

TWO-PARAMETER PROCESS LIMITS FOR AN INFINITE-SERVER QUEUE WITH ARRIVAL DEPENDENT SERVICE TIMES

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ABSTRACT. We study an infinite-server queue with a general arrival process and a large class of general time-varying service time distributions. Specifically, customers' service times are conditionally independent given their arrival times, and each customer's service time, conditional on her arrival time, has a general distribution function. We prove functional limit theorems for the two-parameter processes $X^e(t, y)$ and $X^r(t, y)$ that represent the numbers of customers in the system at time t that have received an amount of service less than or equal to y , and that have a residual amount of service strictly greater than y , respectively. When the arrival process and the initial content process both have continuous Gaussian limits, we show that the two-parameter limit processes are continuous Gaussian random fields. In the proofs, we introduce a new class of sequential empirical processes with conditionally independent variables of non-stationary distributions, and employ the moment bounds resulting from the *method of chaining* for the two-parameter stochastic processes.

1. INTRODUCTION

The objective of this paper is to develop an approach to establish heavy-traffic limit theorems for many-server queues with general time-varying service time distributions. Non-stationary arrival processes have been extensively studied in the literature of many-server queues, and it is standard to assume that the arrival processes satisfy a functional central limit theorem (FCLT) where the limit process has a deterministic time-change with a time-varying arrival rate function (see Assumption 1). For service times, in the exponential case, it is standard to assume that service rates are time-varying; see, e.g., [27, 24, 25, 17, 6, 37, 20, 32]. However, little has been studied for general time-varying service times. For infinite-server queues, it is assumed in [28, 29] that the service time distribution of each customer depends on her arrival time, and in [30, 31] that the service time distribution is phase-type with time-varying rates. Whitt [48] recently provides some direct constructions of general time-varying service times via a general stationary distribution for service requirement and a time-varying service rate for $G_t/G_t/1$ queues. To our best knowledge, many-server queues with general time-varying service times have not been studied.

In this paper, we establish heavy-traffic limits for a general time-varying infinite-server queue, denoted as “ $G_t/G_t/\infty$ queue”. The arrival process is general with time-varying arrival rates satisfying an FCLT with a continuous limit (see Assumption 1). The service process is general with a time-varying service time distribution. Specifically, the service times are conditionally independent given the arrival times and each customer's service time has a

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general distribution that depends on her arrival time (see Assumption 2). That assumption includes a large class of non-stationary models for service times (see more discussions in Remarks 2.1–2.3). We also consider a time-varying initial condition (Assumption 4). For customers initially in the system, their remaining service times are assumed to be conditionally independent given the amount of services they have received at time zero, and each initial customer’s remaining service time has a general distribution depending on the amount of service she has received (or equivalently, the minus of her arrival time, see Assumption 5). We also assume that the arrival and service processes of new customers are independent of the initial conditions of the system as well as the remaining service times of these customers. The non-stationary service time distributions for initial and new customers are allowed to be different.

When the arrival process $A(t)$ is Poisson with an arrival rate function $\lambda(t)$, $t \geq 0$, by Poisson random measure theory, assuming that the system starts from empty, it is shown that the process X counting the number of customers in the system at each time t has a Poisson distribution with mean

$$E[X(t)] = \int_0^t F_u^c(t-u)\lambda(u)du, \quad t \geq 0, \quad (1.1)$$

where $F_u(\cdot)$ is the service-time distribution function conditional on the arrival time u and $F_u^c(\cdot) = 1 - F_u(\cdot)$; see, e.g., [28, 29]. Although the convergence of finite dimensional distributions of the process X is established for the $M_t/G_t/\infty$ model in heavy traffic in Section 9 of [28], the proof of tightness has remained open. The recent work on heavy-traffic analysis of infinite-server queues has focused on service times that are identically distributed, either independent or weakly dependent [45, 5, 10, 18, 7, 33, 34, 35, 36, 40]. When the arrival process $A(t)$ is non-Poisson and has an integrable arrival rate function $\lambda(t)$, $t \geq 0$, satisfying $E[A(t_2) - A(t_1)] = \int_{t_1}^{t_2} \lambda(u)du$ for $0 \leq t_1 \leq t_2$, it is shown in [2] that the expected number of customers in the system at time t has the same formula in (1.1), referred to as *the time-varying (non-stationary) Little’s Law*. See also Remark 2.3 in [28] and discussions in Section 2 of [19]. However, stochastic approximations of the process $X(t)$ are unknown when the arrival process $A(t)$ is non-Poisson. Our result is the first to establish a stochastic approximation for $X(t)$ when the arrival process $A(t)$ is general and satisfies an FCLT (e.g., with a Gaussian limit).

We consider the two-parameter processes $X^e(t, y)$ and $X^r(t, y)$ to describe the system dynamics, where $X^e(t, y)$ and $X^r(t, y)$ represent the numbers of customers in the system (in service) at time t that have received an amount of service less than or equal to y , and that have a residual amount of service strictly greater than y , respectively. The process counting the total number of customers $X(t) = X^e(t, \infty) = X^r(t, 0)$ for each $t \geq 0$. We study the system in a heavy-traffic asymptotic regime, where the arrival rates get large such that the arrival process satisfies an FCLT, while the associated conditional service time distribution functions are unscaled. We show a functional weak law of large number (FWLLN) and an FCLT for the two-parameter processes $X^e(t, y)$ and $X^r(t, y)$ in this asymptotic regime. The components resulting from the service dynamics are continuous two-parameter Gaussian processes in the limit (Definition 3.2). When the limits for the diffusion-scaled arrival and initial content processes are continuous Gaussian processes, the limits for the processes $X^e(t, y)$ and $X^r(t, y)$ are continuous two-parameter Gaussian processes (Theorem 3.2 and Proposition 3.1).

We develop a new approach to show the convergence of the diffusion-scaled two-parameter processes. Recall that when the service times are i.i.d. with a general distribution, the queue-length process can be represented via the sequential empirical processes driven by the service times, and consequently, the limiting queue-length process can be then represented as the mean-square integral of the corresponding Kiefer processes. A key component in that proof is to use the semi-martingale decomposition of the sequential empirical processes, which results in a corresponding decomposition for the queue-length process, and thus techniques of martingale convergence can be applied. That approach was first developed by Krichagina and Puhalskii [18] for $G_t/GI/\infty$ queues, and has recently further developed to study two-parameter processes in [33] for $G_t/GI/\infty$ queues and in [35, 36] for $G_t/G/\infty$ queues with weakly dependent service times satisfying the ϕ -mixing condition. It has also been used to study $G/GI/N(+GI)$ queues in [38, 39, 26, 23], $G_t/M/N_t + GI$ queues in [22] and overloaded $G/M/N + GI$ queues in [12]. Although it has been by far regarded as a standard approach to study infinite-server (many-server) queues, *it fails to work for the $G_t/G_t/\infty$ queues with arrival dependent service times*, due to the dependence structure among the service times. In particular, the dependence of service times upon arrival times does not lead to a semi-martingale decomposition via standard sequential empirical processes, and as a consequence, that approach does not apply to the $G_t/G_t/\infty$ model.

In our approach, we introduce a new class of sequential empirical processes with conditionally independent random variables of non-stationary (time-dependent) distributions given a non-stationary arrival process (see Section 5), which may be of independent interest. The advantage of studying this process is that the service components of the processes $X^e(t, y)$ and $X^r(t, y)$ can be expressed as simple functionals of it (see equations (6.11) and (6.12)). One of the major difficulties in proving the convergence of two-parameter processes for non-Markovian many-server queues has centered around tightness. Through the study of the new sequential empirical process, the procedure for proving tightness has become much simplified (see Section 6). The novelty in proving weak convergence of the new sequential empirical process lies in employing the moment bounds for stochastic processes resulting from the *method of chaining* [4, 42, 43, 44]. It enables us to obtain important moment bounds for two-parameter stochastic processes provided some moment conditions on their increments (see Theorem 4.3 and Propositions 4.1 and 5.1). Such moment bounds are necessary to verify the convergence criteria for two-parameter stochastic processes, especially for processes in the lack of the martingale property (see Theorems 13.5–13.6 in [4], and Theorems 4.1–4.2 in Section 4).

Notably, the method of chaining, originating from Kolmogorov, has been an extremely powerful tool to obtain probability and moment bounds for stochastic processes [42, 43]. For a stochastic process $X : \mathbb{T} \rightarrow \mathbb{R}$ defined on \mathbb{T} , Kolmogorov's chaining idea is to use successive approximations of a point $t \in \mathbb{T}$ and in each successive approximation step, the variation of the process is controlled uniformly over all possible chains. It led to the celebrated Dudley's entropy bound for Gaussian processes. Talagrand [42, 43] has used it to prove many important results on probability and moment bounds of stochastic processes. They turn out to be extremely useful when the stochastic processes lack the martingale property. Such powerful results have not been explored in queueing applications up to date.

One difficulty in applying such moment bounds (see Theorem 4.3) is that they require the diameter and covering number associated with a semimetric defined on the domain of the stochastic processes. Fortunately, for the $G_t/G_t/\infty$ queues, the covariance structures of the two-parameter Gaussian limit processes resulting from the service dynamics enable us

to find a proper semimetric and the associated diameter and covering number, and obtain useful moment bounds for their supremum norm (see Propositions 4.1 and 5.1). These moment bounds also provide important insights on proving the probability bounds in order to verify the convergence criteria for the two-parameter stochastic processes; see the proof of Theorem 5.1.

It is worth noting that the new methodology exploring the moment bounds resulting from the method of chaining may be potentially applied and further developed to study stationary and non-stationary non-Markovian many-server queues. It is evident that our work provides a new proof for the two-parameter process limits of the $G_t/GI/\infty$ queues with i.i.d. service times with any general distribution in [33]; see the discussions in Section 8.2. It will be interesting to see how this approach can be developed to study many-server queues with i.i.d. service times or non-stationary service times, which have been an active research area (see, e.g., [25, 47, 21, 22, 14, 15, 16, 49] and references therein).

1.1. Organization of the paper. In the next subsection, we summarize the notation used in this paper. In Section 2, we present the model and assumptions in detail. The main results are stated in Section 3. We provide some preliminary results for the proofs in Section 4, in particular, criteria for weak convergence and existence of two-parameter processes, moment bounds for two-parameter stochastic processes resulting from the method of chaining, and a sample path property for two-parameter Gaussian processes. In Section 5, we introduce the new class of sequential empirical processes and state the FCLTs, and a moment bound for the two-parameter Gaussian limit process, and the proof of convergence is given in Section 7. The proof of the main result is given in Section 6. We discuss the application of the moment bounds to some relevant processes and the implications of our new method to the study of $G_t/GI/\infty$ queues in Section 8. Some additional results and proofs are collected in the Appendix.

1.2. Notation. Throughout the paper, \mathbb{N} denotes the set of natural numbers. \mathbb{R}^k (\mathbb{R}_+^k) denotes the space of real-valued (nonnegative) k -dimensional vectors, and we write \mathbb{R} (\mathbb{R}_+) for $k = 1$. Let $\mathbb{D}^k = \mathbb{D}(\mathbb{R}_+, \mathbb{R}^k)$ denote \mathbb{R}^k -valued function space of all càdlàg functions on \mathbb{R}_+ . Denote $\mathbb{D} \equiv \mathbb{D}^1$. (\mathbb{D}, J_1) denotes space \mathbb{D} equipped with Skorohod J_1 topology with the metric d_{J_1} [4, 9, 46]. Note that the space (\mathbb{D}, J_1) is complete and separable. Let $\mathbb{D}_{\mathbb{D}} = \mathbb{D}(\mathbb{R}_+, \mathbb{D})$ denote the \mathbb{D} -valued function space of all càdlàg functions on \mathbb{R}_+ with both \mathbb{D} spaces equipped with J_1 topology. Let \mathbb{C} be the subset of \mathbb{D} for continuous functions, and similarly for \mathbb{C}^k and $\mathbb{C}_{\mathbb{C}}$. $\mathbb{D}_2 \equiv \mathbb{D}(\mathbb{R}_+^2, \mathbb{R})$ denotes the space of all “continuous from above with limits from below” functions on \mathbb{R}_+^2 , and is endowed with the same metric $d_{\mathbb{D}_2}$ as in [3]. Let \mathbb{C}_2 be the subset of \mathbb{D}_2 for continuous functions. It is worth noting that $\mathbb{D}_2 \subset \mathbb{D}_{\mathbb{D}}$, and $\mathbb{D}_2 \equiv \mathbb{D}_{\mathbb{D}}$ provided the second \mathbb{D} in $\mathbb{D}_{\mathbb{D}}$ is equipped with uniform norm [3], and thus, we have $\mathbb{C}_2 \equiv \mathbb{C}_{\mathbb{C}}$. When considering functions defined on finite intervals, we write $\mathbb{D}([0, T], \mathbb{R})$, $\mathbb{D}([0, T], \mathbb{D}([0, T'], \mathbb{R}))$ and $\mathbb{D}([0, T] \times [0, T'], \mathbb{R})$ for $T, T' > 0$. For any two complete separable metric spaces S_1 and S_2 , we denote $S_1 \times S_2$ as their product space equipped with the product topology (section 11.4 in [46]).

All random variables and processes are defined in a common complete probability space (Ω, \mathcal{F}, P) . Notations \rightarrow and \Rightarrow mean convergence of real numbers and convergence in distribution, respectively. The abbreviation *a.s.* means *almost surely*. We use small- o notation for real-valued function f and non-zero g , we write $f(x) = o(g(x))$ if $\limsup_{x \rightarrow \infty} |f(x)/g(x)| = 0$.

2. MODEL AND ASSUMPTIONS

We consider a sequence of the time-varying infinite-server queues with arrival dependent service times, denoted as $G_t/G_t/\infty$ queues, with an index $n \in \mathbb{N}$ and let $n \rightarrow \infty$. There is an infinite number of servers, and customers will be served immediately upon arrival. Let $A^n := \{A^n(t) : t \geq 0\}$ be the arrival process with arrival times $\{\tau_i^n : i \in \mathbb{N}\}$. (We use the convention that $A^n(t) \equiv 0$ for $t < 0$ and similarly for other processes.) Let $\eta_i(\tau_i^n)$ be the corresponding service time of the i^{th} customer, given her arrival time τ_i^n , $i \in \mathbb{N}$. Let $X^{n,e}(t, y)$ and $X^{n,r}(t, y)$ represent the numbers of customers at time t that have received an amount of service less than or equal to y , and that have a residual amount of service times strictly greater than y , respectively. Then $X^n(t) = X^{n,e}(t, \infty) = X^{n,r}(t, 0)$ represents the total number of customers in the system at time $t \geq 0$. Note that $X^n(0)$ is the initial amount of customers in the system at time 0. For the initial customers, we let $\check{\tau}_i^n$ be their arrival times, or equivalently, $\tilde{\tau}_i^n = -\check{\tau}_i^n$, be the amount of service the initial customer i has received by time 0, $i = 1, \dots, X^n(0)$. Let $\tilde{\eta}_i^{n,0} := \tilde{\eta}_i(\tilde{\tau}_i^n)$ be the residual service time at time 0 associated with the initial customer i that arrived at time $\check{\tau}_i^n$, $i = 1, \dots, X^n(0)$. Let $D^n := \{D^n(t) : t \geq 0\}$ be the departure process. We assume that the arrival and service processes of new arrivals are independent of the initial content of customers and the service process of initial customers.

The two-parameter processes $X^{n,e}(t, y)$ and $X^{n,r}(t, y)$ can be written as

$$X^{n,e}(t, y) = \sum_{i=1}^{X^n(0)} \mathbf{1}(\tilde{\eta}_i^{n,0} > t) \mathbf{1}(\tilde{\tau}_i^n + t \leq y) + \sum_{i=A^n((t-y)-)+1}^{A^n(t)} \mathbf{1}(\tau_i^n + \eta_i(\tau_i^n) > t), \quad (2.1)$$

$$X^{n,r}(t, y) = \sum_{i=1}^{X^n(0)} \mathbf{1}(\tilde{\eta}_i^{n,0} > t + y) + \sum_{i=1}^{A^n(t)} \mathbf{1}(\tau_i^n + \eta_i(\tau_i^n) > t + y), \quad (2.2)$$

for each $t, y \geq 0$, where $A^n(t-)$ is the left limit of A^n at $t > 0$. Note that the sample paths of the processes $X^{n,e}(t, y)$ and $X^{n,r}(t, y)$ are in \mathbb{D} but not in \mathbb{D}_2 (see Remark 3.3 in [33] for a detailed discussion). The departure process $D^n(t) = X^n(0) + A^n(t) - X^n(t)$ for each $t \geq 0$.

We state the assumptions on the primitives of the system, including the arrival and service processes of new arrivals and initial customers.

Assumption 1. (*Arrival Process*) *The sequence of arrival processes A^n satisfies an FCLT:*

$$\hat{A}^n := \sqrt{n}(\bar{A}^n - \Lambda) \Rightarrow \hat{A} \quad \text{in } (\mathbb{D}, J_1) \quad \text{as } n \rightarrow \infty \quad (2.3)$$

where $\bar{A}^n := n^{-1}A^n$, $\Lambda := \{\Lambda(t) : t \geq 0\}$ is a deterministic nondecreasing continuous function with $\Lambda(0) = 0$, and \hat{A} is a stochastic process with continuous sample paths.

