int_{\mathbb{R}^d\times\mathbb{U}} \mathcal{L}_u f(x)\, \pi(\mathrm{d}x, \mathrm{d}u),\ \forall f \in \mathcal{C}_c^\infty(\mathbb{R}^d)\right\}.$$

This characterization of ergodic occupation measures follows by [22, Lemma 3.2.2].

Theorem 4.1. *$\hat{V}_\vartheta$ is the minimal nonnegative solution in $\mathcal{C}^2(\mathbb{R}^d)$ of*

$$\min_{u\in\mathbb{U}} \Big[\mathcal{L}_u \hat{V}_\vartheta(x) + \mathcal{R}(x,u)\Big] \;=\; \vartheta \hat{V}_\vartheta(x)\,.$$

Moreover, $v \in \mathfrak{U}_{\mathrm{SM}}$ is optimal for the ϑ-discounted problem if and only if

$$\langle b(x, v(x)), \nabla \hat{V}_\vartheta(x)\rangle + \mathcal{R}(x, v(x)) \;=\; H(x, \nabla \hat{V}_\vartheta(x))\,,$$

where

$$H(x,p) \;:=\; \min_{u\in\mathbb{U}} \left[\langle b(x,u), p\rangle + \mathcal{R}(x,u)\right].$$

Proof. The result follows directly from Theorem 3.5.6 and Remark 3.5.8 in [22, Section 3.5.2]. ∎

Theorem 4.2. *There exists $\hat{V} \in \mathcal{C}^2(\mathbb{R}^d)$ satisfying*

$$\min_{u\in\mathbb{U}} \Big[\mathcal{L}_u \hat{V}(x) + \mathcal{R}(x,u)\Big] \;=\; \varrho_*\,.$$

Also, $v \in \mathfrak{U}_{\mathrm{SM}}$ is optimal for the ergodic control problem associate with $\mathcal{R}$ if and only if it satisfies

$$\langle b(x, v(x)), \nabla \hat{V}(x)\rangle + \mathcal{R}(x, v(x)) \;=\; H(x, \nabla \hat{V}(x))\,.$$

Moreover, for an optimal $v \in \mathfrak{U}_{\mathrm{SM}}$, it holds that

$$\lim_{T\to\infty} \frac{1}{T}\, \mathbb{E}_x^v \left[\int_0^T \mathcal{R}\big(\hat{X}(s), v(\hat{X}(s))\big)\right] \;=\; \varrho_*\,, \quad \forall x \in \mathbb{R}^d\,.$$

Proof. This follows directly from Theorem 3.4 in [8]. ∎

If we restrict the ergodic control problem in (2.12) to stable stationary Markov controls, then the problem is equivalent to

$$\min_{\pi\in\mathcal{G}} \int_{\mathbb{R}^d\times\mathbb{U}} \mathcal{R}(x,u)\, \pi(\mathrm{d}x, \mathrm{d}u)$$

(see, for example, [22, Section 3.2 and 3.4]). The next theorem shows the existence of an ϵ-optimal control, for any $\epsilon > 0$. This is proved via the spatial truncation technique.

Theorem 4.3. *For any $\epsilon > 0$, there exists a ball B_R with $R = R(\epsilon) > 0$, a continuous precise control $v_\epsilon \in \mathfrak{U}_{\mathrm{SSM}}$ which agrees with e_d on B_R^c, and an associated invariant measure ν_ϵ satisfying*

$$\int_{\mathbb{R}^d} \mathcal{R}(x, v_\epsilon(x))\, \nu_\epsilon(\mathrm{d}x) \;\le\; \varrho_* + \epsilon\,.$$

Proof. This result follows from the proof of claim (5.14) in [8]. □

4.2. **Asymptotic optimality of the discounted cost problem.** In this subsection, we first establish an estimate for $\hat{X}^n$ by using an auxiliary process. Then, following a similar approach as in [7], we prove asymptotic optimality for the discounted problem.

Given the admissible scheduling policy $\hat{U}^n$, let $\breve{X}^n$ be a d-dimensional process defined by

$$\begin{aligned}\breve{X}^n_i(t) \;:=\; \hat{X}^n_i(0) + \hat{\ell}^n_i(t) + \hat{W}^n_i(t) - \int_0^t \bar{\mu}^n_i\big(\breve{X}^n_i(s) - \langle e, \breve{X}^n(s)\rangle^+ \hat{U}^n_i(s)\big)\,\mathrm{d}s \\ - \int_0^t \bar{\gamma}^n_i \langle e, \breve{X}^n(s)\rangle^+ \hat{U}^n_i(s)\,\mathrm{d}s\end{aligned} \tag{4.2}$$

for $i \in \mathcal{I}$, where $\hat{W}^n_i$ is defined in (2.4). Recall the representations in (2.5). $\breve{X}^n$ is a simpler process (compare with $\hat{X}^n$ in (2.3)). Here we replace the state-dependent rates in the last two terms of (2.3) by their averaged version. It is also worth noting that $\breve{X}^n$ can be viewed as a continuous integral mapping with inputs $\hat{X}^n(0)$, $\hat{\ell}^n$, $\hat{W}^n$ and $\hat{U}^n$ (see, for example, [23, Lemma 5.2]), while $\hat{X}^n$ may not have this representation. This auxiliary process $\breve{X}^n$ is useful in showing Theorem 2.1, the proof of which is given in Appendix A, and relies on the following two lemmas.

Lemma 4.1. *As $n \to \infty$, $\breve{X}^n$ and $\hat{X}^n$ are asymptotically equivalent, that is, $\breve{X}^n - \hat{X}^n$ converges to the zero process uniformly on compact sets in probability.*

The proof of Lemma 4.1 is given in Appendix B.

Lemma 4.2. *We have*

$$\mathbb{E}\left[\|\hat{X}^n(t)\|^m\right] \;\le\; C_1(1 + t^{m_0})(1 + \|x\|^{m_0}) \tag{4.3}$$

for some positive constants C_1 and m_0, with m defined in (2.10).

Proof. Recall $\breve{X}^n$ defined in (4.2). For $t \ge 0$, $\breve{X}^n(t) - \langle e, \breve{X}^n(t)\rangle^+ \hat{U}^n(t)$ satisfies the work-conserving condition. Thus, following the same method in [7, Lemma 3], we have

$$\mathbb{E}\left[\|\breve{X}^n(t)\|^m\right] \;\le\; C_2(1 + t^{m_0})(1 + \|x\|^{m_0})$$

for some positive constants C_2 and m_0. As a consequence, (4.3) holds by Lemma 4.1. □

Proof of Theorem 2.2. (Sketch) We first show that

$$\liminf_{n\to\infty}\; \hat{V}^n_\vartheta(\hat{X}^n(0)) \;\ge\; \hat{V}_\vartheta(x)\,. \tag{4.4}$$

Lemma 4.2 corresponds to [7, Lemma 3], and Theorem 2.1 corresponds [7, Lemma 4]. By using Theorem 4.1, we can get the same result as in [7, Proposition 5]. Thus, we can prove (4.4) by following the proof of [7, Theorem 4 (i)].

Next, we show that there exists a sequence of admissible scheduling polices $\hat{U}^n$ which attains optimality (asymptotically). Observe from [7] that the partial derivatives of $\hat{V}_\vartheta$ in Theorem 4.1 up to order two are locally Hölder continuous (see also [22, Lemma 3.5.4]), and the optimal value $\hat{V}_\vartheta$ has polynomial growth. By [7, Theorem 1], there exists an optimal control $v_h \in \mathfrak{U}_{\mathrm{SM}}$ for the discounted problem. Recall ω defined in Definition 3.6. Let

$$\mathfrak{A}^n_h \;:=\; \{x \in \mathbb{R}^d_+ \colon \langle e, x\rangle \le x_i,\ \forall i \in \mathcal{I}\}\,, \quad \text{and} \quad \mathfrak{X}^n_h \;:=\; \{\hat{x}^n(x) \colon x \in \mathfrak{A}^n_h\}\,.$$

Given X^n, we construct a sequence of scheduling policies as follows:

$$Q^n(t) \;:=\; \begin{cases} \omega\big((\langle e, X^n(t)\rangle - n)^+ v_h(\hat{X}^n(t))\big) & \text{for } \hat{X}^n(t) \in \mathfrak{X}^n_h\,, \\ \tilde{z}^n(X^n(t)) & \text{otherwise}\,, \end{cases} \tag{4.5}$$

where $\tilde{z}^n$ is the static priority policy defined in Definition 3.1. Here the value of the scheduling policy outside $\mathfrak{X}_h^n$ is irrelevant for our purpose. For $n \in \mathbb{N}$, let $\bigl(X_h^n, Q_h^n, Z_h^n, \hat{U}_h^n\bigr)$ be a sequence of queueing systems constructed by using (4.5), and K^n be the process defined by

$$K^n(t) \coloneqq \bigl\langle b(\hat{X}_h^n(t), \hat{U}_h^n(t)), \nabla \hat{V}_\vartheta(\hat{X}_h^n(t))\bigr\rangle + \mathcal{R}\bigl(\hat{X}_h^n(t), \hat{U}_h^n(t)\bigr) - H\bigl(\hat{X}_h^n(t), \nabla \hat{V}_\vartheta(\hat{X}_h^n(t))\bigr)\,.$$