We remark that Assumption 1 implies an FWLLN for the fluid-scaled arrival process \bar{A}^n :

$$\bar{A}^n \Rightarrow \Lambda \quad \text{in } (\mathbb{D}, J_1) \quad \text{as } n \rightarrow \infty. \quad (2.4)$$

A typical example of the limit process \hat{A} is a Brownian motion, $\hat{A}(t) = c_a B(\Lambda(t))$ for some constant $c_a > 0$ capturing the stochastic variability in the arrival process. When the arrival process is renewal, c_a represents the coefficient of variation for the interarrival times.

Assumption 2. (*Service Times*) *The service times are conditionally independent given the arrival times, that is, given the arrival times $\{\tau_i : i \in \mathbb{N}\}$, $\eta_j(\tau_j^n)$ and $\eta_k(\tau_k^n)$ are independent*

for any distinct $j, k \in \mathbb{N}$ and for each $n \in \mathbb{N}$. For each $i \in \mathbb{N}$ and $n \in \mathbb{N}$, given the arrival time $\tau_i^n = t$ for $t \geq 0$, the regular conditional distribution of the variable $\eta_i(\tau_i^n)$ is

$$P(\eta_i(\tau_i^n) \leq x | \tau_i^n = t) = F_t(x), \quad x \geq 0, \quad (2.5)$$

where for each $t \geq 0$, $F_t(\cdot)$ is a continuous cumulative distribution function (c.d.f.) on \mathbb{R}_+ . The conditional mean service times are finite a.s., i.e., $E[\eta_i(\tau_i^n) | \tau_i^n] < \infty$ a.s. Let $F_t^c := 1 - F_t$ for each $t \geq 0$.

We also impose a finite-variation regularity condition on the function $F_t(x)$.

Assumption 3. (A Regularity Condition on $F_t(x)$) For each $x \geq 0$, let the function $\tilde{F}_x(\cdot) := F_t(x - \cdot)$. Assume that

$$\sup_{x \in [0, T]} V_0^T(\tilde{F}_x) < \infty, \quad \text{for any } T > 0, \quad (2.6)$$

where $V_0^T(\tilde{F}_x)$ is the total variation of \tilde{F}_x on $[0, T]$.

Remark 2.1. We give two explicit examples of the c.d.f. $F_t(\cdot)$. The simplest one is the exponential distribution with time-varying rate $\mu(t)$, that is,

$$F_t(x) = 1 - e^{-\mu(t)x}, \quad t, x \geq 0.$$

The second example is a time-varying hyper-exponential distribution, denoted as an $H_2(t)$ distribution, being defined by the mixture of two independent time-varying exponential distributions. The c.d.f. $F_t(\cdot)$ is

$$F_t(x) = p(t)(1 - e^{-\mu_1(t)x}) + (1 - p(t))(1 - e^{-\mu_2(t)x}), \quad t, x \geq 0,$$

where $p(t) \in [0, 1]$ for each $t \geq 0$, and $\mu_1(t)$ and $\mu_2(t)$ are the corresponding time-varying service rates. The condition on $F_t(\cdot)$ in Assumption 2 requires that the rate function $\mu(t)$ is of bounded variation in the first example, and the rate functions $\mu_1(t)$ and $\mu_2(t)$ and the probability function $p(t)$ are of bounded variation in the second example. \square

Remark 2.2. We note that the total variation condition (2.6) in Assumption 3 excludes certain classes of time-varying service time distributions. For example, consider the c.d.f.

$$F_t(x) = x \sin(1/t) \mathbf{1}(0 \leq x \leq \sin(1/t)) + \mathbf{1}(x > \sin(1/t)), \quad \text{for } x \geq 0, \quad (2.7)$$

for each $t > 0$. This function is jointly continuous in t and x . It is easy to verify that for each $t > 0$, $F_t(x)$ in (2.7) is a Lipschitz continuous c.d.f. with a piecewise constant density function

$$f_t(x) = \sin(1/t) \mathbf{1}(0 \leq x \leq \sin(1/t)), \quad \text{for } x \geq 0.$$

However, it is evident that for this example, $\sup_{x \in [0, T]} V_0^T(\tilde{F}_x) = \infty$.

Nevertheless, there are many classes of time-varying service time distributions that satisfy the conditions in Assumptions 2–3. We provide such a class of distributions which may be applicable in many situations. Suppose that there exist a finite number of times $\{t_i, i = 0, \dots, k\}$ satisfying $0 = t_0 < t_1 < \dots < t_k = T < \infty$ and a sequence of continuous c.d.f.'s $\{F_{t_i}(\cdot) : i = 0, \dots, k-1\}$ such that for $t \in [t_i, t_{i+1})$, $F_t(\cdot) = F_{t_i}(\cdot)$ for each $i = 0, \dots, k-1$. For each $i = 0, \dots, k-1$, $F_{t_i}(\cdot)$ is of bounded variation on any finite interval since it is monotone, and thus the conditions in Assumptions 2–3 are satisfied.

Regarding how the conditions in Assumptions 2–3 are used in the proofs, we refer the readers to Remark 6.1. \square

Remark 2.3. We remark that our results can be applied to the explicit constructions of time-varying service times in [48]. Let $\{\tilde{\eta}_i : i \geq 1\}$ be a sequence of i.i.d. random variables with a c.d.f. F that represents the service requirement of each customer. Let $\mu(\cdot)$ be a time-dependent service rate function. Unlike the single-server setting in [48], we need to assume that all the busy servers in our model are serving at this same time-varying rate $\mu(t)$ simultaneously at each time. Let $\{\eta_i : i \geq 1\}$ be the sequence of associated service durations. Then we can write

$$\tilde{\eta}_i = \int_{\tau_i^n}^{\tau_i^n + \eta_i} \mu(s) ds, \quad i \geq 1,$$

which results in the c.d.f.

$$F_t(x) = F\left(\int_t^{x+t} \mu(s) ds\right), \quad t, x \geq 0.$$

It is easy to see that this $F_t(x)$ satisfies the conditions in Assumptions 2–3. By employing Taylor approximation, we can get

$$\eta_i \approx \frac{\tilde{\eta}_i}{\mu(\tau_i^n)}, \quad i \geq 1,$$

which gives us a very simple construction of general time-varying service times:

$$F_t(x) = F(x\mu(t)), \quad t, x \geq 0.$$

Therefore, our model includes such specific constructions as special cases, and in fact, covers a large class of general time-varying service times. \square

We make the following assumption on the initial conditions.

Assumption 4. (*Initial Conditions*) The process $X^{n,e}(0, y)$ for initial content of customers satisfies: (i) there exists $0 < \bar{y} < \infty$ such that $X^n(0) = X^{n,e}(0, \bar{y})$ a.s.; (ii) there exists a deterministic continuous nondecreasing function $\bar{X}^e(0, y)$ for $y \geq 0$ with $\bar{X}^e(0, 0) = 0$ such that $\bar{X}^{n,e}(0, y) := n^{-1}X^{n,e}(0, y) \Rightarrow \bar{X}^e(0, y)$ in (\mathbb{D}, J_1) as $n \rightarrow \infty$; and (iii) there exists a stochastic process with continuous sample paths $\hat{X}^e(0, y)$ for $y \geq 0$ such that $\hat{X}^{n,e}(0, y) := n^{1/2}(\bar{X}^{n,e}(0, y) - \bar{X}^e(0, y)) \Rightarrow \hat{X}^e(0, y)$ in (\mathbb{D}, J_1) as $n \rightarrow \infty$.

It is worth noting that, by definition, $X^{n,e}(0, y)$ is the number of customers at time 0 that have received an amount of service less than or equal to y . Thus, $X^{n,e}(0, y)$ describes the amount of received services of each customer initially in the system, which is equivalent to providing the arrival times for the initial customers. However, this arrival process of initial customers may have different rates from that of new customers under our assumptions. It may be possible that the initial and new customers come from the same arrival process, in which case the processes $X^{n,e}(0, y)$ and $A^n(t)$ will be correlated. We do not consider that scenario in this paper, but instead assume that they are independent as in [33]. The correlated case is considered in [1], in which these two processes are assumed to be asymptotically independent (and the new customers' service times are assumed to be i.i.d. as in [33]).

Assumption 4 implies that the total initial content $X^n(0)$ at time 0 satisfies: (i) $\bar{X}^n(0) \Rightarrow \bar{X}(0) = \bar{X}^e(0, \infty)$ in \mathbb{R} as $n \rightarrow \infty$ and (ii) $\hat{X}^n(0) \Rightarrow \hat{X}(0) = \hat{X}^e(0, \infty)$ in \mathbb{R} as $n \rightarrow \infty$. It also implies that the process $X^{n,r}(0, y)$ for the initial customers at time 0 satisfies: (i) $\bar{X}^{n,r}(0, y) \Rightarrow \bar{X}^r(0, y) := \bar{X}(y) - \bar{X}^e(y, y)$ in (\mathbb{D}, J_1) as $n \rightarrow \infty$ and (ii) $\hat{X}^{n,r}(0) := n^{1/2}(\bar{X}^{n,r}(0, y) - \bar{X}^r(0, y)) \Rightarrow \hat{X}^r(0, y) := \hat{X}(y) - \hat{X}^e(y, y)$ in (\mathbb{D}, J_1) as $n \rightarrow \infty$.

Assumption 5. (*Service Times of Initial Customers*) The remaining service times are conditionally independent given the amount of received services (or equivalently, the arrival times), that is, given $\{\tilde{\tau}_i^n = -\check{\tau}_i^n : i \in \mathbb{N}\}$, $\tilde{\eta}_j(\tilde{\tau}_j^n)$ and $\tilde{\eta}_k(\tilde{\tau}_k^n)$ are independent for any distinct $j, k \in \mathbb{N}$ and for each $n \in \mathbb{N}$. For each $i = 1, \dots, X^n(0)$ and $n \in \mathbb{N}$, given the amount of received service $\tilde{\tau}_i^n = -\check{\tau}_i^n = t$, the regular conditional distribution of the variable $\tilde{\eta}_i(\tilde{\tau}_i^n)$ is

$$P(\tilde{\eta}_i(\tilde{\tau}_i^n) \leq x | \tilde{\tau}_i^n = t) = G_t(x), \quad t, x \geq 0, \quad (2.8)$$

where for each $t \geq 0$, $G_t(\cdot)$ is a continuous c.d.f. on \mathbb{R}_+ . The conditional mean remaining service times of initial customers are finite a.s., i.e., $E[\tilde{\eta}_i(\tilde{\tau}_i^n) | \tilde{\tau}_i^n] < \infty$ a.s. Let $G_t^c := 1 - G_t$ for each $t \geq 0$.

Similar to Assumption 3, we also impose a finite-variation regularity condition on the function $G_t(x)$.

Assumption 6. (*A Regularity Condition on $G_t(x)$*) For each $x \geq 0$, let $\tilde{G}_x(t) := G_t(x)$. Assume that

$$\sup_{x \in [0, T]} V_0^T(\tilde{G}_x) < \infty, \quad \text{for any } T > 0, \quad (2.9)$$

where $V_0^T(\tilde{G}_x)$ is the total variation of \tilde{G}_x on $[0, T]$.

The discussions on the distribution functions $F_t(x)$ in Remarks 2.1–2.3 also apply to the distribution functions $G_t(x)$ for the initial customers. Note that if the service times of the initial customers are i.i.d. with c.d.f. G , conditional on the amount of service received at time 0 being equal to t , the probability that the remaining service time is greater than x is equal to $G_t^c(x) = G^c(t+x)/G^c(t)$ for each $t, x \geq 0$. Namely, in that case, one may assume that $G_t(x) = 1 - G^c(t+x)/G^c(t)$ for each $t, x \geq 0$, for a given continuous c.d.f. G . This type of distributions is considered for the initial content of $G_t/GI/\infty$ queues in [1].

3. MAIN RESULTS

In this section we state the main results of this paper. Define the fluid-scaled processes $\bar{X}^{n,e} = \{\bar{X}^{n,e}(t, y) : t, y \geq 0\}$, $\bar{X}^{n,r} = \{\bar{X}^{n,r}(t, y) : t, y \geq 0\}$, $\bar{X}^n = \{\bar{X}^n(t) : t \geq 0\}$ and $\bar{D}^n = \{\bar{D}^n(t) : t \geq 0\}$ by

$$\bar{X}^{n,e} := n^{-1}X^{n,e}, \quad \bar{X}^{n,r} := n^{-1}X^{n,r}, \quad \bar{X}^n := n^{-1}X^n, \quad \bar{D}^n := n^{-1}D^n.$$

Theorem 3.1. (*FWLLN*) Under Assumptions 1–6,

$$(\bar{A}^n, \bar{X}^{n,e}, \bar{X}^{n,r}, \bar{X}^n, \bar{D}^n) \Rightarrow (\Lambda, \bar{X}^e, \bar{X}^r, \bar{X}, \bar{D}) \quad \text{in } \mathbb{D} \times (\mathbb{D}_{\mathbb{D}})^2 \times \mathbb{D}^2 \quad \text{as } n \rightarrow \infty,$$

where Λ is given in Assumption 1, the fluid limits $\bar{X}^e = \{\bar{X}^e(t, y) : t, y \geq 0\}$, $\bar{X}^r = \{\bar{X}^r(t, y) : t, y \geq 0\}$, $\bar{X} = \{\bar{X}(t) : t \geq 0\}$ and $\bar{D} = \{\bar{D}(t) : t \geq 0\}$ are given by

$$\bar{X}^e(t, y) = \int_0^{(y-t)^+} G_s^c(t) d\bar{X}^e(0, s) + \int_{(t-y)^+}^t F_s^c(t-s) d\Lambda(s), \quad t, y \geq 0, \quad (3.1)$$

$$\bar{X}^r(t, y) = \int_0^\infty G_s^c(t+y) d\bar{X}^e(0, s) + \int_0^t F_s^c(t+y-s) d\Lambda(s), \quad t, y \geq 0, \quad (3.2)$$

$$\bar{X}(t) = \bar{X}^e(t, \infty) = \bar{X}^r(t, 0), \quad t \geq 0, \quad (3.3)$$

and

$$\bar{D}(t) = \int_0^\infty G_s(t) d\bar{X}^e(0, s) + \int_0^t F_s(t-s) d\Lambda(s), \quad t \geq 0. \quad (3.4)$$

Define the diffusion scaled processes $\hat{X}^{n,e} = \{\hat{X}^{n,e}(t, y) : t, y \geq 0\}$, $\hat{X}^{n,r} = \{\hat{X}^{n,r}(t, y) : t, y \geq 0\}$, $\hat{X}^n = \{\hat{X}^n(t) : t \geq 0\}$ and $\hat{D}^n = \{\hat{D}^n(t) : t \geq 0\}$ by $\hat{X}^{n,e} := \sqrt{n}(\bar{X}^{n,e} - \bar{X}^e)$, $\hat{X}^{n,r} := \sqrt{n}(\bar{X}^{n,r} - \bar{X}^r)$, $\hat{X}^n := \sqrt{n}(\bar{X}^n - \bar{X})$, $\hat{D}^n := \sqrt{n}(\bar{D}^n - \bar{D})$, where \bar{X}^e , \bar{X}^r , \bar{X} and \bar{D} are given in (3.1), (3.2), (3.3) and (3.4), respectively.

Definition 3.1. Define the two-parameter processes $\hat{X}_{0,1}^e = \{\hat{X}_{0,1}^e(t, y) : t, y \geq 0\}$, $\hat{X}_{0,1}^r = \{\hat{X}_{0,1}^r(t, y) : t, y \geq 0\}$, $\hat{X}_1^e = \{\hat{X}_1^e(t, y) : t, y \geq 0\}$ and $\hat{X}_1^r = \{\hat{X}_1^r(t, y) : t, y \geq 0\}$ by

$$\hat{X}_{0,1}^e(t, y) := \int_0^{(y-t)^+} G_u^c(t) d\hat{X}^e(0, u), \quad (3.5)$$

$$\hat{X}_{0,1}^r(t, y) := \int_0^\infty G_u^c(t+y) d\hat{X}^e(0, u) = - \int_0^\infty G_u^c(t+y) d\hat{X}^r(0, u), \quad (3.6)$$

$$\hat{X}_1^e(t, y) := \int_{(t-y)^+}^t F_u^c(t-u) d\hat{A}(u), \quad (3.7)$$

$$\hat{X}_1^r(t, y) := \int_0^t F_u^c(t+y-u) d\hat{A}(u), \quad (3.8)$$

for each $t \geq 0$ and $y \geq 0$. They are well-defined as stochastic integrals with “integration by parts” (that is, a pathwise construction via integration by parts).

The existence and continuity of these processes are proved via the continuous mapping theorem (Lemmas 6.2–6.3). When the limit $\hat{X}^e(0, y)$ and $\hat{A}(t)$ are Brownian motions, these stochastic integrals are defined as Itô integrals.