It is easy to see that, for any $y \in \mathbb{R}^d$, $|\omega(y) - y| \le 2d$. Then, using Theorem 2.1 and Lemma 4.2, and following the same proof as in [7, Theorem 2 (i)], we have

$$\int_0^{\cdot} \mathrm{e}^{-\vartheta s} K^n(s)\,\mathrm{d}s \;\Rightarrow\; 0\,. \tag{4.6}$$

Note that (4.6) corresponds to the claim (49) in [7]. Then, we follow the method in [7, Theorem 4 (ii)] and obtain that

$$\lim_{n\to\infty} \mathfrak{J}_\vartheta^n\bigl(\hat{X}_h^n(0), \hat{U}_h^n\bigr) \;\le\; \hat{V}_\vartheta(x)\,.$$

This completes the proof. □

4.3. **Proof of Theorem 2.3.** In this section, we prove the asymptotic optimality for the ergodic control problem by establishing the lower and upper bounds in Theorems 4.4 and 4.5, respectively. The techniques used in the proofs differ from previous works. In the proofs of lower and upper bounds for the diffusion-scaled process $\hat{X}^n$ in (2.3), it is essential to analyze the term $\hat{L}^n$, which in the presence of modulation is not a martingale. Thus, the approach in [8] may not be applied directly, since the proofs there rely on the martingale property. So we construct a martingale by adding a process to $\hat{L}^n$, and we establish results for this auxiliary process. Then, we show the same results hold for $\hat{X}^n$ by asymptotic equivalence. On the other hand, in the proof of convergence of mean empirical measures (these are formally defined later), we need to consider the convergence of scaled Markov-modulated rates, while non-modulated systems do not have this issue.

4.3.1. *Proof of the lower bound for the ergodic problem.* We have the following theorem concerning the lower bound.

Theorem 4.4 (lower bound)**.** *It holds that*

$$\liminf_{n\to\infty} \hat{V}^n(\hat{X}^n(0)) \;\ge\; \varrho_*(x)\,.$$

We first assert that $\hat{X}^n$ is a semi-martingale. The proof of the following lemma is given in Appendix B.

Lemma 4.3. *Under any admissible policy Z^n, $\hat{X}^n$ is a semi-martingale with respect to the filtration $\mathbb{F}^n \coloneqq \{\mathcal{F}_t^n \colon t \ge 0\}$, where $\mathcal{F}_t^n$ is defined in Section 2.1.*

Definition 4.1. Define the family of operators $\mathcal{A}_k^n \colon \mathcal{C}^2(\mathbb{R}^d \times \mathbb{U}) \mapsto \mathcal{C}^2(\mathbb{R}^d \times \mathbb{U} \times \mathcal{K})$ by

$$\mathcal{A}_k^n f(x,u) \;\coloneqq\; \sum_{i\in\mathcal{I}} \Bigl(b_i^n(x,u,k)\partial_i f(x) + \frac{1}{2}\sigma_i^n(x,u,k)\partial_{ii} f(x)\Bigr)\,,$$

where the functions $b_i^n, \sigma_i^n \colon \mathbb{R}^d \times \mathbb{U} \times \mathcal{K} \mapsto \mathbb{R}$ are defined by

$$b_i^n(x,u,k) \;\coloneqq\; \ell_i^{n,k} - \mu_i^n(k)(x - \langle e, x\rangle^+ u_i) - \gamma_i^n(k)\langle e, x\rangle^+ u_i\,,$$

with

$$\ell_i^{n,k} \;\coloneqq\; n^{-\beta}\bigl[\bigl(\lambda_i^n(k) - n\lambda_i(k)\bigr) - n\rho_i\bigl(\mu_i^n(k) - \mu_i(k)\bigr)\bigr]\,,$$

and

$$\sigma_i^n(x,u,k) \;\coloneqq\; n^{1-2\beta}\mu_i^n(k)\rho_i + \frac{\lambda_i^n(k)}{n^{2\beta}} + \frac{\mu_i^n(k)(x_i - \langle e, x\rangle^+ u_i) + \gamma_i^n(k)\langle e, x\rangle^+ u_i}{n^\beta}$$

for $i \in \mathcal{I}$ and $k \in \mathcal{K}$, respectively.

Let G^n denote the k-dimensional process

$$G^n_k(t) := \mathbb{1}(J^n(t) = k) - \mathbb{1}(J^n(0) = k)\,, \quad t \geq 0\,,$$

and B^n denote the d-dimensional processes defined by

$$B^n_i(t) := n^{-\alpha/2+\delta_0} \sum_{k\in\mathcal{K}} \bigl(\lambda_i(k) - \rho_i\mu_i(k)\bigr) \left[(G^n(t))'\Upsilon\right]_k\,, \quad t \geq 0\,, \tag{4.7}$$

for $i \in \mathcal{I}$, $k \in \mathcal{K}$, where $\delta_0 := (1-\beta) - \frac{\alpha}{2}$. Then we have the following result, which shows that all the long-run average absolute moments of the diffusion-scaled process are finite. The proof is given in Appendix B.

Lemma 4.4. *Under any sequence of admissible scheduling polices $\{\hat{U}^n : n \in \mathbb{N}\}$ such that $\sup_n \mathfrak{J}^n(\hat{X}^n(0), \hat{U}^n) < \infty$, we have*

$$\sup_n \limsup_{T\to\infty} \frac{1}{T}\,\mathbb{E}^{\hat{U}^n}\left[\int_0^T |\hat{X}^n(s)|^m \,\mathrm{d}s\right] < \infty \tag{4.8}$$

for m defined in (2.10).

Definition 4.2. Define the mean empirical measure $\zeta^n_T \in \mathcal{P}(\mathbb{R}^d \times \mathbb{U})$ associated with $\hat{X}^n$ and $\hat{U}^n$ by

$$\zeta^n_T(A\times B) := \frac{1}{T}\,\mathbb{E}\left[\int_0^T \mathbb{1}_{A\times B}\bigl(\hat{X}^n(s), \hat{U}^n(s)\bigr)\,\mathrm{d}s\right]$$

for any Borel sets $A \subset \mathbb{R}^d$ and $B \subset \mathbb{U}$.

Note that the sequence $\{\zeta^n_T\}$ is tight by Lemma 4.4. The next lemma shows that the sequence $\{\zeta^n_T\}$ converges, along some subsequence, to an ergodic occupation measure associated with the limiting diffusion process under some stationary stable Markov control.

Lemma 4.5. *Suppose under some sequence of admissible scheduling polices $\{\hat{U}^n : n \in \mathbb{N}\}$, (4.8) holds. Then $\uppi$ is in $\mathcal{G}$, where $\uppi \in \mathcal{P}(\mathbb{R}^d \times \mathbb{U})$ is any limit point of ζ^n_T as $(n,T) \to \infty$.*

Proof. We construct a related stochastic process $\tilde{X}^n$ to prove this lemma. Let $\tilde{X}^n$ be the d-dimensional process defined by

$$\tilde{X}^n := \hat{X}^n + B^n\,, \tag{4.9}$$

where B^n is defined in (4.7). Applying Lemma 3.1 in [1] and Lemma 4.3, $\tilde{X}^n$ is also a semi-martingale. We first consider the case with $\alpha \leq 1$. Using the Kunita–Watanabe formula for semi-martingales (see, e.g., [24], Theorem II.33) with $\mathbb{E} = \mathbb{E}^{\hat{U}^n}$, we obtain

$$\begin{aligned}
&\frac{\mathbb{E}\bigl[f(\tilde{X}^n(T)) - f(\tilde{X}^n(0))\bigr]}{T}\\
&\quad= \frac{1}{T}\,\mathbb{E}\left[\int_0^T \sum_{k\in\mathcal{K}} \mathcal{A}^n_k\, f\bigl(\tilde{X}^n(s), \hat{U}^n(s)\bigr)\mathbb{1}(J^n(s) = k)\,\mathrm{d}s\right]\\
&\qquad+ \frac{1}{T}\,\mathbb{E}\left[\sum_{i\in\mathcal{I}} \int_0^T \partial_i f(\tilde{X}^n(s))\,\mathrm{d}\hat{L}^n_i(s)\right] + \frac{1}{T}\,\mathbb{E}\left[\sum_{i\in\mathcal{I}} \int_0^T \partial_i f(\tilde{X}^n(s))\,\mathrm{d}B^n_i(s)\right]\\
&\qquad+ \frac{1}{T}\,\mathbb{E}\left[\sum_{i,i'\in\mathcal{I}} \int_0^T \partial_{ii'} f(\tilde{X}^n(s))\,\mathrm{d}\,[B^n_i, B^n_{i'}](s)\right] + \frac{1}{T}\,\mathbb{E}\Biggl[\sum_{s\leq T} \mathcal{D}f(\tilde{X}^n, s)\Biggr]
\end{aligned} \tag{4.10}$$