Definition 3.2. Define the processes $\hat{X}_{0,2}^e = \{\hat{X}_{0,2}^e(t, y) : t, y \geq 0\}$ and $\hat{X}_2^e = \{\hat{X}_2^e(t, y) : t, y \geq 0\}$ to be two-parameter Gaussian processes with mean zero and covariance functions: for $t, s \geq 0$ and $y, x \geq 0$,

$$\text{Cov}(\hat{X}_{0,2}^e(t, y), \hat{X}_{0,2}^e(s, x)) = \int_0^{(y-t)^+ \wedge (x-s)^+} (G_u(t \wedge s) - G_u(t)G_u(s)) d\bar{X}^e(0, u), \quad (3.9)$$

$$\text{Cov}(\hat{X}_2^e(t, y), \hat{X}_2^e(s, x)) = \int_{(t-y)^+ \vee (s-x)^+}^{t \wedge s} (F_u(t \wedge s - u) - F_u(t-u)F_u(s-u)) d\Lambda(u). \quad (3.10)$$

Define the processes $\hat{X}_{0,2}^r = \{\hat{X}_{0,2}^r(t, y) : t, y \geq 0\}$ and $\hat{X}_2^r = \{\hat{X}_2^r(t, y) : t, y \geq 0\}$ be two-parameter Gaussian processes with mean zero and covariance functions: for $t, s \geq 0$ and $y, x \geq 0$,

$$\text{Cov}(\hat{X}_{0,2}^r(t, y), \hat{X}_{0,2}^r(s, x)) = \int_0^\infty (G_u^c((t+y) \wedge (s+x)) - G_u^c(t+y)G_u^c(s+x)) d\bar{X}^e(0, u), \quad (3.11)$$

$$\text{Cov}(\hat{X}_2^r(t, y), \hat{X}_2^r(s, x)) = \int_0^{t \wedge s} (F_u^c((t+y) \wedge (s+x) - u) - F_u^c(t+y-u)F_u^c(s+x-u)) d\Lambda(u). \quad (3.12)$$

Theorem 3.2. (FCLT) Under Assumptions 1–6,

$$(\hat{A}^n, \hat{X}^{n,e}, \hat{X}^{n,r}, \hat{X}^n, \hat{D}^n) \Rightarrow (\hat{A}, \hat{X}^e, \hat{X}^r, \hat{X}, \hat{D}) \quad \text{in } \mathbb{D} \times (\mathbb{D}_{\mathbb{D}})^2 \times \mathbb{D}^2 \quad \text{as } n \rightarrow \infty,$$

where \hat{A} is given in Assumption 1, and the limit processes are specified below. The limit process $\hat{X}^e = \{\hat{X}^e(t, y) : t, y \geq 0\}$ and $\hat{X}^r = \{\hat{X}^r(t, y) : t, y \geq 0\}$ are given by

$$\hat{X}^e(t, y) = \hat{X}_{0,1}^e(t, y) + \hat{X}_{0,2}^e(t, y) + \hat{X}_1^e(t, y) + \hat{X}_2^e(t, y), \quad t, y \geq 0, \quad (3.13)$$

$$\hat{X}^r(t, y) = \hat{X}_{0,1}^r(t, y) + \hat{X}_{0,2}^r(t, y) + \hat{X}_1^r(t, y) + \hat{X}_2^r(t, y), \quad t, y \geq 0, \quad (3.14)$$

where the processes $\hat{X}_{0,1}^e$, $\hat{X}_{0,1}^r$, \hat{X}_1^e and \hat{X}_1^r are given in Definition 3.1 and the processes $\hat{X}_{0,2}^e$, $\hat{X}_{0,2}^r$, \hat{X}_2^e and \hat{X}_2^r are given in Definition 3.2, the four processes for \hat{X}^e in (3.13) are mutually independent and so are the four processes for \hat{X}^r in (3.14). The limit process $\hat{X} = \{\hat{X}(t) : t \geq 0\}$ is given by $\hat{X}(t) = \hat{X}^e(t, \infty) = \hat{X}^r(t, 0)$ for $t \geq 0$. The limit departure process $\hat{D} = \{\hat{D}(t) : t \geq 0\}$ is given by $\hat{D}(t) = \hat{X}(0) + \hat{A}(t) - \hat{X}(t)$ for $t \geq 0$. All the limit processes have continuous sample paths.

The continuity of the two-parameter Gaussian processes $\hat{X}_{0,2}^e$, $\hat{X}_{0,2}^r$, \hat{X}_2^e and \hat{X}_2^r will be part of the proof. We next present the characterization of the limit processes when the arrival and initial limits are continuous Gaussian processes. The results follow directly from applying properties of Gaussian processes, and thus we omit the proof.

Proposition 3.1. (*Characterization of the Limit Processes*) Under Assumptions 1–6, if the arrival and initial limit processes $\hat{A}(t)$ and $\hat{X}^e(0, y)$ are continuous Gaussian processes with mean zero and covariance functions $\Phi^a(t, s) = \text{Cov}(\hat{A}(t), \hat{A}(s))$ for $t, s \geq 0$ and $\Phi^e(x, y) = \text{Cov}(\hat{X}^e(0, x), \hat{X}^e(0, y))$ for $x, y \geq 0$, then the limit processes can be characterized as follows. The limit process \hat{X}^e in (3.13) is a continuous two-parameter Gaussian process (random field) and has mean $E[\hat{X}^e(t, y)] = 0$ and covariance function: for $t, s \geq 0$ and $y, x \geq 0$,

$$\begin{aligned} \text{Cov}(\hat{X}^e(t, y), \hat{X}^e(s, x)) &= \text{Cov}(\hat{X}_{0,1}^e(t, y), \hat{X}_{0,1}^e(s, x)) + \text{Cov}(\hat{X}_{0,2}^e(t, y), \hat{X}_{0,2}^e(s, x)) \\ &\quad + \text{Cov}(\hat{X}_1^e(t, y), \hat{X}_1^e(s, x)) + \text{Cov}(\hat{X}_2^e(t, y), \hat{X}_2^e(s, x)), \end{aligned} \quad (3.15)$$

where

$$\text{Cov}(\hat{X}_{0,1}^e(t, y), \hat{X}_{0,1}^e(s, x)) = \int_0^{(y-t)^+} \int_0^{(x-s)^+} G_u^c(t) G_v^c(s) d\Phi^e(u, v), \quad (3.16)$$

$$\text{Cov}(\hat{X}_1^e(t, y), \hat{X}_1^e(s, x)) = \int_{(t-y)^+}^t \int_{(s-x)^+}^s F_u^c(t-u) F_v^c(s-v) d\Phi^a(u, v), \quad (3.17)$$

$\text{Cov}(\hat{X}_{0,2}^e(t, y), \hat{X}_{0,2}^e(s, x))$ and $\text{Cov}(\hat{X}_2^e(t, y), \hat{X}_2^e(s, x))$ are given in (3.9) and (3.10), respectively. The limit process \hat{X}^r in (3.14) is a continuous two-parameter Gaussian process (random field) and has mean $E[\hat{X}^r(t, y)] = 0$ and covariance function: for $t, s \geq 0$ and $y, x \geq 0$,

$$\begin{aligned} \text{Cov}(\hat{X}^r(t, y), \hat{X}^r(s, x)) &= \text{Cov}(\hat{X}_{0,1}^r(t, y), \hat{X}_{0,1}^r(s, x)) + \text{Cov}(\hat{X}_{0,2}^r(t, y), \hat{X}_{0,2}^r(s, x)) \\ &\quad + \text{Cov}(\hat{X}_1^r(t, y), \hat{X}_1^r(s, x)) + \text{Cov}(\hat{X}_2^r(t, y), \hat{X}_2^r(s, x)), \end{aligned} \quad (3.18)$$

where

$$\text{Cov}(\hat{X}_{0,1}^r(t, y), \hat{X}_{0,1}^r(s, x)) = \int_0^\infty \int_0^\infty G_u^c(t+y) G_v^c(s+x) d\Phi^e(u, v), \quad (3.19)$$

$$\text{Cov}(\hat{X}_1^r(t, y), \hat{X}_1^r(s, x)) = \int_0^t \int_0^s F_u^c(t+y-u) F_v^c(s+x-v) d\Phi^a(u, v), \quad (3.20)$$

$\text{Cov}(\hat{X}_{0,2}^r(t, y), \hat{X}_{0,2}^r(s, x))$ and $\text{Cov}(\hat{X}_2^r(t, y), \hat{X}_2^r(s, x))$ are given in (3.11) and (3.12), respectively.

Remark 3.1. It is worth noting that the mean and covariance formulas can be obtained from the corresponding formulas in [33] by simply inserting a time subscript on the service time distribution functions, e.g., the u and v subscripts in (3.9), (3.10), (3.16) and (3.17). That is evidently what should be expected for the $G_t/G_t/\infty$ queue.

If the arrival limit process $\hat{A}(t) = c_a B(\Lambda(t))$, $t \geq 0$, for a standard Brownian motion B , the covariance formulas in (3.16) and (3.19) become

$$\begin{aligned} \text{Cov}(\hat{X}_{0,1}^e(t, y), \hat{X}_{0,1}^e(s, x)) &= c_a^2 \int_{(t-y)^+ \vee (s-x)^+}^{t \wedge s} F_u^c(t-u) F_u^c(s-u) d\Lambda(u), \\ \text{Cov}(\hat{X}_{0,1}^r(t, y), \hat{X}_{0,1}^r(s, x)) &= c_a^2 \int_0^{t \wedge s} F_u^c(t+y-u) F_u^c(s+x-u) d\Lambda(u). \end{aligned}$$

Thus we obtain that the total count limit process \hat{X} is Gaussian with mean zero and covariance function:

$$\begin{aligned} \text{Cov}(\hat{X}(t), \hat{X}(s)) &= \int_0^\infty \int_0^\infty G_u^c(t) G_v^c(s) d\Phi^e(u, v) + \int_0^\infty (G_u^c(t \wedge s) - G_u^c(t) G_u^c(s)) d\bar{X}^e(0, u) \\ &\quad + \int_0^{t \wedge s} F_u^c(t \wedge s - u) d\Lambda(s) + (c_a^2 - 1) \int_0^{t \wedge s} F_u^c(t-u) F_u^c(s-u) d\Lambda(u), \end{aligned}$$

for each $t \geq 0$. The departure limit process \hat{D} is a continuous Gaussian process with mean zero and covariance function: for $t, s \geq 0$,

$$\begin{aligned} \text{Cov}(\hat{D}(t), \hat{D}(s)) &= \int_0^\infty \int_0^\infty G_u(t) G_v(s) d\Phi^e(u, v) + \int_0^\infty (G_u^c(t \wedge s) - G_u^c(t) G_u^c(s)) d\bar{X}^e(0, u) \\ &\quad + \int_0^{t \wedge s} F_u(t \wedge s - u) d\Lambda(u) + (c_a^2 - 1) \int_0^{t \wedge s} F_u^c(t-u) F_u^c(s-u) d\Lambda(u). \end{aligned}$$

□

4. PRELIMINARIES

In this section, we state some preliminary results for the proofs.

4.1. Criteria for weak convergence and existence of two-parameter processes.

We first state a criterion on weak convergence of stochastic processes in $\mathbb{D}([0, T], \mathcal{S})$ endowed with the Skorohod J_1 topology, where \mathcal{S} is a metric space with metric m . The criterion is stated for $\mathbb{D}([0, 1], \mathbb{R})$ in Theorem 13.5 in [4]. However, as remarked in the beginning of Chapter 3 in [4] “with very little change, the theory can be extended to functions on $[0, 1]$ taking values in metric spaces other than \mathbb{R} .”

Theorem 4.1. ([4, Theorem 13.5])

Let X^n and X be stochastic processes with sample paths in $\mathbb{D}([0, T], \mathcal{S})$ where (\mathcal{S}, m) is a metric space. Suppose that

$$\begin{aligned} \text{(i)} \quad &\text{for any } 0 \leq t_1 \leq t_2 \leq \dots \leq t_k \leq T \text{ and } k \geq 1, \\ &(X^n(t_1), \dots, X^n(t_k)) \Rightarrow (X(t_1), \dots, X(t_k)) \quad \text{in } \mathcal{S}^k \quad \text{as } n \rightarrow \infty, \end{aligned} \quad (4.1)$$

$$\begin{aligned} \text{(ii)} \quad &m(X(T), X(T - \delta)) \Rightarrow 0 \quad \text{in } \mathbb{R} \quad \text{as } \delta \rightarrow 0, \end{aligned} \quad (4.2)$$

(iii) for $r \leq s \leq t$, $n \geq 1$ and $\epsilon > 0$,

$$P(\mathfrak{m}(X^n(r), X^n(s)) \wedge \mathfrak{m}(X^n(s), X^n(t)) \geq \epsilon) \leq \frac{1}{\epsilon^4} (H(t) - H(r))^2, \quad (4.3)$$

where H is a nondecreasing and continuous function on $[0, T]$.

Then $X^n \Rightarrow X$ in $\mathbb{D}([0, T], \mathcal{S})$ as $n \rightarrow \infty$.

We then state a criterion on the existence of a stochastic process with sample paths in $\mathbb{D}([0, T] \times [0, T'], \mathbb{R})$ given its finite dimensional distributions for $T, T' > 0$. The criterion is adapted from Theorem 4 in [3], which is a generalization of Theorem 13.6 in [4], from $\mathbb{D}([0, T], \mathbb{R})$ to $\mathbb{D}([0, T] \times [0, T'], \mathbb{R})$.

We need to first introduce the following concepts defined in [3]. A block B in $[0, T] \times [0, T']$ is a subset of $[0, T] \times [0, T']$ of the form $(s, t] \times (x, y]$; two blocks B and C in $[0, T] \times [0, T']$ are said to be neighboring blocks if they share a common edge. Note that there are only two kinds of neighboring blocks in $[0, T] \times [0, T']$, (i) the first kind: $B = (s, t] \times (x, y]$ and $C = (s, t] \times (y, z]$ and (ii) the second kind: $B = (s, t] \times (x, y]$ and $C = (r, s] \times (x, y]$, for $r < s < t$ and $x < y < z$. For each block $B = (s, t] \times (x, y]$, define $X(B) := X(t, y) - X(t, x) - X(s, y) + X(s, x)$ be the increment of X around B for stochastic process X .

Theorem 4.2. ([3, Theorem 4])

There exists a stochastic process with sample paths in $\mathbb{D}([0, T] \times [0, T'], \mathbb{R})$, whose consistent finite dimensional distributions are given by $(X(t_1, y_1), \dots, X(t_k, y_k))$ for each k -tuple $(t_1, y_1), \dots, (t_k, y_k)$ in $[0, T] \times [0, T']$ for some stochastic process X if

- (i) $P(X(t, 0) = 0) = 1$ and $P(X(0, y) = 0) = 1$ for each $t \in [0, T]$ and $y \in [0, T']$,
- (ii) for each $\epsilon > 0$,

$$\lim_{h_1, h_2 \rightarrow 0+} P(|X(t + h_1, y + h_2) - X(t, y)| \geq \epsilon) = 0, \quad 0 \leq t < T, \quad 0 \leq y \leq T', \quad (4.4)$$

- (iii) for each $\epsilon > 0$,

$$\lim_{t \rightarrow T-} P(|X(t, y) - X(T, y)| \geq \epsilon) = 0, \quad 0 \leq y \leq T', \quad (4.5)$$

and

$$\lim_{y \rightarrow T'-} P(|X(t, y) - X(t, T')| \geq \epsilon) = 0, \quad 0 \leq t \leq T, \quad (4.6)$$

- (iv) there exists a measure μ on $[0, T] \times [0, T']$ with continuous marginals such that

$$E[X(B)^2 X(C)^2] \leq \mu(B)\mu(C), \quad (4.7)$$

for all pairs of neighboring blocks B and C in $[0, T] \times [0, T']$.

We remark that the condition (4.7) we provide here is more restrictive than that in Theorem 4 in [3]; see inequalities (2) and (3) in [3] for details.

We next discuss some properties of functions in \mathbb{D} and $\mathbb{D}_{\mathbb{D}}$ and make some observations for their convergence in the Skorohod J_1 topology and the associated weak convergence (following the notation in [4]). Recall that the modulus of continuity of a function $x(\cdot)$ on $[0, T]$ is defined by

$$\omega_x(\delta, T) := \sup_{|s-t| \leq \delta, s, t \in [0, T]} |x(s) - x(t)|. \quad (4.8)$$

For any $T > 0$, call a set $\{t_i : 1 \leq i \leq v\}$ δ -sparse if it satisfies $0 = t_0 < t_1 < \dots < t_v = T$ and $\min_{1 \leq i \leq v} (t_i - t_{i-1}) > \delta$. Define for $T > 0$ and $0 < \delta < T$,

$$\omega'_x(\delta, T) := \inf_{\{t_i\}} \max_{1 \leq i \leq v} \sup_{r, s \in [t_{i-1}, t_i)} |x(r) - x(s)|, \quad (4.9)$$

where the infimum is taken over all δ -sparse set. Define the maximum (absolute) jump in x on $[0, T]$ for any $x \in \mathbb{D}$:

$$j(x, T) := \sup_{0 < t \leq T} |x(t) - x(t-)|; \quad (4.10)$$

the supremum is achieved because only finitely many jumps can exceed a given positive number for functions in \mathbb{D} .

For a two-parameter function $x(t, y) \in \mathbb{D}([0, T], \mathbb{D}([0, T'], \mathbb{R}))$, similarly as (4.8), (4.9) and (4.10), define

$$\omega_x(\delta, T, T') := \sup_{|s-t| \leq \delta, s, t \in [0, T]} d_{J_1}^{T'}(x(s, \cdot), x(t, \cdot)), \quad (4.11)$$

$$\omega'_x(\delta, T, T') := \inf_{\{t_i\}} \max_{1 \leq i \leq v} \sup_{r, s \in [t_{i-1}, t_i)} d_{J_1}^{T'}(x(r, \cdot), x(s, \cdot)), \quad (4.12)$$

and

$$j(x, T, T') := \sup_{0 < t \leq T} d_{J_1}^{T'}(x(t, \cdot), x(t-, \cdot)), \quad (4.13)$$

where for $x, z \in \mathbb{D}([0, T'], \mathbb{R})$, $d_{J_1}^{T'}(x, z)$ is the standard Skorohod J_1 metric on $\mathbb{D}([0, T'], \mathbb{R})$.

We make the following observations:

(i) (*Comparison of the moduli ω and ω' .*) For $x \in \mathbb{D}$,

$$\begin{aligned} \omega'_x(\delta, T) &\leq \omega_x(2\delta, T) \quad \text{if } \delta < T/2, \\ \omega_x(\delta, T) &\leq 2\omega'_x(\delta, T) + j(x, T) \quad \text{if } \delta > 0. \end{aligned} \quad (4.14)$$

See Section 12 in [4]. By a similar argument, for $x \in \mathbb{D}_{\mathbb{D}}$, we obtain that

$$\begin{aligned} \omega'_x(\delta, T, T') &\leq \omega_x(2\delta, T, T') \quad \text{if } \delta < T/2, \\ \omega_x(\delta, T, T') &\leq 2\omega'_x(\delta, T, T') + j(x, T, T') \quad \text{if } \delta > 0. \end{aligned} \quad (4.15)$$

(ii) (*Equivalence of the moduli ω and ω' for continuous functions.*) Since $j(x, T) = 0$ for $x \in \mathbb{C}$, the moduli $\omega_x(\delta, T)$ in (4.8) and $\omega'_x(\delta, T)$ in (4.9) are essentially the same for $x \in \mathbb{C}$ (page 123 in [4]). By a similar argument, since $j(x, T, T') = 0$ for $x \in \mathbb{C}_{\mathbb{C}}$, the moduli $\omega_x(\delta, T, T')$ in (4.11) and $\omega'_x(\delta, T, T')$ in (4.12) are also essentially the same for $x \in \mathbb{C}_{\mathbb{C}}$ (and \mathbb{C}_2).

(iii) (*Criteria for weak convergence.*) The sufficient and necessary conditions for weak convergence of processes $X^n \Rightarrow X$ in $\mathbb{D}(\mathbb{R}_+, \mathcal{S})$ are given by 1) convergence of finite dimensional distributions and 2)

$$\lim_{\delta \rightarrow 0} \limsup_n P(\omega'_{X^n}(\delta) \geq \epsilon) = 0 \quad \text{as } n \rightarrow \infty, \quad (4.16)$$

where $\omega'_x(\delta)$ is equal to $\omega'_x(\delta, T)$ in (4.9) for $\mathcal{S} = \mathbb{R}$ and $\omega'_x(\delta, T, T')$ in (4.12) for $\mathcal{S} = \mathbb{D}$. See, e.g., Theorem 13.1-13.2 in [4] and Corollary 3.7.4 in [9]. When the limit X is continuous, the condition in (4.16) can be replaced by

$$\lim_{\delta \rightarrow 0} \limsup_n P(\omega_{X^n}(\delta) \geq \epsilon) = 0 \quad \text{as } n \rightarrow \infty, \quad (4.17)$$

where $\omega_x(\delta)$ is equal to $\omega_x(\delta, T)$ in (4.8) for $\mathcal{S} = \mathbb{R}$ and $\omega_x(\delta, T, T')$ in (4.11) for $\mathcal{S} = \mathbb{D}$. Moreover, when $X \in \mathbb{C}_{\mathbb{C}}$ (and in \mathbb{C}_2), the metric $d_{J_1}^{T'}$ in (4.11) can be

replaced by the uniform metric to prove the weak convergence. See Theorem 13.1-13.4 in [4] for a detailed discussion, and the argument can be easily extended to functions in $\mathbb{D}_{\mathbb{D}}$.

4.2. Moment bounds for two-parameter processes with the method of chaining.

We introduce an important result on the moments of the supremum norm of two-parameter stochastic processes in any finite time interval provided some moment conditions on their increments. This is obtained by employing the moment bounds for general stochastic processes resulting from the *method of chaining* (see a good review in [42, 43]).

Recall a semimetric satisfies all conditions of a metric except (possibly) the triangle inequality. For a semimetric space (\mathbb{T}, d) , define the covering number $N(\epsilon, d)$ as the minimal number of balls of radius ϵ needed to cover \mathbb{T} . In this paper, we use $\mathbb{T} = [0, T]$ for $T > 0$. The following theorem is obtained by choosing the function $\phi(x) = x^p$ for $p > 1$ in Theorem 2.2.4 and Corollary 2.2.5 in [44].

Theorem 4.3. *Let X be a real-valued, separable stochastic process on $[0, T]$. Suppose that*

$$E[|X(t) - X(s)|^p] \leq C(d(s, t))^p, \quad s, t \in [0, T], \quad (4.18)$$

for some semimetric d on $[0, T]$ and some positive constants $p > 1$ and $C > 0$. Let $d(T) := \sup_{s, t \in [0, T]} d(s, t)$ denote the diameter of $[0, T]$ under the semimetric d . Then, for any $\zeta, \delta > 0$,

$$E \left[\sup_{\substack{d(s, t) \leq \delta, \\ s, t \in [0, T]}} |X(t) - X(s)|^p \right] \leq \hat{K} \left(\int_0^\zeta (N(\epsilon/2, d))^{1/p} d\epsilon + \delta (N(\zeta/2, d))^{2/p} \right)^p, \quad (4.19)$$

for a positive constant \hat{K} depending only on p and C . Moreover, the constant \hat{K} (depending only on p and C) can be chosen such that

$$E \left[\sup_{s, t \in [0, T]} |X(t) - X(s)|^p \right] \leq \hat{K} \left(\int_0^{d(T)} (N(\epsilon/2, d))^{1/p} d\epsilon \right)^p. \quad (4.20)$$

Theorem 4.3 would be more convenient for our purpose if we could calculate the diameter and the covering number explicitly for some given semimetric d . It turns out that for our model, a proper semimetric can be chosen for the new class of sequential empirical processes and its limit (see Definition 5.1, Proposition 5.1 and the proof of Theorem 5.1, and also see more discussions in Section 8).

Proposition 4.1. *Let $X(t, y)$ be a real-valued, separable two-parameter stochastic process on $[0, T] \times [0, T']$. For $0 \leq s < t \leq T$, define $Z_{s,t}(y) := X(t, y) - X(s, y)$ for $y \in [0, T']$. Suppose that*

$$E[|Z_{s,t}(y) - Z_{s,t}(x)|^p] \leq C(d_{s,t}(x, y))^p, \quad \text{for } x, y \in [0, T'], \quad (4.21)$$

where C is a positive constant, $d_{s,t}(x, y)$ is a semimetric on $[0, T']$ such that the diameter $d_{s,t}(T')$ of $[0, T']$ under this semimetric is equal to $d_{s,t}(0, T')$, and the covering number

$$N(\epsilon, d_{s,t}) \leq \left\lceil \frac{d_{s,t}(0, T')}{2\epsilon} \right\rceil + 1. \quad (4.22)$$

Then,

$$E \left[\sup_{x, y \in [0, T']} |Z_{s,t}(y) - Z_{s,t}(x)|^p \right] \leq K(d_{s,t}(0, T'))^p, \quad (4.23)$$

for some constant $K > 0$ depending only on p and C . The same bound holds for $0 \leq t < s \leq T$ by defining a semimetric $d_{t,s}$ symmetrically.

Proof. By (4.20) in Theorem 4.3, we have

$$\begin{aligned} & E \left[\sup_{x,y \in [0,T']} |Z_{s,t}(y) - Z_{s,t}(x)|^p \right] \\ & \leq \left(\hat{K} \int_0^{d_{s,t}(0,T')} (N(\epsilon/2, d_{s,t}))^{1/p} d\epsilon \right)^p \\ & \leq \left(\hat{K} \int_0^{d_{s,t}(0,T')} ([d_{s,t}(0,T')/\epsilon] + 1)^{1/p} d\epsilon \right)^p \\ & \leq \left(\hat{K} \int_0^{d_{s,t}(0,T')} (2d_{s,t}(0,T')/\epsilon)^{1/p} d\epsilon \right)^p \\ & = \frac{2\hat{K}^p}{(1-1/p)^p} (d_{s,t}(0,T'))^p, \end{aligned}$$

where \hat{K} depends on p and C , and the claim follows by defining $K = 2\hat{K}^p/(1-1/p)^p$. \square

4.3. A sample path property of two-parameter Gaussian processes. Recall that a real-valued stochastic process X defined on \mathbb{R}_+ is *stochastic continuous* if for any $\epsilon > 0$,

$$\lim_{h \rightarrow 0} P(|X(t+h) - X(t)| \geq \epsilon) = 0, \quad t \geq 0 \tag{4.24}$$

and is *continuous in quadratic mean* if (and only if) for all $t > 0$,

$$\lim_{s \rightarrow t} E[|X(t) - X(s)|^2] = 0. \tag{4.25}$$

It is well known that a real-valued Gaussian process is continuous in quadratic mean if and only if it is stochastically continuous. We quote the following lemma in [11].

Lemma 4.1. ([11, Theorem 1]) *If a real-valued Gaussian process with sample paths in \mathbb{D} is stochastically continuous, then it has sample paths in \mathbb{C} a.s.*

We now consider real-valued two-parameter Gaussian processes defined on \mathbb{R}_+^2 . Recall that a two-parameter Gaussian process X defined on \mathbb{R}_+^2 is *continuous in quadratic mean* if (4.25) holds for $s, t \in \mathbb{R}_+^2$. Also, recall that a function x is in \mathbb{D}_2 if at each $t \in \mathbb{R}_+^2$, $x_{\mathcal{Q}}(t) := \lim_{s \in \mathcal{Q}} x(s)$ exists for each of the four quadrants \mathcal{Q} at t and $x(t) = x_{\mathcal{Q}_{\geq, \geq}}(t)$ where $\mathcal{Q}_{\geq, \geq}$ represents the upper right quadrant, and $X(t) \equiv 0$ if either coordinate of t equals to zero. If $x(t) = x_{\mathcal{Q}}(t)$ for each of the four quadrants \mathcal{Q} , then $x \in \mathbb{C}_2$. The next lemma generalizes Lemma 4.1 to two-parameter Gaussian processes. Its proof is similar to that of Theorem 1 in [11], and can be found in the Appendix.

Lemma 4.2. *Let X be a separable mean-zero Gaussian process with sample paths in \mathbb{D}_2 . If X is continuous in quadratic mean, then it has sample paths in \mathbb{C}_2 a.s.*

5. A NEW CLASS OF SEQUENTIAL EMPIRICAL PROCESSES

Define the two-parameter process $\hat{V}^n := \{\hat{V}^n(t, x) : t, x \geq 0\}$ by

$$\begin{aligned} \hat{V}^n(t, x) &:= \frac{1}{\sqrt{n}} \sum_{i=1}^{A^n(t)} (\mathbf{1}(\eta_i(\tau_i^n) > x - \tau_i^n) - F_{\tau_i^n}^c(x - \tau_i^n)) \\ &= -\frac{1}{\sqrt{n}} \sum_{i=1}^{A^n(t)} (\mathbf{1}(\eta_i(\tau_i^n) \leq x - \tau_i^n) - F_{\tau_i^n}(x - \tau_i^n)), \quad t, x \geq 0. \end{aligned} \quad (5.1)$$

The process \hat{V}^n is convenient to prove the convergence of the two-parameter processes $\hat{X}^{n,e}$ and $\hat{X}^{n,r}$, since the service components of those two processes can be expressed as simple functionals of \hat{V}^n ; see equations (6.11) and (6.12).

Let $\hat{V} := \{\hat{V}(t, x) : t, x \geq 0\}$ be a Gaussian process with mean zero and covariance function

$$\text{Cov}(\hat{V}(t, y), \hat{V}(s, x)) = \int_0^{t \wedge s} F_u(x \wedge y - u) F_u^c(x \vee y - u) d\Lambda(u), \quad (5.2)$$

for $t, s \geq 0$ and $x, y \geq 0$. We show the weak convergence of \hat{V}^n to \hat{V} in the following theorem, whose proof can be found in Section 7. It is worth noting that the bounded variation condition on $\tilde{F}_x(\cdot)$ in (2.6) of Assumption 3 is not required in the proof of the theorem.

Theorem 5.1. *Under Assumptions 1–2, $\hat{V}^n \Rightarrow \hat{V}$ in $\mathbb{D}_{\mathbb{D}}$ as $n \rightarrow \infty$, and \hat{V} has continuous sample paths.*

Note that the sample paths of \hat{V}^n lie in both $\mathbb{D}_{\mathbb{D}}$ and \mathbb{D}_2 . We choose to work with the space $\mathbb{D}_{\mathbb{D}}$ in the proof of Theorem 5.1. Our proof relies on the convergence criterion for processes in the space $\mathbb{D}([0, T], \mathcal{S})$ stated in Theorem 4.1, for which the moment bounds for the two-parameter processes discussed in Section 4.2 play a key role. Moreover, in the proof of Theorem 3.2, we use the weak convergence criterion for processes in $\mathbb{D}_{\mathbb{D}}$ via the modulus of continuity in (4.17) when the limit process is continuous. Specifically, in the proof of Lemma 6.4 for the convergence of the processes $(\hat{X}_2^{n,r}, \hat{X}_2^{n,e})$, we rely on the relationship between \hat{V}^n and $(\hat{X}_2^{n,r}, \hat{X}_2^{n,e})$ in (6.11) and (6.12), and the property of \hat{V}^n in (6.17) as a result of the convergence of $\hat{V}^n \Rightarrow \hat{V}$ in $\mathbb{D}_{\mathbb{D}}$. One might also prove the convergence $\hat{V}^n \Rightarrow \hat{V}$ in \mathbb{D}_2 by employing a convergence criterion for processes in the space \mathbb{D}_2 (see, e.g., [41]).

Another relevant sequential empirical process, which is a variation of the process \hat{V}^n , may be of independent interest, although not directly applied to the queueing model. Define a sequential empirical process $\hat{W}^n := \{\hat{W}^n(t, x) : t, x \geq 0\}$ by

$$\hat{W}^n(t, x) := \frac{1}{\sqrt{n}} \sum_{i=1}^{A^n(t)} (\mathbf{1}(\eta_i(\tau_i^n) \leq x) - F_{\tau_i^n}(x)), \quad t, x \geq 0. \quad (5.3)$$

Let $\hat{W} := \{\hat{W}(t, x) : t, x \geq 0\}$ be a two-parameter Gaussian process with mean zero and covariance function

$$\text{Cov}(\hat{W}(t, y), \hat{W}(s, x)) = \int_0^{t \wedge s} F_u(x \wedge y) F_u^c(x \vee y) d\Lambda(u), \quad (5.4)$$

for $t, s \geq 0$ and $x, y \geq 0$. By a slight modification of the proof of Theorem 5.1, we also obtain the following FCLT.

Theorem 5.2. *Under Assumptions 1–2, $\hat{W}^n \Rightarrow \hat{W}$ in $\mathbb{D}_{\mathbb{D}}$ as $n \rightarrow \infty$, and \hat{W} has continuous sample paths.*

Given the conditions on $\{\eta_i(\tau_i^n) : i \geq 1\}$ in Assumptions 1–2, we regard the process \hat{V}^n and \hat{W}^n as a new class of sequential empirical processes with conditionally independent variables of non-stationary (time-dependent) distributions given a non-stationary arrival process. To our best knowledge, this is the first time that such non-stationary sequential empirical processes are introduced in the literature.

We apply Proposition 4.1 to obtain a moment bound for the two-parameter Gaussian process $\hat{V}(t, x)$. A similar bound can be also obtained for $\hat{W}(t, x)$.

Definition 5.1. For any $0 \leq s < t \leq T$, define a semimetric $d_{s,t}(x, y)$ on $[0, T']$ as follows: for $0 \leq x \leq y \leq T'$, let

$$d_{s,t}(x, y) := \left((t-s) \wedge (y-x) + \int_s^t [F_u(y-u) - F_u(x-u)] d\Lambda(u) \right)^{1/2}, \quad (5.5)$$

and for $0 \leq y \leq x \leq T'$, by symmetry, let

$$d_{s,t}(x, y) := d_{s,t}(y, x). \quad (5.6)$$

It is easy to check that $d_{s,t}(x, y)$ defined in (5.5)–(5.6) is indeed a semimetric on $[0, T']$ for any $T' > 0$. First, $d_{s,t}(x, y) \geq 0$. Second, $d_{s,t}(x, y) = d_{s,t}(y, x)$. Third, for each $0 \leq s < t \leq T$, $d_{s,t}(x, y) = 0$ if and only if $x = y$, since $t - s$ is strictly positive. We further observe that the diameter of $[0, T']$ under $d_{s,t}$ is equal to

$$d_{s,t}(0, T') = \left((t-s) \wedge T' + \int_s^t F_u(T'-u) d\Lambda(u) \right)^{1/2}, \quad (5.7)$$

and the covering number satisfies (4.22).

Proposition 5.1. *The two-parameter Gaussian process \hat{V} satisfies*

$$E \left[\sup_{x \in [0, T']} |\hat{V}(t, x) - \hat{V}(s, x)|^p \right] \leq \check{K} |t-s + (\Lambda(t) - \Lambda(s))|^{p/2}, \quad (5.8)$$

for $p = 2, 4$, and some constant $\check{K} > 0$.