for any $f \in \mathcal{C}^\infty_c(\mathbb{R}^d)$, where

$$\mathcal{D}f(\tilde{X}^n, s) := \Delta f(\tilde{X}^n(s)) - \sum_{i\in\mathcal{I}} \partial_i f(\tilde{X}^n(s-))\Delta\tilde{X}^n_i(s) - \frac{1}{2}\sum_{i,i'\in\mathcal{I}} \partial_{ii'} f(\tilde{X}^n(s-))\Delta\tilde{X}^n_i(s)\Delta\tilde{X}^n_{i'}(s)$$

for $s \geq 0$. Using [1, Lemma 3.1], $B_i^n(s) + \hat{L}_i^n(s)$ is a martingale, and hence the sum of the second and third terms of (4.10) is equal to zero. By equation (8) in [1] and the same calculation as in equation (10) of [1], the fourth term on the r.h.s. of (4.10) can be written as

$$\frac{2}{T}\,\mathbb{E}\left[\sum_{i,i'\in\mathcal{I}}\int_0^T \sum_{k\in\mathcal{K}}\sum_{k'\in\mathcal{K}} \bigl(\lambda_i(k)-\rho_i\mu_i(k)\bigr)\bigl(\lambda_{i'}(k')-\rho_{i'}\mu_{i'}(k')\bigr)\Upsilon_{kk'}\partial_{ii'}f(\tilde{X}^n(s))\mathbb{1}(J^n(s)=k)\,\mathrm{d}s\right].$$

Note that for any $f \in \mathcal{C}_c^\infty(\mathbb{R}^d)$,

$$\begin{aligned}
&\limsup_{(n,T)\to\infty}\frac{1}{T}\,\mathbb{E}\left[\int_0^T f(\tilde{X}^n(s),\hat{U}^n(s))\bigl(\mathbb{1}(J^n(s)=k)-\pi_k\bigr)\,\mathrm{d}s\right]\\
&\qquad = \limsup_{(n,T)\to\infty}\frac{1}{T}\,\mathbb{E}\left[\int_0^T n^{-\alpha/2} f(\tilde{X}^n(s),\hat{U}^n(s))\,\mathrm{d}\left(\int_0^s n^{\alpha/2}\bigl(\mathbb{1}(J^n(u)=k)-\pi_k\bigr)\,\mathrm{d}u\right)\right] = 0
\end{aligned}$$

by the boundedness of f, [1, Proposition 3.2] and [3, Theorem 5.2]. Thus, we can replace $\mathbb{1}(J^n(s)=k)$ by π_k for all $k\in\mathcal{K}$ in (4.10), when we let $(n,T)\to\infty$.

We next prove that the last term on the r.h.s. of (4.10) vanishes as $(n,T)\to\infty$. Let

$$\|f\|_{\mathcal{C}^3} := \sup_{x\in\mathbb{R}^d}\Bigl(|f(x)| + \sum_{i,j\in\mathcal{I}}|\partial_{ij}f(x)| + \sum_{i,j,k\in\mathcal{I}}|\partial_{ijk}f(x)|\Bigr).$$

Since the jump size of $\tilde{X}^n$ is of order $n^{-\alpha/2+\delta_0}$ or $n^{-\beta}$, then by Taylor's formula, we have

$$\bigl|\mathcal{D}f(\tilde{X}^n,s)\bigr| \leq \frac{\hat{c}_0\|f\|_{\mathcal{C}^3}}{n^{\alpha/2}}\sum_{i,i'\in\mathcal{I}}\bigl|\Delta\tilde{X}_i^n(s)\Delta\tilde{X}_{i'}^n(s)\bigr| \tag{4.11}$$

for some positive constant $\hat{c}_0$ independent of n. By equation (2) in [1], and the independence of Poisson processes, we obtain

$$\begin{aligned}
&\frac{1}{T}\,\mathbb{E}\Biggl[\sum_{s\leq T}\sum_{i,i'\in\mathcal{I}}\bigl|\Delta\tilde{X}_i^n(s)\Delta\tilde{X}_{i'}^n(s)\bigr|\Biggr]\\
&\qquad = \frac{1}{T}\,\mathbb{E}\Biggl[\int_0^T\sum_{k\in\mathcal{K}}\sum_{k\neq k',k'\in\mathcal{K}}\hat{c}_k\Bigl(q_{kk'}\mathbb{1}(J^n(s)=k)+q_{k'k}\mathbb{1}(J^n(s)=k')\Bigr)\\
&\qquad\qquad + \sum_{i\in\mathcal{I}}\biggl(\frac{\lambda_i(J^n(s))}{n^{2\beta}} + \frac{\mu_i^n(J^n(s))Z_i^n(s)}{n^{2\beta}} + \frac{\gamma_i^n(J^n(s))Q_i^n(s)}{n^{2\beta}}\biggr)\,\mathrm{d}s\Biggr],
\end{aligned} \tag{4.12}$$

where $\{\hat{c}_k : k\in\mathcal{K}\}$ are determined by the constants in (4.7). Using (4.8), the r.h.s. of (4.12) is uniformly bounded over $n\in\mathbb{N}$ and $T>0$. Therefore, by (4.7) and (4.11), the last term on the r.h.s. of (4.10) converges to 0 as $(n,T)\to\infty$.

As in Definition 4.2, let $\tilde{\zeta}_T^n \in \mathcal{P}(\mathbb{R}^d\times\mathbb{U})$ denote the mean empirical measure associated with $\tilde{X}^n$ and $\hat{U}^n$, that is,

$$\tilde{\zeta}_T^n(A\times B) := \frac{1}{T}\,\mathbb{E}\left[\int_0^T \mathbb{1}_{A\times B}\bigl(\tilde{X}^n(s),\hat{U}^n(s)\bigr)\,\mathrm{d}s\right]$$

for any Borel sets $A\subset\mathbb{R}^d$ and $B\subset\mathbb{U}$. Then, by (4.10) and the above analysis, for $f\in\mathcal{C}_c^\infty(\mathbb{R}^d)$, we have

$$\limsup_{(n,T)\to\infty}\int_{\mathbb{R}^d\times\mathbb{U}}\biggl(\sum_{k\in\mathcal{K}}\mathcal{A}_k^n f(x,u)\pi_k + \mathbb{1}(\alpha\leq 1)\sum_{i,i'\in\mathcal{I}}\theta_{ii'}\partial_{ii'}f(x)\biggr)\tilde{\zeta}_T^n(\mathrm{d}x,\mathrm{d}u) = 0, \tag{4.13}$$

where $\{\theta_{ii'} : i,i'\in\mathcal{I}\}$ is defined in (2.8). Note that for each $i\in\mathcal{I}$, the sums $\sum_{k\in\mathcal{K}} b_i^n(x,u,k)\pi_k$ and $\sum_{k\in\mathcal{K}}\sigma_i^n(x,u,k)\pi_k$ converge uniformly over compact sets in $\mathbb{R}^d\times\mathbb{U}$, to b_i (see (2.9)) and $2\mathbb{1}(\alpha\geq 1)\lambda_i^\pi$, respectively.

On the other hand, by the definition of B^n, we have that

$$\sup_t \bigl|\hat{X}^n(t) - \tilde{X}^n(t)\bigr| \,\le\, n^{-\alpha/2}\bar{C}_0 \tag{4.14}$$

for some positive constant $\bar{C}_0$. By (4.8) and (4.14), we deduce that $\bigl\{\tilde{\zeta}^n_T\bigr\}$ is tight. Let (n_l, T_l) be any sequence such that $\tilde{\zeta}^n_T$ converges to $\tilde{\uppi}$, as $(n_l, T_l) \to \infty$. Hence, for any $f \in \mathcal{C}^\infty_c(\mathbb{R}^d)$, we have

$$\int_{\mathbb{R}^d\times\mathbb{U}} \mathcal{L}_u f(x)\,\tilde{\uppi}(\mathrm{d}x \times \mathrm{d}u) = 0 \quad \text{for } \alpha \le 1\,,$$

with $\mathcal{L}_u$ defined in (4.1). Using (4.14), we deduce that ζ^n_T and $\tilde{\zeta}^n_T$ have same limit points. Therefore, as $(n, T) \to \infty$, any limit point $\uppi$ of ζ^n_T satisfies

$$\int_{\mathbb{R}^d\times\mathbb{U}} \mathcal{L}_u f(x)\,\uppi(\mathrm{d}x \times \mathrm{d}u) = 0 \quad \text{for } \alpha \le 1\,.$$