Proof. We prove the case when $p = 4$. The case when $p = 2$ follows from a similar argument. Without loss of generality, we only prove the bound for $0 \leq s < t \leq T$. Let $Z_{s,t}[\hat{V}](y) := \hat{V}(t, y) - \hat{V}(s, y)$ for $y \in [0, T']$. By Proposition 4.1, it suffices to show that

$$E[|Z_{s,t}[\hat{V}](y) - Z_{s,t}[\hat{V}](x)|^4] \leq \tilde{C}(d_{s,t}(x, y))^4, \quad (5.9)$$

for some constant $\tilde{C} > 0$ and $d_{s,t}(x, y)$ given in Definition 5.1. By direct calculation, we obtain that

$$\begin{aligned} & E[|Z_{s,t}[\hat{V}](y) - Z_{s,t}[\hat{V}](x)|^4] \\ &= 3 \left(\int_s^t [F_u(y-u) - F_u(x-u)] [1 - F_u(y-u) + F_u(x-u)] d\Lambda(u) \right)^2 \\ &\leq 3 \left(\int_s^t [F_u(y-u) - F_u(x-u)] d\Lambda(u) \right)^2 \end{aligned}$$

$$\begin{aligned}
&\leq 3 \left((t-s) \wedge (y-x) + \int_s^t [F_u(y-u) - F_u(x-u)] d\Lambda(u) \right)^2 \\
&= 3(d_{s,t}(x,y))^4.
\end{aligned} \tag{5.10}$$

Thus, by Proposition 4.1, and the fact that $\hat{V}(t,0) = 0$ a.s., we obtain

$$\begin{aligned}
&E \left[\sup_{x \in [0, T^t]} |\hat{V}(t,x) - \hat{V}(s,x)|^4 \right] \\
&\leq \check{K}(d_{s,t}(x,y))^4 \\
&\leq \check{K} \left(t-s + \int_s^t F_u(T^t - u) d\Lambda(u) \right)^2 \\
&\leq \check{K}(t-s + (\Lambda(t) - \Lambda(s)))^2,
\end{aligned} \tag{5.11}$$

for some $\check{K} > 0$. This completes the proof. \square

6. PROOF OF THEOREM 3.2

In this section we prove Theorem 3.2. Theorem 3.1 follows from Theorem 3.2 and thus its proof is omitted. We first provide a representation of the diffusion-scaled processes $\hat{X}^{n,r}$ and $\hat{X}^{n,e}$, which follows from simple calculations.

Lemma 6.1. *The diffusion-scaled processes $\hat{X}^{n,r}$ and $\hat{X}^{n,e}$ can be represented as*

$$\hat{X}^{n,r}(t,y) = \hat{X}_{0,1}^{n,r}(t,y) + \hat{X}_{0,2}^{n,r}(t,y) + \hat{X}_1^{n,r}(t,y) + \hat{X}_2^{n,r}(t,y), \quad t, y \geq 0, \tag{6.1}$$

$$\hat{X}^{n,e}(t,y) = \hat{X}_{0,1}^{n,e}(t,y) + \hat{X}_{0,2}^{n,e}(t,y) + \hat{X}_1^{n,e}(t,y) + \hat{X}_2^{n,e}(t,y), \quad t, y \geq 0, \tag{6.2}$$

where

$$\hat{X}_{0,1}^{n,r}(t,y) = \int_0^\infty G_u^c(t+y) d\hat{X}^{n,e}(0,u), \quad \hat{X}_{0,1}^{n,e}(t,y) = \int_0^{(y-t)^+} G_u^c(t) d\hat{X}^{n,e}(0,u), \tag{6.3}$$

$$\hat{X}_1^{n,r}(t,y) = \int_0^t F_u^c(t+y-u) d\hat{A}^n(u), \quad \hat{X}_1^{n,e}(t,y) = \int_{(t-y)^+}^t F_u^c(t-u) d\hat{A}^n(u), \tag{6.4}$$

$$\hat{X}_{0,2}^{n,r}(t,y) = -\frac{1}{\sqrt{n}} \sum_{i=1}^{X^{n,e}(0,\infty)} (\mathbf{1}(\tilde{\eta}_i(\tilde{\tau}_i^n) \leq t+y) - G_{\tilde{\tau}_i^n}(t+y)), \tag{6.5}$$

$$\hat{X}_{0,2}^{n,e}(t,y) = -\frac{1}{\sqrt{n}} \sum_{i=1}^{X^{n,e}(0,(y-t)^+)} (\mathbf{1}(\tilde{\eta}_i(\tilde{\tau}_i^n) \leq t) - G_{\tilde{\tau}_i^n}(t)), \tag{6.6}$$

$$\hat{X}_2^{n,r}(t,y) = -\frac{1}{\sqrt{n}} \sum_{i=1}^{A^n(t)} (\mathbf{1}(\eta_i(\tau_i^n) \leq t+y-\tau_i^n) - F_{\tau_i^n}(t+y-\tau_i^n)), \tag{6.7}$$

$$\hat{X}_2^{n,e}(t,y) = -\frac{1}{\sqrt{n}} \sum_{i=A^n((t-y)^-)+1}^{A^n(t)} (\mathbf{1}(\eta_i(\tau_i^n) \leq t-\tau_i^n) - F_{\tau_i^n}(t-\tau_i^n)). \tag{6.8}$$

It is evident that the sample paths of $\hat{X}^{n,r}(t, y)$ and $\hat{X}^{n,e}(t, y)$ are in $\mathbb{D}_{\mathbb{D}}$ (but not in \mathbb{D}_2) by definition and by the explicit representations in Lemma 6.1. Note that we will show that the limit processes have continuous sample paths a.s. and thus are in the space $\mathbb{C}_{\mathbb{C}}$ (and \mathbb{C}_2).

To prove Theorem 3.2, we take the following roadmap:

- (i) Joint convergence of $\hat{X}_{0,1}^{n,r}$, $\hat{X}_1^{n,r}$, $\hat{X}_{0,1}^{n,e}$ and $\hat{X}_1^{n,e}$ in $\mathbb{D}_{\mathbb{D}}$. (Lemmas 6.2–6.3)
- (ii) Joint convergence of $\hat{X}_2^{n,r}$ and $\hat{X}_2^{n,e}$ in $\mathbb{D}_{\mathbb{D}}$. (Lemma 6.4)
- (iii) Joint convergence of $\hat{X}_{0,2}^{n,r}$ and $\hat{X}_{0,2}^{n,e}$ in $\mathbb{D}_{\mathbb{D}}$. (Lemmas 6.5–6.6)
- (iv) Completing the proof of Theorem 3.2.

To prove the convergence of $\hat{X}_{0,1}^{n,r}$, $\hat{X}_1^{n,r}$, $\hat{X}_{0,1}^{n,e}$ and $\hat{X}_1^{n,e}$, we apply the continuous mapping theorem. For that, define four mappings: $\phi^r, \psi^r, \phi^e, \psi^e : \mathbb{D} \rightarrow \mathbb{D}_{\mathbb{D}}$ by

$$\phi^r(z)(t, y) = \int_0^{\infty} G_s^c(t+y) dz(s), \quad \phi^e(z)(t, y) = \int_0^{(y-t)^+} G_s^c(t) dz(s), \quad (6.9)$$

$$\psi^r(z)(t, y) = \int_0^t F_s^c(t+y-s) dz(s), \quad \psi^e(z)(t, y) = \int_{(t-y)^-}^t F_s^c(t-s) dz(s), \quad (6.10)$$

for $t, y \geq 0$. The continuity of the four mappings is stated in the following lemma, whose proof can be found in Section 9.

Lemma 6.2. *Suppose that Assumptions 2–3 and 5–6 hold, and that $z^n \rightarrow z$ in \mathbb{D} as $n \rightarrow \infty$ and $z \in \mathbb{C}$. Then $(\phi^e(z^n), \psi^r(z^n), \psi^e(z^n)) \rightarrow (\phi^e(z), \psi^r(z), \psi^e(z))$ in $(\mathbb{D}_{\mathbb{D}})^3$ as $n \rightarrow \infty$. If, in addition, there exists \bar{y} such that for all $y \geq \bar{y}$, $z^n(y) = z^n(\bar{y})$ for $n \geq 1$ and $z(y) = z(\bar{y})$, then $\phi^r(z^n) \rightarrow \phi^r(z)$ in $\mathbb{D}_{\mathbb{D}}$ as $n \rightarrow \infty$, jointly with the convergence of the other three mappings. Moreover, $\phi^r(z), \psi^r(z), \phi^e(z), \psi^e(z) \in \mathbb{C}_{\mathbb{C}}$ (and \mathbb{C}_2).*

Lemma 6.3. *Under Assumptions 1–6,*

$$(\hat{X}_{0,1}^{n,r}, \hat{X}_1^{n,r}, \hat{X}_{0,1}^{n,e}, \hat{X}_1^{n,e}) \Rightarrow (\hat{X}_{0,1}^r, \hat{X}_1^r, \hat{X}_{0,1}^e, \hat{X}_1^e) \quad \text{in } (\mathbb{D}_{\mathbb{D}})^4 \quad \text{as } n \rightarrow \infty.$$

Proof. First recall the representations in (6.3) and (6.4). By the convergence of \hat{A}^n in Assumption 1, applying the continuous mapping theorem and Lemma 6.2, we obtain the joint convergence $(\hat{X}_1^{n,r}, \hat{X}_1^{n,e}) \Rightarrow (\hat{X}_1^r, \hat{X}_1^e)$ in $(\mathbb{D}_{\mathbb{D}})^2$ as $n \rightarrow \infty$. By the convergence of $\hat{X}^{n,e}(0, \cdot)$ in Assumption 4, similarly, we obtain the joint convergence $(\hat{X}_{0,1}^{n,r}, \hat{X}_{0,1}^{n,e}) \Rightarrow (\hat{X}_{0,1}^r, \hat{X}_{0,1}^e)$ in $(\mathbb{D}_{\mathbb{D}})^2$ as $n \rightarrow \infty$. Since the processes \hat{A}^n and $\hat{X}^{n,e}(0, \cdot)$ are independent, we obtain the joint convergence result in the lemma. \square

We next prove the joint convergence of $\hat{X}_2^{n,r}$ and $\hat{X}_2^{n,e}$ in $\mathbb{D}_{\mathbb{D}}$. Recall the discussions on the weak convergence of the processes in $\mathbb{D}_{\mathbb{D}}$ in Section 4.1. Recall the modulus $\omega_x(\delta, T, T')$ in (4.11).

Lemma 6.4. *Under Assumptions 1–2,*

$$(\hat{X}_2^{n,r}, \hat{X}_2^{n,e}) \Rightarrow (\hat{X}_2^r, \hat{X}_2^e) \quad \text{in } (\mathbb{D}_{\mathbb{D}})^2 \quad \text{as } n \rightarrow \infty.$$

Proof. Observe that

$$\hat{X}_2^{n,r}(t, y) = \hat{V}^n(t, t+y), \quad \text{a.s.}, \quad (6.11)$$

and

$$\hat{X}_2^{n,e}(t, y) = \hat{V}^n(t, t) - \hat{V}^n((t-y)^-, t), \quad \text{a.s.} \quad (6.12)$$

By comparing the covariance functions, it is easy to verify that $\hat{X}_2^r(t, y)$ and $\hat{V}(t, t + y)$ represent the same Gaussian process, and so do $\hat{X}_2^e(t, y)$ and $\hat{V}(t, t) - \hat{V}(t - y, t)$. Since $\hat{V}(t, y) \in \mathbb{C}_2$, we have $\hat{V}(t, t + y) \in \mathbb{C}_2$ and $\hat{V}(t, t) - \hat{V}(t - y, t) \in \mathbb{C}_2$, and thus, $\hat{X}_2^r(t, y) \in \mathbb{C}_2$ and $\hat{X}_2^e(t, y) \in \mathbb{C}_2$. We can then deduce the weak convergence of $\hat{X}_2^{n,r}$ and $\hat{X}_2^{n,e}$ from that of \hat{V}^n . By (6.11) and (6.12), it suffices to prove their convergence separately.

We consider the processes on $[0, T] \times [0, T']$ for $T, T' > 0$. Since the finite dimensional distributions of \hat{V}^n converge to those of \hat{V} , by (6.11), it is easy to see that the finite dimensional distributions of $\hat{X}_2^{n,r}$ converge to those of \hat{X}_2^r . Since the limit $\hat{X}_2^r \in \mathbb{C}_{\mathbb{C}}$, it suffices to show for each $\epsilon > 0$,

$$\lim_{\delta \rightarrow 0} \limsup_n P \left(\omega_{\hat{X}_2^{n,r}}(\delta, T, T') \geq \epsilon \right) = 0, \quad (6.13)$$

For $\delta > 0$, define

$$\mathcal{R}_\delta := \{r, s \in [0, T], x, y \in [0, T'] : |r - s|^2 + |x - y|^2 \leq 2\delta^2\} \subset [0, T] \times [0, T'].$$

By definition in (4.11) and the relation in (6.11),

$$\begin{aligned} \omega_{\hat{X}_2^{n,r}}(\delta, T, T') &= \sup_{|r-s| \leq \delta, r, s \in [0, T]} d_{J_1}^{T'}(\hat{V}^n(r, r + \cdot), \hat{V}^n(s, s + \cdot)) \\ &\leq \sup_{|r-s| \leq \delta, r, s \in [0, T]} \sup_{x \in [0, T']} |\hat{V}^n(r, r + x) - \hat{V}^n(s, s + x)| \\ &\leq \sup_{\mathcal{R}_\delta} |\hat{V}^n(r, x) - \hat{V}^n(s, y)|. \end{aligned} \quad (6.14)$$

By Theorem 5.1 and $\hat{V} \in \mathbb{C}_2$, we have (see, e.g., section 13.4 in [46], the Lipschitz property of the supremum function with the uniform norm) that for each $\delta > 0$,

$$\sup_{\mathcal{R}_\delta} |\hat{V}^n(r, x) - \hat{V}^n(s, y)| \Rightarrow \sup_{\mathcal{R}_\delta} |\hat{V}(r, x) - \hat{V}(s, y)| \quad \text{as } n \rightarrow \infty. \quad (6.15)$$

Combining the weak convergence above with (6.13) and (6.14), we see that it suffices to show

$$\lim_{\delta \rightarrow 0} P \left(\sup_{\mathcal{R}_\delta} |\hat{V}(r, x) - \hat{V}(s, y)| \geq \epsilon \right) = 0. \quad (6.16)$$

By Theorem 5.1 and the fact that $\hat{V} \in \mathbb{C}_{\mathbb{C}}$, we have for each $\epsilon > 0$,

$$\lim_{\delta \rightarrow 0} \limsup_n P \left(\sup_{|r-s| \leq \delta, r, s \in [0, T]} \sup_{x \in [0, T']} |\hat{V}^n(r, x) - \hat{V}^n(s, x)| \geq \epsilon \right) = 0. \quad (6.17)$$

As the similar argument for (6.15) and (6.16), since $\hat{V}(t, y) \in \mathbb{C}_{\mathbb{C}}$, we have

$$\lim_{\delta \rightarrow 0} P \left(\sup_{|r-s| \leq \delta, r, s \in [0, T]} \sup_{x \in [0, T']} |\hat{V}(r, x) - \hat{V}(s, x)| \geq \epsilon \right) = 0. \quad (6.18)$$

The proof of (6.18) implying (6.16) follows from the same argument as in the proof of the equivalence of \mathbb{C}_2 and $\mathbb{C}_{\mathbb{C}}$ (i.e., $\mathbb{C}_2 \equiv \mathbb{C}_{\mathbb{C}}$) (first considering step functions in the second argument of $\hat{V}(\cdot, \cdot)$ and then taking their uniform limits; see, e.g., [3] or [41]).

We next prove the convergence of $\hat{X}_2^{n,e}$. By (6.12), we have that as in (6.14),

$$\begin{aligned} &\omega_{\hat{X}_2^{n,e}}(\delta, T, T') \\ &= \sup_{|r-s| \leq \delta, r, s \in [0, T]} d_{J_1}^{T'}(\hat{V}^n(r, r) - \hat{V}^n((r - \cdot) -, r), \hat{V}^n(s, s) - \hat{V}^n((s - \cdot) -, s)) \end{aligned}$$

$$\begin{aligned}
&\leq \sup_{|r-s|\leq\delta, r,s\in[0,T]} |\hat{V}^n(r,r) - \hat{V}^n(s,s)| \\
&\quad + \sup_{|r-s|\leq\delta, r,s\in[0,T]} \sup_{x\in[0,T']} |\hat{V}^n((r-x)-,r) - \hat{V}^n((s-x)-,s)| \\
&\leq \sup_{|r-s|\leq\delta, r,s\in[0,T]} |\hat{V}^n(r,r) - \hat{V}^n(s,s)| \\
&\quad + \sup_{|r-s|\leq\delta, r,s\in[0,T]} \sup_{x\in[0,T']} |\hat{V}^n((r-x),r) - \hat{V}^n((s-x),s)| \\
&\quad + \sup_{|r-s|\leq\delta, r,s\in[0,T]} \sup_{x\in[0,T']} |\hat{V}^n((r-x)-,r) - \hat{V}^n(r-x,r)| \\
&\quad + \sup_{|r-s|\leq\delta, r,s\in[0,T]} \sup_{x\in[0,T']} |\hat{V}^n((s-x)-,s) - \hat{V}^n(s-x,s)| \\
&\leq 2 \sup_{\mathcal{R}_\delta} |\hat{V}^n(r,x) - \hat{V}^n(s,y)| + \frac{2}{\sqrt{n}}. \tag{6.19}
\end{aligned}$$

Therefore, the weak convergence of $\hat{X}_2^{n,e}$ reduces to (6.14), and thus the proof of (6.16), which has already been done. Now, the proof is complete. \square

We next prove the joint convergence of the processes $\hat{X}_{0,2}^{n,r}$ and $\hat{X}_{0,2}^{n,e}$ for the initial customers. We define another auxiliary two-parameter process $\hat{V}_0^n = \{\hat{V}_0^n(t,x) : t,x \geq 0\}$:

$$\begin{aligned}
\hat{V}_0^n(t,x) &:= \frac{1}{\sqrt{n}} \sum_{i=1}^{X^{n,e}(0,t)} (\mathbf{1}(\tilde{\eta}_i(\tilde{\tau}_i^n) > x) - G_{\tilde{\tau}_i^n}^c(x)) \\
&= -\frac{1}{\sqrt{n}} \sum_{i=1}^{X^{n,e}(0,t)} (\mathbf{1}(\tilde{\eta}_i(\tilde{\tau}_i^n) \leq x) - G_{\tilde{\tau}_i^n}(x)), \quad t,x \geq 0. \tag{6.20}
\end{aligned}$$

This process \hat{V}_0^n is also a nonstandard sequential empirical process. Let $\hat{V}_0 := \{\hat{V}(t,x) : t,x \geq 0\}$ be a two-parameter Gaussian process with mean zero and covariance function

$$Cov(\hat{V}_0(t,y), \hat{V}_0(s,x)) = \int_0^{t \wedge s} G_u(x \wedge y) G_u^c(x \vee y) d\bar{X}^e(0,u), \tag{6.21}$$

for $t,s \geq 0$ and $x,y \geq 0$. Similar to Definition 5.1, we can define a semimetric $d_{s,t}(x,y)$ on $[0, T']$ for any $0 \leq s < t \leq T$: for $0 \leq x \leq y \leq T'$,

$$d_{s,t}^0(x,y) := \left((t-s) \wedge (y-x) + \int_s^t (G_u(y) - G_u(x)) d\bar{X}^e(0,u) \right)^{1/2}, \tag{6.22}$$

and for $0 \leq y \leq x \leq T'$, by symmetry, let $d_{s,t}^0(x,y) = d_{s,t}^0(y,x)$. Thus, similar to Proposition 5.1, we obtain

$$E \left[\sup_{x \in [0, T']} |\hat{V}_0(t,x) - \hat{V}_0(s,x)|^p \right] \leq \check{K}_0 |t-s + (\bar{X}^e(0,t) - \bar{X}^e(0,s))|^{p/2}, \tag{6.23}$$

for $p = 2, 4$, and some constant $\check{K}_0 > 0$.

Lemma 6.5. *Under Assumptions 4-5, $\hat{V}_0^n \Rightarrow \hat{V}_0$ in $\mathbb{D}_{\mathbb{D}}$ as $n \rightarrow \infty$ and \hat{V}_0 has continuous sample paths.*

Proof. This follows from the same steps as in the proof of convergence $\hat{V}^n \Rightarrow \hat{V}$ in $\mathbb{D}_{\mathbb{D}}$ in Section 5. We only provide a sketch proof here.

Step 1. As in Lemma 7.1, we show that the two-parameter Gaussian process $\hat{V}_0(t, x)$ has sample paths in \mathbb{C}_2 and thus in $\mathbb{C}_{\mathbb{C}}$. The proof follows a similar argument as that of Lemma 7.1, by choosing the measure

$$\mu_0(B) := \sqrt{3} \int_s^t (G_u(y) - G_u(x)) d\bar{X}^e(0, u)$$

for any block $B = (s, t] \times (x, y] \subset [0, T] \times [0, T']$.

Step 2. As in Lemma 7.2, we show that for each $t \geq 0$, $\hat{V}_0^n(t) \Rightarrow \hat{V}_0(t)$ in \mathbb{D} as $n \rightarrow \infty$. The proof follows a similar argument by choosing

$$H_0^t(y) := C_{H,0}^t \int_0^t G_u(y) d\bar{X}^e(0, u) \quad \text{for } y \in [0, T']$$

for some constant $C_{H,0}^t > 0$.

Step 3. As in Theorem 5.1, we show that $\hat{V}_0^n \Rightarrow \hat{V}_0$ in $\mathbb{D}_{\mathbb{D}}$ as $n \rightarrow \infty$ by applying Theorem 4.1. The argument and calculations are similar, by using the semimetric $d_{s,t}^0(x, y)$ defined in (6.22) and choosing the nondecreasing and continuous function

$$H_0(t) := \tilde{K}_0^{1/2}(t + \bar{X}^e(0, t))$$

for some constant $\tilde{K}_0 > 0$.

This completes the proof. \square

Lemma 6.6. *Under Assumptions 4–5,*

$$(\hat{X}_{0,2}^{n,r}, \hat{X}_{0,2}^{n,e}) \Rightarrow (\hat{X}_{0,2}^r, \hat{X}_{0,2}^e) \quad \text{in } (\mathbb{D}_{\mathbb{D}})^2 \quad \text{as } n \rightarrow \infty.$$

Proof. Observe that

$$\hat{X}_{0,2}^{n,r}(t, y) = \hat{V}_0^n(\infty, t + y) = \hat{V}_0^n(\bar{y}, t + y), \quad a.s. \quad (6.24)$$

$$\hat{X}_{0,2}^{n,e}(t, y) = \hat{V}_0^n((y - t)^+, t), \quad a.s. \quad (6.25)$$

Given the convergence of \hat{V}_0^n to \hat{V}_0 in $\mathbb{D}_{\mathbb{D}}$, the joint convergence of $(\hat{X}_{0,2}^{n,r}, \hat{X}_{0,2}^{n,e})$ can be proved similarly as in Lemma 6.4. By (6.24) and (6.25), similarly to (6.14) and (6.19), we obtain

$$\begin{aligned} \omega_{\hat{X}_{0,2}^{n,r}}(\delta, T, T') &= \sup_{|r-s| \leq \delta, r, s \in [0, T]} d_{J_1}^{T'}(\hat{V}_0^n(\bar{y}, r + \cdot), \hat{V}_0^n(\bar{y}, s + \cdot)) \\ &\leq \sup_{|r-s| \leq \delta, r, s \in [0, T]} \sup_{x \in [0, T']} |\hat{V}_0^n(\bar{y}, r + x) - \hat{V}_0^n(\bar{y}, s + x)| \\ &\leq \sup_{|r-s| \leq \delta, r, s \in [0, 2T]} |\hat{V}_0^n(\bar{y}, r) - \hat{V}_0^n(\bar{y}, s)| \end{aligned} \quad (6.26)$$

and

$$\begin{aligned} \omega_{\hat{X}_{0,2}^{n,e}}(\delta, T, T') &= \sup_{|r-s| \leq \delta, r, s \in [0, T]} d_{J_1}^{T'}(\hat{V}_0^n((\cdot - r)^+, r), \hat{V}_0^n((\cdot - s)^+, s)) \\ &\leq \sup_{|r-s| \leq \delta, r, s \in [0, T]} \sup_{x \in [0, T']} |\hat{V}_0^n((x - r)^+, r) - \hat{V}_0^n((x - s)^+, s)| \\ &\leq \sup_{\mathcal{R}_{\delta}} |\hat{V}_0^n(r, x) - \hat{V}_0^n(s, y)|. \end{aligned} \quad (6.27)$$

We then need to show that for any $\epsilon > 0$,

$$\lim_{\delta \rightarrow 0} P \left(\sup_{|r-s| \leq \delta, r, s \in [0, T]} |\hat{V}_0(\bar{y}, r) - \hat{V}_0(\bar{y}, s)| \geq \epsilon \right) = 0, \quad (6.28)$$

$$\lim_{\delta \rightarrow 0} P \left(\sup_{|r-s| \leq \delta, r, s \in [0, T]} \sup_{x \in [0, T']} |\hat{V}_0(r, x) - \hat{V}_0(s, x)| \geq \epsilon \right) = 0. \quad (6.29)$$

Thus, the convergence follows from Lemma 6.5 and a similar argument in Lemma 6.4. \square

Remark 6.1. It is worth noting that the total variation conditions on the functions \tilde{F}_x and \tilde{G}_x in Assumptions 3 and 6, respectively, are only used in the proof of Lemma 6.2, and thus in the proof of Lemma 6.3 for the convergences of the processes $(\hat{X}_{0,1}^{n,r}, \hat{X}_1^{n,r}, \hat{X}_{0,1}^{n,e}, \hat{X}_1^{n,e})$. As already noted above, the total variation condition in (2.6) of Assumption 3 is not required for the proof of the convergence of $\hat{V}^n \Rightarrow \hat{V}$ in \mathbb{D} in Theorem 5.1. As a consequence, in the proofs of the processes $(\hat{X}_2^{n,r}, \hat{X}_2^{n,e})$ in Lemma 6.4 and $(\hat{X}_{0,2}^{n,r}, \hat{X}_{0,2}^{n,e})$ in Lemma 6.6, those total variation conditions in (2.6) and (2.9) of Assumptions 3 and 6 are not required. In fact, in the proofs of Lemmas 6.4–6.6, we only require the functions $F_t(x)$ and $G_t(x)$ to be continuous in x for each t . \square

We are now ready to complete the proof of Theorem 3.2.

Completing the Proof of Theorem 3.2. We first show the joint convergence

$$(\hat{X}_{0,1}^{n,r}, \hat{X}_{0,2}^{n,r}, \hat{X}_1^{n,r}, \hat{X}_2^{n,r}) \Rightarrow (\hat{X}_{0,1}^r, \hat{X}_{0,2}^r, \hat{X}_1^r, \hat{X}_2^r) \quad \text{in } (\mathbb{D})^4 \quad \text{as } n \rightarrow \infty. \quad (6.30)$$

We begin by defining an auxiliary process. Define the two-parameter process $\tilde{X}_2^{n,r} = \{\tilde{X}_2^{n,r}(t, y) : t, y \geq 0\}$ by

$$\tilde{X}_2^{n,r}(t, y) = -\frac{1}{\sqrt{n}} \sum_{i=1}^{[n\Lambda(t)]} (\mathbf{1}(\eta_i(u_i^n) \leq t + y - u_i^n) - F_{u_i^n}(t + y - u_i^n)), \quad (6.31)$$

where $u_i^n = \Lambda^{-1}(\frac{i}{n})$ for $i = 1, \dots, [n\Lambda(t)]$, and Λ^{-1} is the inverse function of Λ . Note that, comparing with the definition of $\hat{X}_2^{n,r}(t, y)$ in (6.7), we replace $A^n(t)$ by $[n\Lambda(t)]$ and τ_i^n by u_i^n in the definition of $\tilde{X}_2^{n,r}$ here. By a similar argument in Lemma 6.4, we have

$$\tilde{X}_2^{n,r} \Rightarrow \hat{X}_2^r \quad \text{in } \mathbb{D} \quad \text{as } n \rightarrow \infty. \quad (6.32)$$

Moreover, since $\tilde{X}_2^{n,r}$ and A^n are independent, we obtain the joint convergence

$$(\hat{X}_1^{n,r}, \tilde{X}_2^{n,r}) \Rightarrow (\hat{X}_1^r, \hat{X}_2^r) \quad \text{in } (\mathbb{D})^2 \quad \text{as } n \rightarrow \infty.$$

Then, by Lemma 6.4 and (6.32), we obtain that for any $\zeta > 0$ and $T, T' > 0$,

$$\lim_{n \rightarrow \infty} P \left(\sup_{t \in [0, T]} \sup_{y \in [0, T']} |\hat{X}_2^{n,r}(t, y) - \tilde{X}_2^{n,r}(t, y)| > \zeta \right) = 0,$$

which completes the proof of the joint convergence

$$(\hat{X}_1^{n,r}, \hat{X}_2^{n,r}) \Rightarrow (\hat{X}_1^r, \hat{X}_2^r) \quad \text{in } (\mathbb{D})^2 \quad \text{as } n \rightarrow \infty.$$

Similarly, we obtain the joint convergence

$$(\hat{X}_{0,1}^{n,r}, \hat{X}_{0,2}^{n,r}) \Rightarrow (\hat{X}_{0,1}^r, \hat{X}_{0,2}^r) \quad \text{in } (\mathbb{D})^2 \quad \text{as } n \rightarrow \infty.$$

Since the initial quantities are mutually independent with new arrival processes and service processes, we have proved the joint convergence in (6.30). By continuity of addition in space $\mathbb{D}_{\mathbb{D}}$ and continuous mapping theorem, we obtain that

$$\hat{X}^{n,r} \Rightarrow \hat{X}^r \quad \text{in } \mathbb{D}_{\mathbb{D}} \quad \text{as } n \rightarrow \infty.$$

Similarly, we also have

$$\hat{X}^{n,e} \Rightarrow \hat{X}^e \quad \text{in } \mathbb{D}_{\mathbb{D}} \quad \text{as } n \rightarrow \infty.$$

In addition, by the joint convergence in Lemmas 6.3, 6.4 and 6.6 and by the above arguments, we also obtain

$$(\hat{X}^{n,e}, \hat{X}^{n,r}) \Rightarrow (\hat{X}^e, \hat{X}^r) \quad \text{in } (\mathbb{D}_{\mathbb{D}})^2 \quad \text{as } n \rightarrow \infty.$$

Recall that $\hat{X}^n(t) = \hat{X}^{n,r}(t, 0)$ for each $t \geq 0$, we obtain that $\hat{X}^n \Rightarrow \hat{X}$ in \mathbb{D} as $n \rightarrow \infty$. Recall that $\hat{D}^n(t) = \hat{X}^n(0) - \hat{X}^n(t) + \hat{A}^n(t)$ for each $t \geq 0$, we obtain that $\hat{D}^n \Rightarrow \hat{D}$ in \mathbb{D} as $n \rightarrow \infty$. The proof of Theorem 3.2 is now complete. \square

7. PROOF OF THEOREM 5.1

In this section we prove Theorem 5.1. We start with a lemma on the sample path property of the limiting two-parameter process \hat{V} . For each fixed $t \geq 0$, we denote $\hat{V}^n(t) = \{\hat{V}^n(t, x) : x \geq 0\}$ and it is an element of \mathbb{D} . Similarly, for each fixed $t \geq 0$, we denote $\hat{V}(t) = \{\hat{V}(t, x) : x \geq 0\}$ and the following lemma shows that $\hat{V}(t, x)$ has sample paths in \mathbb{C}_2 and thus in $\mathbb{C}_{\mathbb{C}}$, and also implies that for each $t \geq 0$, $\hat{V}(t)$ is also an element of \mathbb{D} (actually \mathbb{C}).

Lemma 7.1. *Under Assumptions 1–2, the two-parameter Gaussian process $\hat{V}(t, x)$ is continuous, i.e., it has sample paths in \mathbb{C}_2 , and thus in $\mathbb{C}_{\mathbb{C}}$.*

Proof. Consider $[0, T] \times [0, T']$. Since the covariance function of \hat{V} is continuous, by Lemma 4.2, it suffices to show that $\hat{V}(t, x) \in \mathbb{D}([0, T] \times [0, T'], \mathbb{R})$.

We apply Theorem 4.2. By (5.2), we have $\text{Var}(\hat{V}(t, 0)) = \text{Var}(\hat{V}(0, y)) = 0$ for $0 \leq t \leq T$ and $0 \leq y \leq T'$. Thus, condition (i) of Theorem 4.2 is satisfied. It is easy to see that conditions (ii) and (iii) are also satisfied since the covariance function of \hat{V} is continuous. So we focus on condition (iv). Recall that there are only two kinds of neighboring blocks in $[0, T] \times [0, T']$, we consider the first kind with $B = (s, t] \times (x, y]$ and $C = (r, s] \times (x, y]$ for $r < s < t$ and $x < y$. By Cauchy-Schwarz inequality, it suffices to show that there exists some finite measure μ with continuous marginals on $[0, T] \times [0, T']$ such that

$$E[\hat{V}(B)^4] \leq \mu(B)^2. \tag{7.1}$$

By definition, the left hand side of (7.1) is

$$\begin{aligned} & E[|\hat{V}(t, y) - \hat{V}(s, y) - \hat{V}(t, x) + \hat{V}(s, x)|^4] \\ &= 3(E[|\hat{V}(t, y) - \hat{V}(s, y) - \hat{V}(t, x) + \hat{V}(s, x)|^2])^2 \\ &= 3\left(\int_s^t [F_u(y-u) - F_u(x-u)][1 - F_u(y-u) + F_u(x-u)]d\Lambda(u)\right)^2 \\ &\leq \left(\sqrt{3}\int_s^t [F_u(y-u) - F_u(x-u)]d\Lambda(u)\right)^2, \end{aligned} \tag{7.2}$$

where the first equation holds since the kurtosis of any normal random variable is 3. It is easy to verify that the measure μ on $[0, T] \times [0, T']$ defined by

$$\mu(B) := \sqrt{3} \int_s^t [F_u(y-u) - F_u(x-u)] d\Lambda(u), \quad \forall B = (s, t] \times (x, y] \subset [0, T] \times [0, T'], \quad (7.3)$$

is finite and has continuous marginals. Thus, the condition (7.1) is verified for the first kind of neighboring blocks in $[0, T] \times [0, T']$. A similar argument also verifies it for the second kind of neighboring blocks. This completes the proof. \square

Lemma 7.2. *Under Assumptions 1-2, for each $t \geq 0$, $\hat{V}^n(t) \Rightarrow \hat{V}(t)$ in \mathbb{D} as $n \rightarrow \infty$.*

Proof. It suffices to prove the convergence in $\mathbb{D}[0, T']$ for each $T' > 0$. Fix $t \geq 0$. We apply Theorem 4.1 with $\mathcal{S} = \mathbb{R}$.

We first prove that the condition (4.1) holds, that is, the l -dimensional random variables

$$(\hat{V}^n(t, y_j), 1 \leq j \leq l) \Rightarrow (\hat{V}(t, y_j), 1 \leq j \leq l) \quad \text{in } \mathbb{R}^l \quad \text{as } n \rightarrow \infty, \quad (7.4)$$

for any $0 \leq y_1 \leq \dots \leq y_l \leq T'$ and $l \geq 1$. We first consider the case when $l = 1$ (removing subscript 1 in y_1 for brevity below) and it is easy to generalize to $l > 1$.