When $\alpha > 1$, the proof is the same as above. This completes the proof. □

Proof of Theorem 4.4. Without loss of generality, suppose $\hat{V}^{n_l}(\hat{X}^{n_l}(0))$ for some increasing sequence $\{n_l\} \subset \mathbb{N}$ converges to a finite value, as $l \to \infty$, and $\hat{U}^{n_l} \in \widehat{\mathfrak{U}}^{n_l}$. By the definitions of $\hat{V}^n$, and the mean empirical measure ζ^n_T in Definition 4.2, there exists a sequence of $\{T_l\} \subset \mathbb{R}_+$ with $T_l \to \infty$, such that

$$\hat{V}^{n_l}(\hat{X}^{n_l}(0)) + \frac{1}{l} \,\ge\, \int_{\mathbb{R}^d\times\mathbb{U}} \mathcal{R}(x,u)\,\zeta^{n_l}_{T_l}(\mathrm{d}x, \mathrm{d}u)\,.$$

By Lemma 4.4 and Lemma 4.5, $\{\zeta^{n_l}_{T_l} : l \in \mathbb{N}\}$ is tight and any limit point of $\zeta^{n_l}_{T_l}$ is in $\mathcal{G}$. Thus

$$\lim_{l\to\infty} \hat{V}^{n_l}(\hat{X}^{n_l}(0)) \,\ge\, \int_{\mathbb{R}^d\times\mathbb{U}} \mathcal{R}(x,u)\,\uppi(\mathrm{d}x, \mathrm{d}u) \,\ge\, \varrho_*\,.$$

This completes the proof. □

4.3.2. *Proof of the upper bound for the ergodic problem.* We have the following theorem concerning the upper bound.

Theorem 4.5 (upper bound)**.** *It holds that*

$$\limsup_{n\to\infty}\, \hat{V}^n(\hat{X}^n(0)) \,\le\, \varrho_*(x)\,.$$

The following lemma is used in the proof of the upper bound. The lemma shows that under a scheduling policy constructed from the ϵ-optimal control given in Theorem 4.3, any limit of the mean empirical measures of the diffusion-scaled queueing processes is the ergodic occupation measure of the limiting diffusion under that control.

Lemma 4.6. *For any fixed $\epsilon > 0$, let $\{\hat{q}^n : n \in \mathbb{N}\}$ be a sequence of maps such that*

$$\hat{q}^n_i[v](\hat{x}) \;=\; \begin{cases} \upomega\bigl(\langle e, n^\beta \hat{x}\rangle^+ v(\hat{x})\bigr) & \text{for} \quad \sup_{i\in\mathcal{I}} |\hat{x}_i| \le \kappa n^{1-\beta}\,, \\ \tilde{q}^n(n^\beta \hat{x} + n\rho) & \text{for} \quad \sup_{i\in\mathcal{I}} |\hat{x}_i| > \kappa n^{1-\beta}\,. \end{cases}$$

with $\tilde{q}^n$ defined in Definition 3.1, κ in Definition 3.7, and $v \equiv v_\epsilon$ in Theorem 4.3. For $\hat{x} \in \mathbb{R}^d$, let $\hat{z}^n[v](\hat{x}) = n^\beta \hat{x} + n\rho - \hat{q}^n[v](\hat{x})$, and

$$u^n[v](\hat{x}) \;\coloneqq\; \begin{cases} \frac{\hat{q}^n[v](\hat{x})}{\langle e, \hat{q}^n[v](\hat{x})\rangle} & \text{if} \quad \langle e, \hat{q}^n[v](\hat{x})\rangle > 0\,, \\ e_d & \text{otherwise}\,. \end{cases}$$

Let $\hat{\zeta}^n_T$ be the mean empirical measure defined by

$$\hat{\zeta}^n_T(A\times B) \;\coloneqq\; \frac{1}{T}\,\mathbb{E}^{Z^n}\left[\int_0^T \mathbb{1}_{A\times B}\bigl(\hat{X}^n(s), u^n[v](\hat{X}^n(s))\bigr)\,\mathrm{d}s\right]$$

for Borel sets $A \subset \mathbb{R}^d$ and $B \subset \mathbb{U}$, where $\hat{X}^n$ is the queueing process under the admissible scheduling policy $Z^n(t) = \hat{z}^n[v](\hat{X}^n(t))$. Let $\pi_v \in \mathcal{P}(\mathbb{R}^d \times \mathbb{U})$ be the ergodic occupation measure of the controlled diffusion in (2.9) under the control v. Then $\hat{\zeta}^n_T$ has a unique limit point π_v as $(n,T) \to \infty$.

Proof. Applying Lemma 3.3, we obtain that $\hat{\zeta}^n_T$ is tight. Recall the definition of $\tilde{X}^n$ in (4.9). Define the mean empirical measure

$$\breve{\zeta}^n_T(A \times B) := \frac{1}{T}\,\mathbb{E}\left[\int_0^T \mathbb{1}_{A\times B}\bigl(\tilde{X}^n(s), u^n[v](\tilde{X}^n(s))\bigr)\,\mathrm{d}s\right]$$

for Borel sets $A \subset \mathbb{R}^d$ and $B \subset \mathbb{U}$. For any $f \in \mathcal{C}^\infty_c(\mathbb{R}^d \times \mathbb{U})$, we have

$$\frac{1}{T}\,\mathbb{E}\left[\int_0^T f\bigl(\tilde{X}^n(s), u^n[v](\tilde{X}^n(s))\bigr)\,\mathrm{d}s\right] = \int_{\mathbb{R}^d\times\mathbb{U}} f(x,u)\,\breve{\zeta}^n_T(\mathrm{d}x,\mathrm{d}u)\,.$$

By (4.14), it is easy to see $\breve{\zeta}^n_T$ is also tight, and $\hat{\zeta}^n_T$ and $\breve{\zeta}^n_T$ have same limits as $(n,T) \to \infty$. Thus, to prove the lemma, it suffices to show that $\breve{\zeta}^n_T$ has the unique limit point π_v as $(n,T) \to \infty$.

Note that

$$\sup_{\hat{x}\in\mathbb{R}^d\cap D} |u^n[v](\hat{x}) - v(\hat{x})| \;\to\; 0 \quad \text{as } n \to \infty \tag{4.15}$$

for any compact set $D \subset \mathbb{R}^d$. Let π^n be any limit point of $\breve{\zeta}^n_T$ as $T \to \infty$. We have

$$\pi^n(\mathrm{d}x,\mathrm{d}u) = \nu^n(\mathrm{d}x)\,\delta_{u^n[v](x)}(u)\,, \quad \text{where } \nu^n(A) = \lim_{T\to\infty}\frac{1}{T}\,\mathbb{E}\left[\int_0^T \mathbb{1}_A(\tilde{X}^n(s))\,\mathrm{d}s\right]$$

for $A \subset \mathbb{R}^d$. By Lemma 4.5, ν^n exists for all n and $\{\nu^n : n \in \mathbb{N}\}$ is tight. We choose an increasing sequence $n \in \mathbb{N}$ such that $\nu^n \to \nu$ in $\mathcal{P}(\mathbb{R}^d)$. For each n, let $\tilde{\mathcal{A}}^n$ be the operator defined by

$$\tilde{\mathcal{A}}^n f(x) = \sum_{k\in\mathcal{K}} \mathcal{A}^n_k f(x, u^n[v](x))\pi_k + \mathbb{1}(\alpha \le 1)\sum_{i,i'\in\mathcal{I}} \theta_{ii'}\partial_{ii'} f(x)\,.$$

Recall $\mathcal{L}_v$ defined in (4.1) for $v \in \mathfrak{U}_{\mathrm{SM}}$. Therefore, we have

$$\int_{\mathbb{R}^d} \tilde{\mathcal{A}}^n f\,\mathrm{d}\nu^n - \int_{\mathbb{R}^d} \mathcal{L}_v f\,\mathrm{d}\nu = \int_{\mathbb{R}^d}\bigl(\tilde{\mathcal{A}}^n f - \mathcal{L}_v f\bigr)\,\mathrm{d}\nu^n + \int_{\mathbb{R}^d}\mathcal{L}_v f\,\bigl(\mathrm{d}\nu^n - \mathrm{d}\nu\bigr)\,. \tag{4.16}$$

By (4.15) and the convergence of $\tilde{\mathcal{A}}^n$ in (4.13), we have $\tilde{\mathcal{A}}^n f \to \mathcal{L}_v f$ uniformly as $n \to \infty$; thus the first term on the r.h.s. of (4.16) converges to 0. By the convergence of ν^n, the second term of (4.16) also converges to 0. Applying Lemma 4.5, it holds that, for any $f \in \mathcal{C}^\infty_c(\mathbb{R}^d \times \mathbb{U})$,

$$\int_{\mathbb{R}^d} \tilde{\mathcal{A}}^n f\,\mathrm{d}\nu^n \;\to\; 0 \quad \text{as} \quad n \to \infty\,.$$