To show that $\hat{V}^n(t, y) \Rightarrow \hat{V}(t, y)$, by the continuity theorem (see, e.g., [8]), it suffices to show that the characteristic function of $\hat{V}^n(t, y)$, denoted by $\varphi_{t,y}^n(\theta)$, converges pointwisely to that of $\hat{V}(t, y)$, denoted by $\varphi_{t,y}(\theta)$, and $\varphi_{t,y}(\theta)$ is continuous at $\theta = 0$. Recall the covariance function of \hat{V} in (5.2). For each $t, y \geq 0$, $\hat{V}(t, y)$ is a normal random variable with mean zero and variance $\int_0^t F_u(y-u)F_u^c(y-u)d\Lambda(u)$. Thus we have

$$\varphi_{t,y}(\theta) = E \left[\exp \left(i\theta \hat{V}(t, y) \right) \right] = \exp \left(-\frac{\theta^2}{2} \int_0^t F_u(y-u)F_u^c(y-u)d\Lambda(u) \right), \quad (7.5)$$

and $\varphi_{t,y}(\theta)$ is continuous at $\theta = 0$. For $\varphi_{t,y}^n(\theta)$, let $\mathcal{A}^n(t) = \sigma(A^n(s), s \leq t) \vee \mathcal{N}$ where \mathcal{N} is the collection of P -null sets, and we have

$$\begin{aligned} \varphi_{t,y}^n(\theta) &= E \left[\exp \left(i\theta \hat{V}^n(t, y) \right) \right] = E \left[E \left[\exp \left(i\theta \hat{V}^n(t, y) \right) \mid \mathcal{A}^n(t) \right] \right] \\ &= E \left[E \left[\exp \left(i\theta \frac{1}{\sqrt{n}} \sum_{i=1}^{A^n(t)} \left(\mathbf{1}(\eta_i(\tau_i^n) > y - \tau_i^n) - F_{\tau_i^n}^c(y - \tau_i^n) \right) \right) \mid \mathcal{A}^n(t) \right] \right] \\ &= E \left[E \left[\prod_{i=1}^{A^n(t)} \exp \left(i\theta \frac{1}{\sqrt{n}} \left(\mathbf{1}(\eta_i(\tau_i^n) > y - \tau_i^n) - F_{\tau_i^n}^c(y - \tau_i^n) \right) \right) \mid \mathcal{A}^n(t) \right] \right] \\ &= E \left[\prod_{i=1}^{A^n(t)} \left[1 - \frac{\theta^2}{2n} F_{\tau_i^n}^c(y - \tau_i^n) F_{\tau_i^n}^c(y - \tau_i^n) + o(n^{-1}) \right] \right]. \end{aligned} \quad (7.6)$$

Recall that $\tilde{F}_t(u) = F_u(t-u)$ for $u, t \geq 0$ and $\tilde{F}_t^c(u) = 1 - \tilde{F}_t(u)$. Thus, for large enough n (specified below), we have

$$\begin{aligned} &|\varphi_{t,y}^n(\theta) - \varphi_{t,y}(\theta)| \\ &= \left| E \left[\prod_{i=1}^{A^n(t)} \left[1 - \frac{\theta^2}{2n} \tilde{F}_y(\tau_i^n) \tilde{F}_y^c(\tau_i^n) + o(n^{-1}) \right] \right] - \exp \left(-\frac{\theta^2}{2} \int_0^t \tilde{F}_y(u) \tilde{F}_y^c(u) d\Lambda(u) \right) \right| \end{aligned}$$

$$\begin{aligned}
&\leq E \left[\prod_{i=1}^{A^n(t)} \left[1 - \frac{\theta^2}{2n} \tilde{F}_y(\tau_i^n) \tilde{F}_y^c(\tau_i^n) + o(n^{-1}) \right] \right. \\
&\quad \left. - \prod_{i=1}^{A^n(t)} \exp \left(-\frac{\theta^2}{2n} \tilde{F}_y(\tau_i^n) \tilde{F}_y^c(\tau_i^n) \right) \right] \\
&\quad + \left| E \left[\exp \left(-\frac{\theta^2}{2} \int_0^t \tilde{F}_y(u) \tilde{F}_y^c(u) d\bar{A}^n(u) \right) \right] - \exp \left(-\frac{\theta^2}{2} \int_0^t \tilde{F}_y(u) \tilde{F}_y^c(u) d\Lambda(u) \right) \right| \\
&\leq E \left[\sum_{i=1}^{A^n(t)} \left| \exp \left(-\frac{\theta^2}{2n} \tilde{F}_y(\tau_i^n) \tilde{F}_y^c(\tau_i^n) \right) - \left(1 - \frac{\theta^2}{2n} \tilde{F}_y(\tau_i^n) \tilde{F}_y^c(\tau_i^n) \right) \right| \right] + o(1) \\
&\quad + \left| E \left[\exp \left(-\frac{\theta^2}{2} \int_0^t \tilde{F}_y(u) \tilde{F}_y^c(u) d\bar{A}^n(u) \right) \right] - \exp \left(-\frac{\theta^2}{2} \int_0^t \tilde{F}_y(u) \tilde{F}_y^c(u) d\Lambda(u) \right) \right| \\
&\leq \frac{\theta^4}{4n^2} E[A^n(t)] + o(1) \\
&\quad + \left| E \left[\exp \left(-\frac{\theta^2}{2} \int_0^t \tilde{F}_y(u) \tilde{F}_y^c(u) d\bar{A}^n(u) \right) \right] - \exp \left(-\frac{\theta^2}{2} \int_0^t \tilde{F}_y(u) \tilde{F}_y^c(u) d\Lambda(u) \right) \right| \\
&\rightarrow 0, \quad \text{as } n \rightarrow \infty. \tag{7.7}
\end{aligned}$$

Here the first inequality is by subtracting and adding the same term and triangle inequality. The second inequality is by Lemma 9.1. The third inequality is by Lemma 9.2 for large n such that $|\frac{\theta^2}{2n}| < 1$. The final convergence to zero is implied by the facts that $\bar{A}^n \Rightarrow \Lambda$ in \mathbb{D} , the continuous mapping theorem and

$$\left\{ \exp \left(-\frac{\theta^2}{2} \int_0^t \tilde{F}_y(u) \tilde{F}_y^c(u) d\bar{A}^n(u) \right) : n \geq 1 \right\}$$

is uniformly integrable. Therefore, we have shown that for each fixed $t \geq 0$ and $y \geq 0$,

$$\hat{V}^n(t, y) \Rightarrow \hat{V}(t, y) \quad \text{in } \mathbb{R} \quad \text{as } n \rightarrow \infty.$$

A straightforward generalization implies the convergence of finite dimensional distributions of $\hat{V}^n(t)$. Indeed, when $l > 1$, consider $(y_1, \dots, y_l) \in \mathbb{R}^l$ where $0 \leq y_1 < \dots < y_l \leq T'$. We need to show that for any $(\theta_1, \dots, \theta_l) \in \mathbb{R}^l$,

$$E \left[\exp \left(i \sum_{i=1}^l \theta_i \hat{V}^n(t, y_i) \right) \right] \rightarrow E \left[\exp \left(i \sum_{i=1}^l \theta_i \hat{V}(t, y_i) \right) \right] \quad \text{as } n \rightarrow \infty$$

and the limit is continuous at $(0, \dots, 0) \in \mathbb{R}^l$.

By definition, $\sum_{i=1}^l \theta_i \hat{V}(t, y_i)$ is a normal random variable with mean zero and variance

$$\int_0^t \left(\sum_{i=1}^l \sum_{j=1}^l \theta_i \theta_j F_u(y_i \wedge y_j - u) F_u^c(y_i \vee y_j - u) \right) d\Lambda(u).$$

Thus we have

$$E \left[\exp \left(i \sum_{i=1}^l \theta_i \hat{V}(t, y_i) \right) \right] = \exp \left(-\frac{1}{2} \int_0^t \left(\sum_{i=1}^l \sum_{j=1}^l \theta_i \theta_j F_u(y_i \wedge y_j - u) F_u^c(y_i \vee y_j - u) \right) d\Lambda(u) \right), \tag{7.8}$$

and it is continuous at $(0, \dots, 0) \in \mathbb{R}^l$ by bounded convergence theorem.

By definition,

$$\begin{aligned} \sum_{i=1}^l \theta_i \hat{V}^n(t, y_i) &= - \sum_{i=1}^l \theta_i \left(\frac{1}{\sqrt{n}} \sum_{k=1}^{A^n(t)} [\mathbf{1}(\eta_k(\tau_k^n) \leq y_i - \tau_k^n) - F_{\tau_k^n}(y_i - \tau_k^n)] \right) \\ &= - \frac{1}{\sqrt{n}} \sum_{k=1}^{A^n(t)} \left(\sum_{i=1}^l \theta_i [\mathbf{1}(\eta_k(\tau_k^n) \leq y_i - \tau_k^n) - F_{\tau_k^n}(y_i - \tau_k^n)] \right). \end{aligned}$$

Thus, a direct calculation as in (7.6) shows that

$$\begin{aligned} &E \left[\exp \left(i \sum_{i=1}^l \theta_i \hat{V}^n(t, y_i) \right) \right] \\ &= E \left[\prod_{k=1}^{A^n(t)} \left(1 - \frac{1}{2n} \sum_{i=1}^l \sum_{j=1}^l \theta_i \theta_j F_{\tau_k^n}(y_i \wedge y_j - \tau_k^n) F_{\tau_k^n}^c(y_i \vee y_j - \tau_k^n) + o(n^{-1}) \right) \right]. \quad (7.9) \end{aligned}$$

The convergence of (7.9) to (7.8) can be shown in a similar way as in (7.7) by Lemmas 9.1–9.2.

We next show that the limit process $\hat{V}(t) = \{\hat{V}(t, y) : y \geq 0\}$ for each $t \geq 0$ satisfies condition (4.2), that is,

$$\hat{V}(t, T') - \hat{V}(t, T' - \delta) \Rightarrow 0 \quad \text{as } \delta \rightarrow 0.$$

It suffices to show that

$$E \left[|\hat{V}(t, T') - \hat{V}(t, T' - \delta)|^2 \right] \rightarrow 0 \quad \text{as } \delta \rightarrow 0.$$

This directly follows from the continuity of the covariance function of \hat{V} .

We now verify condition (4.3) in Theorem 4.1. Define a function $H^t : [0, T'] \rightarrow \mathbb{R}$ by

$$H^t(y) := C_{H,t} \int_0^t F_u(y - u) d\Lambda(u) \quad \text{for } y \in [0, T'], \quad (7.10)$$

where $C_{H,t}$ is a large positive constant to be specified below. It is easy to see that $H^t(y)$ is nondecreasing and continuous. Thus, it suffices to show that for $0 \leq x \leq y \leq z \leq T'$,

$$P(|\hat{V}^n(t, x) - \hat{V}^n(t, y)| \wedge |\hat{V}^n(t, y) - \hat{V}^n(t, z)| \geq \epsilon) \leq \frac{1}{\epsilon^4} (H^t(z) - H^t(x))^2. \quad (7.11)$$

First we observe that for any $K \in \mathbb{N}$, $nK = A^n(\tau_{nK}^n)$. On $\{A^n(T) \leq nK = A^n(\tau_{nK}^n)\}$, we have $t = t \wedge \tau_{nK}^n$ for $t \leq T$. Thus, $A^n(t) = A^n(t \wedge \tau_{nK}^n)$ and $\hat{V}^n(t, x) = \hat{V}^n(t \wedge \tau_{nK}^n, x)$ on $\{A^n(T) \leq nK\}$.

Now, for $K \in \mathbb{N}$ such that $K > \Lambda(T)$ and $\epsilon > 0$,

$$\begin{aligned} &P(|\hat{V}^n(t, x) - \hat{V}^n(t, y)| \wedge |\hat{V}^n(t, y) - \hat{V}^n(t, z)| \geq \epsilon) \\ &\leq P(A^n(T) \geq nK) \\ &\quad + P(A^n(T) \leq nK, |\hat{V}^n(t, x) - \hat{V}^n(t, y)| \wedge |\hat{V}^n(t, y) - \hat{V}^n(t, z)| \geq \epsilon) \\ &\leq P(\bar{A}^n(T) \geq K) \\ &\quad + \frac{1}{\epsilon^4} E \left[\mathbf{1}(\bar{A}^n(T) \leq K) \cdot |\hat{V}^n(t, x) - \hat{V}^n(t, y)|^2 \cdot |\hat{V}^n(t, y) - \hat{V}^n(t, z)|^2 \right] \end{aligned}$$

$$\begin{aligned}
&\leq P(\bar{A}^n(T) \geq K) \\
&\quad + \frac{1}{\epsilon^4} \left(E \left[|\hat{V}^n(t \wedge \tau_{nK}^n, x) - \hat{V}^n(t \wedge \tau_{nK}^n, y)|^4 \right] \right)^{1/2} \\
&\quad \quad \times \left(E \left[|\hat{V}^n(t \wedge \tau_{nK}^n, y) - \hat{V}^n(t \wedge \tau_{nK}^n, z)|^4 \right] \right)^{1/2}, \tag{7.12}
\end{aligned}$$

where the last inequality is from Cauchy-Schwartz inequality and from the observation that $\hat{V}(t, x) = \hat{V}(t \wedge \tau_{nK}^n, x)$ for $t \leq T$ on $\{A^n(T) \leq nK\}$.

Since $\bar{A}^n(T) \Rightarrow \Lambda(T)$ as $n \rightarrow \infty$ by Assumption 1, we have

$$P(\bar{A}^n(T) \geq K) \rightarrow P(\Lambda(T) \geq K) = 0 \quad \text{as } n \rightarrow \infty$$

for the chosen $K > \Lambda(T)$. Therefore, due to (7.12), (7.11) is implied by

$$E \left[|\hat{V}^n(t \wedge \tau_{nK}^n, x) - \hat{V}^n(t \wedge \tau_{nK}^n, y)|^4 \right] \leq (H^t(y) - H^t(x))^2, \tag{7.13}$$

for $0 \leq x \leq y \leq T'$. Now we calculate the left hand side of (7.13):

$$\begin{aligned}
&E \left[|\hat{V}^n(t \wedge \tau_{nK}^n, x) - \hat{V}^n(t \wedge \tau_{nK}^n, y)|^4 \right] \\
&= E \left[\left| \frac{1}{\sqrt{n}} \sum_{i=1}^{A^n(t) \wedge nK} \mathbf{1}(\eta_i(\tau_i^n) \in (x - \tau_i^n, y - \tau_i^n)) - F_{\tau_i^n}(y - \tau_i^n) + F_{\tau_i^n}(x - \tau_i^n) \right|^4 \right] \\
&= 3E \left[\left(\int_0^{t \wedge \tau_{nK}^n} [F_u(y - u) - F_u(x - u)][1 - F_u(y - u) + F_u(x - u)] d\bar{A}^n(u) \right)^2 \right] \\
&\quad + \frac{1}{n^2} E \left[\sum_{i=1}^{A^n(t) \wedge nK} |\mathbf{1}(\eta_i(\tau_i^n) \in (x - \tau_i^n, y - \tau_i^n)) - F_{\tau_i^n}(y - \tau_i^n) + F_{\tau_i^n}(x - \tau_i^n)|^4 \right] \\
&\quad - \frac{3}{n^2} E \left[\sum_{i=1}^{A^n(t) \wedge nK} [F_{\tau_i^n}(y - \tau_i^n) - F_{\tau_i^n}(x - \tau_i^n)]^2 [1 - F_{\tau_i^n}(y - \tau_i^n) + F_{\tau_i^n}(x - \tau_i^n)]^2 \right] \\
&\leq 3E \left[\left(\int_0^{t \wedge \tau_{nK}^n} [F_u(y - u) - F_u(x - u)] d\bar{A}^n(u) \right)^2 \right] + o(1/n) \\
&\leq 3E \left[\left(K \wedge \int_0^t [F_u(y - u) - F_u(x - u)] d\bar{A}^n(u) \right)^2 \right] + o(1/n). \tag{7.14}
\end{aligned}$$

Since $\bar{A}^n \Rightarrow \Lambda$ in \mathbb{D} as $n \rightarrow \infty$ by Assumption 1, it is easy to see that for each $t \geq 0$, as $n \rightarrow \infty$,

$$K \wedge \int_0^t [F_u(y - u) - F_u(x - u)] d\bar{A}^n(u) \Rightarrow K \wedge \int_0^t [F_u(y - u) - F_u(x - u)] d\Lambda(u) \quad \text{in } \mathbb{R}.$$

Since $\{(K \wedge \int_0^t [F_u(y - u) - F_u(x - u)] d\bar{A}^n(u))^2 : n \geq 1\}$ is uniformly integrable, by Theorem 3.5 in [4], as $n \rightarrow \infty$, we have

$$\begin{aligned}
&E \left[\left(K \wedge \int_0^t [F_u(y - u) - F_u(x - u)] d\bar{A}^n(u) \right)^2 \right] \\
&\rightarrow \left(K \wedge \int_0^t [F_u(y - u) - F_u(x - u)] d\Lambda(u) \right)^2
\end{aligned}$$

$$= \left(\int_0^t [F_u(y-u) - F_u(x-u)] d\Lambda(u) \right)^2,$$

because of the fact that $K > \Lambda(T)$. Thus by choosing $C_{H,t} > 0$ large enough, we have proved (7.13). The proof is complete. \square

Now we are ready to prove Theorem 5.1.