Therefore,

$$\int_{\mathbb{R}^d} \mathcal{L}_v f\,\mathrm{d}\nu = 0\,, \quad \forall f \in \mathcal{C}^\infty_c(\mathbb{R}^d \times \mathbb{U})\,,$$

which implies that ν is the invariant measure of $\hat{X}$ defined in (2.9) under the control v. By (4.15), we obtain $\delta_{u^n[v](\cdot)}(u) \to \delta_{v(\cdot)}(u)$ in the topology of Markov controls. Define the ergodic occupation measure $\pi_v \in \mathcal{P}(\mathbb{R}^d \times \mathbb{U})$ by $\pi_v(\mathrm{d}x,\mathrm{d}u) := \nu(\mathrm{d}x)\delta_{v(x)}(u)$. Then, for $g \in \mathcal{C}^\infty_c(\mathbb{R}^d \times \mathbb{U})$, we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d\times\mathbb{U}} g(x,u)\bigl(\pi_v(\mathrm{d}x,\mathrm{d}u) - \pi^n(\mathrm{d}x,\mathrm{d}u)\bigr)\right| \\
&\qquad\le \left|\int_{\mathbb{U}}\left(\int_{\mathbb{R}^d} g(x,u)\bigl(\nu(\mathrm{d}x) - \nu^n(\mathrm{d}x)\bigr)\right)\delta_{u^n[v](x)}(u)\right| \\
&\qquad\qquad + \left|\int_{\mathbb{U}}\left(\int_{\mathbb{R}^d} g(x,u)\nu(\mathrm{d}x)\right)\bigl(\delta_{u^n[v](x)}(u) - \delta_{v(x)}(u)\bigr)\right|\,.
\end{aligned} \tag{4.17}$$

By the convergence of ν^n, the first term of (4.17) converges to 0 as $n \to \infty$. Since ν has a continuous density, then applying [22, Lemma 2.4.1], we deduce that the second term of (4.17) converges to 0 as $n \to \infty$. Thus, $\pi^n \to \pi_v$ in $\mathcal{P}(\mathbb{R}^d \times \mathbb{U})$. This completes the proof. □

Proof of Theorem 4.5. Let $\tilde{m} = 2m$ with m defined in (2.10). Let Z^n be a scheduling policy such that $Z^n(t) = \hat{z}^n[v_\epsilon](\hat{X}^n(t))$ with v_ϵ (together with a positive constant $R(\epsilon)$) defined in Theorem 4.3 and $\hat{z}^n$ defined in Lemma 4.6. Note that

$$\int_{\mathbb{R}^d \times \mathbb{U}} \mathcal{R}(x, u)\, \pi_{v_\epsilon}(\mathrm{d}x, \mathrm{d}u) \;\le\; \varrho_* + \epsilon\,,$$

where $\pi_{v_\epsilon} \in \mathcal{P}(\mathbb{R}^d \times \mathbb{U})$ is the ergodic occupation measure defined by $\pi_{v_\epsilon}(\mathrm{d}x, \mathrm{d}u) := \nu_\epsilon(\mathrm{d}x)\delta_{v_\epsilon(x)}(u)$. Let $z^n(x) = \hat{z}^n[v_\epsilon](\hat{x}^n(x))$ for $x \in \mathbb{Z}^d_+$, and $c_0 \equiv R(\epsilon)$ in Definition 3.2. Then, by Lemma 3.3, there exits $\hat{n}_0 \in \mathbb{N}$ such that

$$\widehat{\mathcal{L}}_n^{z^n} \widehat{\mathcal{V}}_{\tilde{m},\xi}(\hat{x}, k) \;\le\; C_1 - C_2 \widehat{\mathcal{V}}_{\tilde{m},\xi}(\hat{x}, k) \quad \forall (\hat{x}, k) \in \mathfrak{X}^n \times \mathcal{K}\,, \quad \forall n \ge \hat{n}_0\,, \tag{4.18}$$

for some positive constants C_1 and C_2. Using (4.18), we can select a sequence of $\{T_n : n \in \mathbb{N}\}$ such that $T_n \to \infty$ as $n \to \infty$, and

$$\sup_{n \ge \hat{n}_0} \sup_{T \ge T_n} \int_{\mathbb{R}^d \times \mathbb{U}} \widehat{\mathcal{V}}_{\tilde{m},\xi}(\hat{x}, k)\, \hat{\zeta}^n_T(\mathrm{d}\hat{x}, \mathrm{d}u) \;<\; \infty\,.$$

It follows that $\widetilde{\mathcal{R}}(x - \hat{z}^n[v_\epsilon](x))$ is uniformly integrable. Moreover, by Lemma 4.6, $\hat{\zeta}^n_T$ converges in distribution to π_{v_ϵ}. This completes the proof. □

Appendix A. Proofs of Theorem 2.1 and Lemma 3.1

Proof of Theorem 2.1. To prove (i), we fix $\beta = 1/2$, and first show that $\hat{X}^n$ is stochastically bounded (see Definition 5.4 in [25]). Recall the definition of $\hat{X}^n$ in (2.3). By (2.6) and (2.7), $\{\hat{\ell}^n_i + \hat{L}^n_i : n \in \mathbb{N}\}$ is stochastically bounded in $(\mathbb{D}, \mathcal{J})$. The predictable quadratic variation processes of $\hat{S}^n_i$ and $\hat{R}^n_i$ are defined by

$$\langle \hat{S}^n_i \rangle(t) \;:=\; \int_0^t \mu^n_i(J^n(s)) \bar{Z}^n_i(s)\, \mathrm{d}s\,, \quad \langle \hat{R}^n_i \rangle(t) \;:=\; \int_0^t \gamma^n_i(J^n(s)) \bar{Q}^n_i(s)\, \mathrm{d}s\,,$$

respectively. By (2.2), we have the crude inequality

$$0 \;\le\; \bar{X}^n_i(t) \;\le\; \bar{X}^n_i(0) + n^{-1} A^n_i(t)\,,$$

and thus, by (2.1), the analogous inequalities hold for $\bar{Z}^n_i$ and $\bar{Q}^n_i$. Thus, applying Lemma 5.8 in [25] together with (2.7), we deduce that $\{(\hat{S}^n_i, \hat{R}^n_i) : n \in \mathbb{N}\}$ is stochastically bounded in $(\mathbb{D}, \mathcal{J})^2$, and thus $\{\hat{W}^n_i : n \in \mathbb{N}\}$ is stochastically bounded. For each $u \in \mathbb{U}$, the map

$$x \;\mapsto\; c_1\bigl(x - \langle e, x\rangle^+ u\bigr) + c_2 \langle e, x\rangle^+ u$$

has the Lipschitz property, where c_1 and c_2 are some positive constants. Then, by Assumption 2.1, we obtain

$$\|\hat{X}^n(t)\| \;\le\; \|\hat{X}^n(0)\| + \|\hat{W}^n(t)\| + C \int_0^t \bigl(1 + \|\hat{X}^n(s)\|\bigr)\, \mathrm{d}s$$

for $t \ge 0$ and some constant C. Therefore, applying Gronwall's inequality, and using the assumption on $\hat{X}^n(0)$ and Lemma 5.3 in [25], it follows that $\{\hat{X}^n : n \in \mathbb{N}\}$ is stochastically bounded in $(\mathbb{D}^d, \mathcal{J})$. Then, applying the functional weak law of large numbers (Lemma 5.9 in [25]), we have

$$\frac{\hat{X}^n}{\sqrt{n}} \;=\; \bar{X}^n - \rho \;\Rightarrow\; 0 \quad \text{in} \quad (\mathbb{D}^d, \mathcal{J}) \quad \text{as} \quad n \to \infty\,,$$

for $t \geq 0$. This implies $\bar{X}^n \Rightarrow \rho$ in $(\mathbb{D}^d, \mathcal{J})$ as $n \to \infty$. By (2.1) and Assumption 2.2, we have $\langle e, \bar{Q}^n \rangle = (\langle e, \bar{X}^n \rangle - 1)^+ \Rightarrow 0$ in $(\mathbb{D}, \mathcal{J})$ as $n \to \infty$. Since $\bar{Q}^n \geq 0$, it follows that $\bar{Q}^n \Rightarrow 0$ and $\bar{Z}^n \Rightarrow \rho$, both in $(\mathbb{D}^d, \mathcal{J})$ as $n \to \infty$.

We next prove (ii). For $i \in \mathcal{I}$ and $t \geq 0$, $\hat{A}_i^n$ can be written as

$$\hat{A}_i^n(t) \,=\, n^{-\beta} \left(A_{*,i}^n \left(n \sum_{k \in \mathcal{K}} \frac{\lambda_i^n(k)}{n} \int_0^t \mathbb{1}(J^n(s) = k)\, \mathrm{d}s \right) - n \sum_{k \in \mathcal{K}} \frac{\lambda_i^n(k)}{n} \int_0^t \mathbb{1}(J^n(s) = k)\, \mathrm{d}s \right).$$

By [3, Theorem 5.1] and Assumption 2.1, we have

$$\sum_{k \in \mathcal{K}} \frac{\lambda_i^n(k)}{n} \int_0^\cdot \mathbb{1}(J^n(s) = k)\, \mathrm{d}s \xrightarrow{\text{u.c.p.}} \lambda_i^\pi \mathfrak{e}(\cdot)\,, \quad \text{as} \quad n \to \infty\,,$$

for $i \in \mathcal{I}$, $\xrightarrow{\text{u.c.p.}}$ denotes uniform convergence on compact sets in probability, and $\mathfrak{e}(t) := t$ for all $t \geq 0$. Thus, by the FCLT of Poisson martingales and a random change of time (see, for example, [26, Page 151]), we have

$$\hat{A}^n \;\Rightarrow\; \mathbb{1}(\alpha \geq 1) \frac{\Lambda}{\sqrt{2}} W_1 \quad \text{in} \quad (\mathbb{D}^d, \mathcal{J}) \quad \text{as} \quad n \to \infty\,,$$

where W_1 is a d-dimensional standard Brownian motion. Similarly, applying Theorem 2.1 (i), [3, Theorem 5.1] and Assumption 2.1,

$$\begin{aligned}\sum_{k \in \mathcal{K}} \mu_i^n(k) \int_0^\cdot \bar{Z}^n(s) \mathbb{1}(J^n(s) = k)\, \mathrm{d}s \,&=\, \sum_{k \in \mathcal{K}} \mu_i^n(k) \int_0^\cdot (\bar{Z}^n(s) - \rho_i) \mathbb{1}(J^n(s) = k)\, \mathrm{d}s \\ &\quad + \sum_{k \in \mathcal{K}} \mu_i^n(k) \rho_i \int_0^\cdot \mathbb{1}(J^n(s) = k)\, \mathrm{d}s \xrightarrow{\text{u.c.p.}} \lambda_i^\pi \mathfrak{e}(\cdot)\,, \quad \text{as} \quad n \to \infty\,,\end{aligned}$$

and

$$\sum_{k \in \mathcal{K}} \gamma_i^n(k) \int_0^\cdot \bar{Q}^n(s) \mathbb{1}(J^n(s) = k)\, \mathrm{d}s \xrightarrow{\text{u.c.p.}} 0\,, \quad \text{as} \quad n \to \infty\,,$$

for $i \in \mathcal{I}$ and $t \geq 0$. Thus, we obtain

$$\hat{S}^n \;\Rightarrow\; \mathbb{1}(\alpha \geq 1) \frac{\Lambda}{\sqrt{2}} W_2 \quad \text{in} \quad (\mathbb{D}^d, \mathcal{J}) \quad \text{as} \quad n \to \infty\,,$$

with a d-dimensional standard Brownian motion W_2, and

$$\hat{R}^n \;\Rightarrow\; 0 \quad \text{in} \quad (\mathbb{D}^d, \mathcal{J}) \quad \text{as} \quad n \to \infty\,.$$

Since the Poisson processes are independent and the random time changes converge to deterministic functions, the joint weak convergence of $(\hat{L}^n, \hat{A}^n, \hat{S}^n, \hat{R}^n)$ holds. Note that $\widetilde{W}$, W_1 and W_2 are independent, and thus

$$\hat{W}^n \;\Rightarrow\; \hat{W} \quad \text{in} \quad (\mathbb{D}^d, \mathcal{J}) \quad \text{as} \quad n \to \infty\,.$$

This completes the proof of (ii).

It is easy to see that $\hat{\ell}_i^n$, $\mu_i^n(k)$ and $\gamma_i^n(k)$ are uniformly bounded in i, k and n. The rest of the proof of (iii) is same as [7, Lemma 4(iii)].

Finally, we prove (iv). Note that $\hat{U}^n$ may not have a limit in the space $\mathbb{D}^d$. So to establish the weak limit, we need to assume $\hat{U}^n$ is tight in $\mathbb{D}^d$. By the representation of $\breve{X}^n$ in (4.2) together with Theorem 2.1 (ii) , and the continuity of the integral representation (see [25, Theorem 4.1] for one-dimension and [3, Lemma 4.1] in the multi-dimensional case), any limit of $\breve{X}^n$ is a unique strong solution of (2.9). Applying Lemma 4.1, we deduce that the limit $\hat{X}$ of $\hat{X}^n$ is also a strong solution of (2.9).

Recall that $\tau^n(t)$ is defined in Definition 2.1. For $r \ge 0$, we observe that

$$\begin{aligned}\hat{W}_i^n(t+r) - \hat{W}_i^n(t) &= \hat{W}_i^n(\tau^n(t)+r) - \hat{W}_i^n(\tau^n(t)) \\ &\quad + \hat{W}_i^n(t+r) - \hat{W}_i^n(\tau^n(t)+r) + \hat{W}_i^n(t) - \hat{W}_i^n(\tau^n(t))\,.\end{aligned}$$

It is easy to see that as $n \to \infty$, $\tau^n(t) \Rightarrow t$. By the random change of time lemma in [26, Page 151], we have

$$\hat{W}_i^n(t+r) - \hat{W}_i^n(\tau^n(t)+r) + \hat{W}_i^n(t) - \hat{W}_i^n(\tau^n(t)) \;\Rightarrow\; 0 \quad \text{in} \quad \mathbb{R}\,,$$

and thus

$$\hat{W}_i^n(\tau^n(t)+r) - \hat{W}_i^n(\tau^n(t)) \;\Rightarrow\; \hat{W}_i(t+r) - \hat{W}_i(t) \quad \text{in} \quad \mathbb{R}\,.$$

Thus, by Definition 2.1, and following the proof of Lemma 6 in [7], we deduce that $\hat{U}^n$ is non-anticipative. □

Proof of Lemma 3.1. Note that

$$(a \pm 1)^m - a^m \;=\; \pm m a^{m-1} + \mathcal{O}(a^{m-2})\,, \quad a \in \mathbb{R}\,.$$

Recall the definition of $\tilde{x}^n$ in Definition 3.2. We obtain

$$\begin{aligned}\bar{\mathcal{L}}_n^{\tilde{z}_n} f_n(x) \;=\; &\sum_{i\in\mathcal{I}} \xi_i \Big(\bar{\lambda}_i^n \big(m\tilde{x}_i^n |\tilde{x}_i^n|^{m-2} + \mathcal{O}(|\tilde{x}_i^n|^{m-2})\big) \\ &+ \bar{\mu}_i^n \tilde{z}_i^n \big(-m\tilde{x}_i^n |\tilde{x}_i^n|^{m-2} + \mathcal{O}(|\tilde{x}_i^n|^{m-2})\big) + \bar{\gamma}_i^n \tilde{q}_i^n \big(-m\tilde{x}_i^n |\tilde{x}_i^n|^{m-2} + \mathcal{O}(|\tilde{x}_i^n|^{m-2})\big)\Big)\,.\end{aligned}$$

Let

$$\bar{F}_n^{(1)}(x) \;\coloneqq\; \sum_{i\in\mathcal{I}} \xi_i \Big(\bar{\lambda}_i^n + \bar{\mu}_i^n \tilde{z}_i^n + \bar{\gamma}_i^n \tilde{q}_i^n\Big)\mathcal{O}(|\tilde{x}_i^n|^{m-2})\,,$$

and

$$\bar{F}_n^{(2)}(x) \;\coloneqq\; \sum_{i\in\mathcal{I}} \xi_i \Big(\bar{\lambda}_i^n - \bar{\mu}_i^n \tilde{z}_i^n - \bar{\gamma}_i^n \tilde{q}_i^n\Big) m\tilde{x}_i^n |\tilde{x}_i^n|^{m-2}\,.$$

It is easy to see that

$$\bar{\mathcal{L}}_n^{\tilde{z}_n} f_n(x) \;=\; \bar{F}_n^{(1)}(x) + \bar{F}_n^{(2)}(x)\,.$$

From Definition 3.1 and Assumption 2.1, we have

$$\begin{aligned}\bar{F}_n^{(1)}(x) \;&\le\; \sum_{i\in\mathcal{I}} \xi_i \Big(\bar{\lambda}_i^n + \bar{\mu}_i^n x_i + \bar{\gamma}_i^n x_i\Big)\mathcal{O}(|\tilde{x}_i^n|^{m-2}) \\ &=\; \sum_{i\in\mathcal{I}} \xi_i \Big(\bar{\lambda}_i^n + \bar{\mu}_i^n (\tilde{x}_i^n + n\rho_i) + \bar{\gamma}_i^n (\tilde{x}_i^n + n\rho_i)\Big)\mathcal{O}(|\tilde{x}_i^n|^{m-2}) \\ &\le\; \sum_{i\in\mathcal{I}} \Big(\mathcal{O}(n)\mathcal{O}(|\tilde{x}_i^n|^{m-2}) + \mathcal{O}(|\tilde{x}_i^n|^{m-1})\Big)\,.\end{aligned} \tag{A.1}$$

Next, we consider $\bar{F}_n^{(2)}(x)$. By using the balance equation $\tilde{z}_i^n = \tilde{x}_i^n - \tilde{q}_i^n + \rho_i^n n$, we obtain

$$\bar{F}_n^{(2)}(x) \;=\; \sum_{i\in\mathcal{I}} \xi_i \Big(-\bar{\mu}_i^n \tilde{x}_i^n + \bar{\lambda}_i^n - \bar{\mu}_i^n \rho_i n - (\bar{\gamma}_i^n - \bar{\mu}_i^n)\tilde{q}_i^n\Big) m\tilde{x}_i^n |\tilde{x}_i^n|^{m-2}\,.$$

By Assumption 2.1, we have

$$\bar{\lambda}_i^n - \bar{\mu}_i^n \rho_i n = \mathcal{O}(n^\beta)\,. \tag{A.2}$$

Let $\breve{c}_1 \coloneqq \sup_{i,k,n} |\gamma_i^n(k) - \mu_i^n(k)|$, and $\breve{c}_2$ be some positive constant such that

$$\inf_{i\in\mathcal{I},k\in\mathcal{I},n\in\mathbb{N}} \big\{\mu_i^n(k), \gamma_i^n(k)\big\} \;\ge\; \breve{c}_2 \;>\; 0\,.$$

We choose

$$\xi_1 \;=\; 1\,, \quad \text{and} \quad \xi_i \;=\; \frac{\epsilon_1^m}{d^m} \min_{i'\le i-1} \xi_{i'} \quad \text{for } i \ge 2\,,$$

where $\epsilon_1 := \frac{\breve{c}_1}{8\breve{c}_2}$. Then, by using (3.13) and (A.2), we obtain

$$\begin{aligned}\bar{F}_n^{(2)}(x) \,&\le\, \sum_{i\in\mathcal{I}}\Big[-m\xi_i\big((1-\eta_i(x))\bar{\mu}_i^n + \eta_i(x)\bar{\gamma}_i^n\big)|\tilde{x}_i^n|^m \\ &\qquad\qquad + \xi_i\Big(\mathcal{O}(n^\beta) - (\bar{\gamma}_i^n - \bar{\mu}_i^n)\bar{\eta}_i(x)\sum_{j=1}^{i-1}\tilde{x}_j^n\Big)m\tilde{x}_i^n|\tilde{x}_i^n|^{m-2}\Big] \\ &\le \sum_{i\in\mathcal{I}}\xi_i\mathcal{O}(n^\beta)(\tilde{x}_i^n)^{m-1} - \frac{3m\breve{c}_2}{4}\xi_i|\tilde{x}_i^n|^m\,,\end{aligned} \tag{A.3}$$

where the proof for the second inequality of (A.3) is same as the proof for the claim (5.12) in [8]. Using Young's inequality and since $\beta \ge 1/2$, we have

$$\begin{aligned}\mathcal{O}(n)\mathcal{O}(|\tilde{x}_i^n|^{m-2}) \,&\le\, \epsilon\big(\mathcal{O}(|\tilde{x}_i^n|^{m-2})\big)^{m/m-2} + \epsilon^{1-m/2}\big(\mathcal{O}(n)\big)^{m\beta} \\ \mathcal{O}(n^\beta)\mathcal{O}(|\tilde{x}_i^n|^{m-1}) \,&\le\, \epsilon\big(\mathcal{O}(|\tilde{x}_i^n|^{m-1})\big)^{m/m-1} + \epsilon^{1-m}\big(\mathcal{O}(n^\beta)\big)^{m}\end{aligned} \tag{A.4}$$

for any $\epsilon > 0$. Therefore, by (A.1), (A.3), and (A.4), we have

$$\bar{\mathcal{L}}_n^{\tilde{z}^n} f_n(x) \,\le\, C_1 n^{m\beta} - C_2 f_n(x)\,, \quad \forall x \in \mathbb{Z}_+^d\,.$$

This completes the proof. ☐

Appendix B. Proofs of Lemma 4.1, Lemma 4.3, and Lemma 4.4

Proof of Lemma 4.1. For $i \in \mathcal{I}$ and $t \ge 0$, we have

$$\begin{aligned}\hat{X}_i^n(t) - \breve{X}_i^n(t) \,=\, &-\int_0^t \big(\mu_i^n(J^n(s)) - \bar{\mu}_i^n\big)\hat{X}_i^n(s)\,\mathrm{d}s + \int_0^t \bar{\mu}_i^n\big(\breve{X}_i^n(s) - \hat{X}_i^n(s)\big)\,\mathrm{d}s \\ &+ \int_0^t (\mu_i^n(J^n(s)) - \bar{\mu}_i^n - \gamma_i^n(J^n(s)) + \bar{\gamma}_i^n)\langle e, \hat{X}^n(s)\rangle^+ \hat{U}_i^n(s)\,\mathrm{d}s \\ &- (\bar{\mu}_i^n - \bar{\gamma}_i^n)\int_0^t \big(\langle e, \breve{X}^n(s)\rangle^+ - \langle e, \hat{X}^n(s)\rangle^+\big)\hat{U}_i^n(s)\,\mathrm{d}s\,.\end{aligned} \tag{B.1}$$

For any $a, b \in \mathbb{R}$, $a^+ - b^+ = \eta(a-b)$ with $\eta \in [0,1]$. Then, the last term of (B.1) can be written as

$$\int_0^t \big(\langle e, \breve{X}^n(s)\rangle^+ - \langle e, \hat{X}^n(s)\rangle^+\big)\,\hat{U}_i^n(s)\,\mathrm{d}s \,=\, \int_0^t \tilde{\eta}(\breve{X}^n(s), \hat{X}^n(s))\big\langle e, \breve{X}^n(s) - \hat{X}^n(s)\big\rangle \hat{U}_i^n(s)\,\mathrm{d}s\,,$$

where $\tilde{\eta}(x,y)\colon (x,y) \in \mathbb{R}^2 \mapsto [0,1]$. Note that $\hat{U}_i^n(t) \in [0,1]$ for all $i \in \mathcal{I}$ and $t \ge 0$. By the continuous integral mapping ([23, Lemma 5.2]), if the first and third terms of (B.1) converge to the zero process uniformly on compact sets in probability, then $\hat{X}^n - \breve{X}^n$ must converge to the zero process uniformly on compact sets in probability. The first term of (B.1) can be written as

$$-\sum_{k\in\mathcal{K}}\mu_i^n(k)\int_0^t n^{-\alpha/2}\hat{X}_i^n(s)\,\mathrm{d}\Big(n^{\alpha/2}\int_0^s \big(\mathbb{1}(J^n(u) = k) - \pi_k\big)\,\mathrm{d}u\Big)\,.$$

Note that $1 - \beta - \alpha = \min\{0, (1-\alpha)/2\}$. Applying Theorem 2.1 (i), we have $n^{-\alpha/2}\hat{X}_i^n$ converges to the zero process uniformly on compact sets in probability. Similarly, since $\hat{U}_i^n(s)$ is bounded, by Theorem 2.1 (i), we obtain that $n^{-\alpha/2}\langle e, \hat{X}^n(s)\rangle^+)\hat{U}_i^n(s)$ converges to the zero process uniformly on compact sets in probability. Then, the asymptotic equivalence of $\breve{X}^n$ and $\hat{X}^n$ follows as in the proof in [3, Lemma 4.4]. ☐

Proof of Lemma 4.3. For $n \in \mathbb{N}$ and $i \in \mathcal{I}$, define the processes $M^n_{S_i} = \{M^n_{S_i}(t)\colon t \ge 0\}$ and $M^n_{R_i} = \{M^n_{R_i}(t)\colon t \ge 0\}$ by

$$M^n_{S_i}(t) := S^n_{*,i}(t) - t\,, \quad \text{and} \quad M^n_{R_i} := R^n_{*,i}(t) - t\,,$$

respectively. It is obvious that $M^n_{S_i}$ and $M^n_{R_i}$ are square integrable martingales with respect to the filtration generated by the processes $S^n_{*,i}$ and $R^n_{*,i}$. Define the d-dimensional processes τ^n_1 and τ^n_2 by

$$\tau^n_{1,i}(t) := \int_0^t \mu^n_i(J^n(s))Z^n_i(s)\,\mathrm{d}s\,, \quad \text{and} \quad \tau^n_{2,i}(t) := \int_0^t \gamma^n_i(J^n(s))Q^n_i(s)\,\mathrm{d}s\,,$$

respectively. It is easy to see that $\{\tau^n_{j,i}\colon i \in \mathcal{I}, j \in \{1,2\}\}$ have continuous nondecreasing nonnegative sample paths. For $x_1 \in \mathbb{R}^d_+$ and $x_2 \in \mathbb{R}^d_+$, we obtain

$$(\tau^n_1(t) \le x_1, \tau^n_2(t) \le x_2) \in \mathcal{H}^n(x_1, x_2)\,,$$

where

$$\mathcal{H}^n(x_1,x_2) := \{S^n_i(s_{1,i}), R^n_i(s_{2,i}), X^n_i(0)\colon i \in \mathcal{I}, s_1 \le x_1, s_2 \le x_2\} \\ \vee \sigma\{A^n_i(s), J^n(s), Z^n_i(s)\colon s \ge 0, i \in \mathcal{I}\} \vee \mathcal{N}.$$

This implies that $(\tau^n_1(t), \tau^n_2(t))$ is $\mathbb{H}^n$-stopping time, where $\mathbb{H}^n := \{\mathcal{H}^n(x_1,x_2)\colon x_1 \in \mathbb{R}^d_+, x_2 \in \mathbb{R}^d_+\}$. Since $X^n_i(t) \le X^n_i(0) + A^n_i(t)$ for $i \in \mathcal{I}$, we observe that

$$\mathbb{E}\left[\tau^n_{1,i}(t)\right] \le \max_k\{\mu_i(k)\}t(X^n_i(0) + \mathbb{E}[A^n_i(t)] + n) < \infty\,,$$
$$\mathbb{E}\left[S^n_{*,i}(\tau^n_{1,i}(t))\right] \le \max_k\{\mu_i(k)\}t(X^n_i(0) + \mathbb{E}[A^n_i(t)] + n) < \infty\,.$$

Similarly, we have that $\mathbb{E}[\tau^n_{2,i}(t)]$ and $\mathbb{E}[R^n_{*,i}(\tau^n_{2,i}(t))]$ are finite. Thus, applying Lemma 3.2 in [25] and Theorem 8.7 on page 87 of [27], and using the decomposition in (2.3) and Lemma 3.1 in [3], we conclude that $\hat{X}^n$ is a semi-martingale with respect to the filtration $\widetilde{\mathbb{F}}^n := \{\widetilde{\mathcal{F}}^n_t\colon t \ge 0\}$, where

$$\widetilde{\mathcal{F}}^n_t := \sigma\{S^n_i(s), R^n_i(s), X^n_i(0)\colon i \in \mathcal{I}, s \le t\} \vee \sigma\{A^n_i(s), J^n(s), Z^n_i(s)\colon i \in \mathcal{I}, s \ge 0\} \vee \mathcal{N}\,,$$

and $\mathcal{N}$ is a collection of $\mathbb{P}$-null sets. Since the processes $A^n(t)$, $J^n(t)$ and $Z^n(t)$ are adapted to $\mathcal{F}^n_t$, we can replace $\widetilde{\mathbb{F}}^n$ by the smaller filtration $\mathbb{F}^n$. This completes the proof. □

Proof of Lemma 4.4. Define the function $g \in \mathcal{C}^2(\mathbb{R}^d)$ by $g(x) := \sum_{i\in\mathcal{I}} g_i(x_i)$ with $g_i \in \mathcal{C}^2(\mathbb{R})$ defined by $g_i(x) = |x|^m$ for $|x| \ge 1$ and $i \in \mathcal{I}$. Recall $\mathcal{A}^n_k$ defined in Definition 4.1. Applying the Kunita–Watanabe formula to $\tilde{X}^n$ with $\mathbb{E} = \mathbb{E}^{\hat{U}^n}$ and the fact $\hat{L}^n_i + B^n_i$ is a martingale, we have

$$\begin{aligned}\mathbb{E}\bigl[g(\tilde{X}^n(t))\bigr] &= \mathbb{E}\bigl[g(\tilde{X}^n(0))\bigr] + \sum_{k\in\mathcal{K}} \mathbb{E}\left[\int_0^t \mathcal{A}^n_k\, g\bigl(\tilde{X}^n(s), \hat{U}^n(s)\bigr)\mathbb{1}(J^n(s) = k)\,\mathrm{d}s\right] \\ &\quad + \mathbb{E}\left[\sum_{i,i'\in\mathcal{I}} \int_0^t \partial_{ii'} g(\tilde{X}^n(s))\,\mathrm{d}\,[B^n_i, B^n_{i'}](s)\right] + \mathbb{E}\left[\sum_{s\le t} \mathcal{D}g(\tilde{X}^n, s)\right]\end{aligned} \tag{B.2}$$

for $t \ge 0$, where $\mathcal{D}g(\tilde{X}^n, s)$ is defined analogously to (4.10). By Assumption 2.1 and Young's inequality, we have

$$b^n_i(x,u,k)g'_i(x) \le \frac{\bar{c}_1}{d}\bigl(1 + (\langle e, x\rangle^+)^m\bigr) - \frac{\bar{c}_2}{d}|x|^m\,,$$
$$\sigma^n_i(x,u,k)g''_i(x) \le \frac{2\bar{c}_1}{d}\bigl(1 + (\langle e, x\rangle^+)^m\bigr) + \frac{\bar{c}_2}{4d}|x|^m\,,$$

and thus, for all $k \in \mathcal{K}$, we obtain

$$\mathcal{A}^n_k\, g(x,u) \le 2\bar{c}_1\bigl(1 + (\langle e, x\rangle^+)^m\bigr) - \frac{7}{8}\bar{c}_2|x|^m\,, \tag{B.3}$$

where $\bar{c}_1$ and $\bar{c}_2$ are some positive constants independent of n. Following the same analysis for the fourth term on the r.h.s. of (4.10), and using Young's inequality, we have

$$\begin{aligned}\mathbb{E}\left[\sum_{i,i'\in\mathcal{I}}\int_0^t \partial_{ii'}g(\tilde{X}^n(s))\,\mathrm{d}\,[B_i^n,B_{i'}^n](s)\right] \\ \le\ \mathbb{E}\left[\int_0^t \bar{c}_1\bigl(1+(\langle e,\tilde{X}^n(s)\rangle^+)^m\bigr)+\frac{\bar{c}_2}{8}|\tilde{X}^n(s)|^m\,\mathrm{d}s\right].\end{aligned}\tag{B.4}$$

Since the jump size is of order $n^{-\beta}$ or $n^{-\alpha/2+\delta_0}$, we can find a positive constant $\bar{c}_3$ such that

$$\sup_{|x_i-x_i'|\le 1}|g_i''(x_i')|\ \le\ \bar{c}_3(1+|x_i|^{m-2})$$

for each $x_i\in\mathbb{R}$. Then, applying the Taylor remainder theorem, we obtain

$$\Delta g_i(\tilde{X}_i^n(s)) - g_i'(\tilde{X}_i^n(s))\Delta\tilde{X}_i^n(s)\ \le\ \frac{1}{2}\sup_{|x_i'-\tilde{X}_i^n(s-)|\le 1}|g_i''(x_i')|(\Delta\tilde{X}_i^n(s))^2\,,$$

for each $i\in\mathcal{I}$. Following a similar analysis as in (4.12), and using Young's inequality, we obtain

$$\begin{aligned}\mathbb{E}\left[\sum_{s\le t}\mathcal{D}g_i(\tilde{X}^n,s)\right] &\le\ \mathbb{E}\left[\sum_{s\le t}\bar{c}_3\Bigl(1+|\tilde{X}_i^n(s-)|^{m-2}\Bigr)(\Delta\hat{X}_i^n(s))^2\right] \\ &\le\ \mathbb{E}\left[\int_0^t\Bigl(\bar{c}_4+\bar{c}_5\bigl(\langle e,\tilde{X}^n(s)\rangle^+\bigr)^m+\frac{\bar{c}_2}{2}|\tilde{X}^n(s)|^m\Bigr)\,\mathrm{d}s\right]\end{aligned}\tag{B.5}$$

for some positive constants $\bar{c}_4$ and $\bar{c}_5$. Thus, by (B.2)–(B.5), we obtain

$$\mathbb{E}\left[\int_0^t|\tilde{X}^n(s)|^m\,\mathrm{d}s\right]\ \le\ \bar{c}_6\,\mathbb{E}\bigl[g(\tilde{X}^n(0))\bigr]+\bar{c}_7t+\bar{c}_8\,\mathbb{E}\left[\int_0^t(\langle e,\tilde{X}^n(s)\rangle^+)^m\,\mathrm{d}s\right]\tag{B.6}$$

for some positive constants $\bar{c}_i$, $i\in\{6,7,8\}$. Using (4.14), we see that (B.6) also holds if we replace $\tilde{X}^n$ with $\hat{X}^n$. Therefore, under any sequence satisfying $\sup_n\mathfrak{J}^n(\hat{X}^n(0),\hat{U})<\infty$, we have established (4.8). This completes the proof. □

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