Proof of Theorem 5.1. Consider $[0, T] \times [0, T']$. To prove this, it suffices to verify the following three conditions by Theorem 4.1 with $\mathcal{S} = \mathbb{D}$.

(i) for any $0 \leq t_1 \leq t_2 \leq \dots \leq t_k \leq T$ and $k \geq 1$,

$$(\hat{V}^n(t_1), \dots, \hat{V}^n(t_k)) \Rightarrow (\hat{V}(t_1), \dots, \hat{V}(t_k)) \quad \text{in } \mathbb{D}^k \quad \text{as } n \rightarrow \infty, \quad (7.15)$$

(ii)

$$d_{J_1}(\hat{V}(T), \hat{V}(T - \delta)) \Rightarrow 0 \quad \text{in } \mathbb{R} \quad \text{as } \delta \rightarrow 0, \quad (7.16)$$

(iii) for $r \leq s \leq t \leq T$ and $n \geq 1$,

$$P\left(d_{J_1}(\hat{V}^n(r), \hat{V}^n(s)) \wedge d_{J_1}(\hat{V}^n(s), \hat{V}^n(t)) \geq \epsilon\right) \leq \frac{1}{\epsilon^4} (H(t) - H(r))^2, \quad (7.17)$$

for some nondecreasing and continuous function H on $[0, T]$.

To prove (7.15), we show that for any $0 \leq t_1 \leq t_2 \leq \dots \leq t_k$ and $k \geq 1$,

$$(\hat{V}^n(t_1), \dots, \hat{V}^n(t_k)) \Rightarrow (\hat{V}(t_1), \dots, \hat{V}(t_k)) \quad \text{in } (\mathbb{D}[0, T'])^k \quad \text{as } n \rightarrow \infty. \quad (7.18)$$

Lemma 7.2 implies that the sequence $\{\hat{V}^n(t) : n \geq 1\}$ is tight for each $t \in [0, T]$, and thus, $(\hat{V}^n(t_1), \hat{V}^n(t_2), \dots, \hat{V}^n(t_k))$ is also tight for $0 \leq t_1 < t_2 < \dots < t_k \leq T$. Then it suffices to show the finite dimensional distributions of $(\hat{V}^n(t_1), \hat{V}^n(t_2), \dots, \hat{V}^n(t_k))$ converge to those of $(\hat{V}(t_1), \hat{V}(t_2), \dots, \hat{V}(t_k))$. It suffices to show that for any $0 \leq y_1 < \dots < y_l \leq T'$ and $\{\theta_{ij} \in \mathbb{R} : i = 1, \dots, k, j = 1, \dots, l\}$,

$$E \left[\exp \left(i \sum_{i=1}^k \sum_{j=1}^l \theta_{ij} \hat{V}^n(t_i, y_j) \right) \right] \rightarrow E \left[\exp \left(i \sum_{i=1}^k \sum_{j=1}^l \theta_{ij} \hat{V}(t_i, y_j) \right) \right] \quad \text{as } n \rightarrow \infty,$$

and the limit function is continuous. The proof is essentially the same in Lemma 7.2 in the case $k = 1$. We only highlight the minor difference in calculating the expectations in the case $k > 1$. We can write

$$\begin{aligned} \sum_{i=1}^k \sum_{j=1}^l \theta_{ij} \hat{V}^n(t_i, y_j) &= \sum_{i=1}^k \sum_{j=1}^l \theta_{ij} \hat{V}^n(t_1, y_j) + \sum_{i=2}^k \sum_{j=1}^l \theta_{ij} [\hat{V}^n(t_2, y_j) - \hat{V}^n(t_1, y_j)] \\ &\quad + \dots + \sum_{i=k-1}^k \sum_{j=1}^l \theta_{ij} [\hat{V}^n(t_k, y_j) - \hat{V}^n(t_{k-1}, y_j)]. \end{aligned}$$

Observe that the expression above is the summation of conditionally independent terms, $\hat{V}^n(t_m, \cdot) - \hat{V}^n(t_{m-1}, \cdot)$ for $1 \leq m \leq k$. The expectation can then be easily calculated. We omit the details here for brevity.

Condition (7.16) is simply implied by the fact that $\hat{V} \in \mathbb{C}_{\mathbb{C}}$ proved in Lemma 7.1.

Now we focus on (7.17). For $K \in \mathbb{N}$ such that $K > \Lambda(T)$ and $\epsilon > 0$,

$$P(d_{J_1}(\hat{V}^n(r), \hat{V}^n(s)) \wedge d_{J_1}(\hat{V}^n(s), \hat{V}^n(t)) \geq \epsilon)$$

$$\begin{aligned}
&\leq P\left(\sup_{y \in [0, T']} |\hat{V}^n(r, y) - \hat{V}^n(s, y)| \wedge \sup_{y \in [0, T']} |\hat{V}^n(s, y) - \hat{V}^n(t, y)| \geq \epsilon\right) \\
&\leq P(A^n(T) \geq nK) \\
&\quad + P\left(A^n(T) \leq nK, \sup_{y \in [0, T']} |\hat{V}^n(r, y) - \hat{V}^n(s, y)| \wedge \sup_{y \in [0, T']} |\hat{V}^n(s, y) - \hat{V}^n(t, y)| \geq \epsilon\right) \\
&\leq P(\bar{A}^n(T) \geq K) \\
&\quad + \frac{1}{\epsilon^4} E \left[\mathbf{1}(\bar{A}^n(T) \leq K) \cdot \sup_{y \in [0, T']} |\hat{V}^n(r, y) - \hat{V}^n(s, y)|^2 \cdot \sup_{y \in [0, T']} |\hat{V}^n(s, y) - \hat{V}^n(t, y)|^2 \right] \\
&\leq P(\bar{A}^n(T) \geq K) \\
&\quad + \frac{1}{\epsilon^4} \left(E \left[\sup_{x \in [0, T']} |\hat{V}^n(t \wedge \tau_{nK}^n, x) - \hat{V}^n(s \wedge \tau_{nK}^n, x)|^4 \right] \right)^{1/2} \\
&\quad \quad \times \left(E \left[\sup_{x \in [0, T']} |\hat{V}^n(s \wedge \tau_{nK}^n, x) - \hat{V}^n(r \wedge \tau_{nK}^n, x)|^4 \right] \right)^{1/2}, \tag{7.19}
\end{aligned}$$

where the last inequality is from Cauchy-Schwartz inequality and noting that $\hat{V}(t, x) = \hat{V}(t \wedge \tau_{nK}^n, x)$ for $t \leq T$ on $\{A^n(T) \leq nK\}$.

Since $\bar{A}^n(T) \Rightarrow \Lambda(T)$ as $n \rightarrow \infty$ by Assumption 1, we have

$$P(\bar{A}^n(T) \geq K) \rightarrow P(\Lambda(T) \geq K) = 0 \quad \text{as } n \rightarrow \infty$$

for the chosen $K > \Lambda(T)$. Therefore, due to (7.19), (7.17) is implied by

$$E \left[\sup_{x \in [0, T']} |\hat{V}^n(t \wedge \tau_{nK}^n, x) - \hat{V}^n(s \wedge \tau_{nK}^n, x)|^4 \right] \leq (H(t) - H(s))^2, \tag{7.20}$$

for $0 \leq s \leq t \leq T$ and some nondecreasing and continuous function H on $[0, T]$.

Now we prove (7.20). Define

$$Z_{s,t}^n[\hat{V}^n](x) := \hat{V}^n(t \wedge \tau_{nK}^n, x) - \hat{V}^n(s \wedge \tau_{nK}^n, x),$$

for $0 \leq s \leq t \leq T$ and $x \in [0, T']$. We apply Proposition 4.1 to obtain an upper bound for the supremum norm of $Z_{s,t}^n[\hat{V}^n](x)$. By definition, we first obtain for $0 \leq x < y \leq T'$,

$$\begin{aligned}
&E \left[|Z_{s,t}^n[\hat{V}^n](y) - Z_{s,t}^n[\hat{V}^n](x)|^4 \right] \\
&= E \left[|\hat{V}^n(t \wedge \tau_{nK}^n, y) - \hat{V}^n(s \wedge \tau_{nK}^n, y) - \hat{V}^n(t \wedge \tau_{nK}^n, x) + \hat{V}^n(s \wedge \tau_{nK}^n, x)|^4 \right] \\
&= \frac{1}{n^2} E \left[\left| \sum_{i=A^n(s) \wedge nK}^{A^n(t) \wedge nK} \mathbf{1}(\eta_i(\tau_i^n) \in (x - \tau_i^n, y - \tau_i^n)) - F_{\tau_i^n}(y - \tau_i^n) + F_{\tau_i^n}(x - \tau_i^n) \right|^4 \right] \\
&= 3E \left[\left(\int_{s \wedge \tau_{nK}^n}^{t \wedge \tau_{nK}^n} [F_u(y - u) - F_u(x - u)] [1 - F_u(y - u) + F_u(x - u)] d\bar{A}^n(u) \right)^2 \right] \\
&\quad + \frac{1}{n^2} E \left[\sum_{i=A^n(s) \wedge nK}^{A^n(t) \wedge nK} |\mathbf{1}(\eta_i(\tau_i^n) \in (x - \tau_i^n, y - \tau_i^n)) - F_{\tau_i^n}(y - \tau_i^n) + F_{\tau_i^n}(x - \tau_i^n)|^4 \right]
\end{aligned}$$

$$\begin{aligned}
 & -\frac{3}{n^2} E \left[\sum_{i=A^n(s) \wedge nK}^{A^n(t) \wedge nK} [F_{\tau_i^n}(y - \tau_i^n) - F_{\tau_i^n}(x - \tau_i^n)]^2 [1 - F_{\tau_i^n}(y - \tau_i^n) + F_{\tau_i^n}(x - \tau_i^n)]^2 \right] \\
 & \leq 3E \left[\left(\int_{s \wedge \tau_{nK}^n}^{t \wedge \tau_{nK}^n} [F_u(y - u) - F_u(x - u)] d\bar{A}^n(u) \right)^2 \right] + o(1/n) \\
 & \leq 3E \left[\left(K \wedge \int_s^t [F_u(y - u) - F_u(x - u)] d\bar{A}^n(u) \right)^2 \right] + o(1/n). \tag{7.21}
 \end{aligned}$$

By the similar argument in Lemma 7.2, we have as $n \rightarrow \infty$,

$$\begin{aligned}
 & E \left[\left(K \wedge \int_s^t [F_u(y - u) - F_u(x - u)] d\bar{A}^n(u) \right)^2 \right] \\
 & \rightarrow \left(K \wedge \int_s^t [F_u(y - u) - F_u(x - u)] d\Lambda(u) \right)^2 \\
 & = \left(\int_s^t [F_u(y - u) - F_u(x - u)] d\Lambda(u) \right)^2 \\
 & \leq \left((t - s) \wedge (y - x) + \int_s^t [F_u(y - u) - F_u(x - u)] d\Lambda(u) \right)^2.
 \end{aligned}$$

Thus, we know that there exists a constant $C > 0$ such that for each $n \geq 1$,

$$\begin{aligned}
 & E \left[|Z_{s,t}^n[\hat{V}^n](y) - Z_{s,t}^n[\hat{V}^n](x)|^4 \right] \tag{7.22} \\
 & \leq C^2 \left((t - s) \wedge (y - x) + \int_s^t [F_u(y - u) - F_u(x - u)] d\Lambda(u) \right)^2 = C^2 (d_{s,t}(x, y))^4.
 \end{aligned}$$

where $d_{s,t}(x, y)$ is the semimetric on $[0, T']$ given in Definition 5.1. Therefore, by Proposition 4.1, it is easy to see that (7.20) (thus (7.17)) holds with

$$H(t) = \tilde{K}^{1/2}(t + \Lambda(t)),$$

which is a nondecreasing and continuous function, and $\tilde{K} > 0$ is a large enough constant. Now, the proof is complete. \square

8. DISCUSSIONS

8.1. Application of Proposition 4.1 to the processes \hat{X}_2^e and \hat{X}_2^r . Proposition 5.1 gives a useful upper bound for $E[\sup_{y \in [0, T']} |\hat{V}(t, y) - \hat{V}(s, y)|^p]$ when $p = 2, 4$ by applying the moment bounds resulting from the method of chaining, where the upper bound is of the form $H(t) - H(s)$ for some nondecreasing and continuous function H . As a consequence, the convergence criterion in Theorem 4.1 becomes very convenient to prove the weak convergence of the processes \hat{V}^n , and thus the processes $\hat{X}_2^{n,r}$ and $\hat{X}_2^{n,e}$ by the relationships in (6.11) and (6.12). In this section, we provide similar upper bounds for \hat{X}_2^r and \hat{X}_2^e by applying Proposition 4.1. As we see below, additional conditions are required on the cumulative arrival rate function $\Lambda(t)$ or the conditional distribution function $F_u(x)$ in order to obtain such useful moment bounds. Thus, to prove Theorem 3.2, these additional conditions are required if we were to prove the convergences of $\hat{X}_2^{n,r}$ and $\hat{X}_2^{n,e}$ directly by applying the convergence criteria in Theorem 4.1 with these moment bounds. It is worth noting that the

approach to prove the convergence of $\hat{X}^{n,e}$ and $\hat{X}^{n,r}$ via the convergence of \hat{V}^n requires the minimal conditions on the system primitives. Although not used in the proof of Theorem 3.2, these bounds may be of independent interest. We state the following two propositions and their proofs can be found in Appendix.

Proposition 8.1. *Under Assumptions 1–2, if Λ is Lipschitz continuous with Lipschitz constant $L > 0$, the two-parameter Gaussian process $\hat{X}_2^e(t, y)$ has the following upper bound for $p = 2, 4$:*

$$E \left[\sup_{y \in [0, T']} |\hat{X}_2^e(t, y) - \hat{X}_2^e(s, y)|^p \right] \leq K_e \left(2L(t-s) + \int_0^T [F_u(t-u) - F_u(s-u)] d\Lambda(u) \right)^{p/2}, \quad (8.1)$$

for some constant $K_e > 0$ and $0 \leq s < t \leq T$.

Proposition 8.2. *Under Assumptions 1–2, if $F_u(\cdot - u)$ is Lipschitz continuous with Lipschitz constant $L(u)$ such that $L_T := \int_0^T L(u) d\Lambda(u) < +\infty$ for each $T > 0$, the two-parameter Gaussian process $\hat{X}_2^r(t, y)$ has the following upper bound for $p = 2, 4$:*

$$E \left[\sup_{y \in [0, T']} |\hat{X}_2^r(t, y) - \hat{X}_2^r(s, y)|^p \right] \leq K_r \left((t-s)^{1/4} + (t-s)^{1/2} + (t-s) + \Lambda(t) - \Lambda(s) \right)^{p/2}, \quad (8.2)$$

for some constant $K_r > 0$ and $0 \leq s < t \leq T$.

8.2. Comments on the new methodology for $G_t/GI/\infty$ queues. We make the following comments on the application of our new methodology for $G_t/GI/\infty$ queues.

(i) Assumption 2 can be relaxed to allow the service time distribution function F to be general, without any continuity condition. From the proofs of Lemmas 7.1–7.2 (see equations (7.3) and (7.10)), we see that the marginal continuity of μ and the continuity of H^t are evidently satisfied in the i.i.d. case with any general c.d.f. F . Similarly, Assumption 5 can be relaxed to allow the remaining service time distribution G to be general without any continuity condition.

(ii) The upper bound in Proposition 8.1 holds for any general distribution function F , while the upper bound in Proposition 8.2 holds for any Lipschitz continuous distribution function F .

9. APPENDIX

In this section, we collect some auxiliary results that are used in the proofs, and proofs of Lemmas 4.2 and 6.2, and Propositions 8.1–8.2. We first state two technical lemmas, whose proofs can be found in [8].

Lemma 9.1. *Let z_1, \dots, z_n and w_1, \dots, w_n be complex numbers of modulus less than 1. Then*

$$\left| \prod_{i=1}^n z_i - \prod_{i=1}^n w_i \right| \leq \sum_{i=1}^n |z_i - w_i|.$$

Lemma 9.2. *If b is a complex number with $|b| \leq 1$, then $|e^b - (1+b)| \leq |b|^2$.*

Proof of Lemma 4.2. By Theorem 2 in [13], for each $t \in \mathbb{R}_+^2$, there exists a deterministic function $\alpha(t)$ such that

$$P \left(\limsup_{s \rightarrow t} X(s) = X(t) + \frac{1}{2}\alpha(t), \quad \liminf_{s \rightarrow t} X(s) = X(t) - \frac{1}{2}\alpha(t) \right) = 1. \quad (9.1)$$

(9.1) yields that

$$\limsup_{u,v \rightarrow t} |X(u) - X(v)| = \alpha(t), \quad a.s.$$

Since $X(t) \in \mathbb{D}_2$, this further implies that

$$|X_{\mathcal{Q}_i}(t) - X_{\mathcal{Q}_{i'}}(t)| = \alpha(t), \quad a.s., \quad (9.2)$$

where $|X_{\mathcal{Q}_i}(t) - X_{\mathcal{Q}_{i'}}(t)|$ is the maximum over all the six possible differences of X over any two quadrants of the four quadrants at t .

Let $t_0 \in \mathbb{R}_+^2$. Then the difference $X_{\mathcal{Q}_i}(t_0) - X_{\mathcal{Q}_{i'}}(t_0)$ is a mean zero Gaussian random variable. However, by (9.2), it equals either $+\alpha(t_0)$ or $-\alpha(t_0)$, which is impossible for a mean zero Gaussian random variable unless $\alpha(t_0) = 0$. Thus, t_0 is a continuity point of X , and the sample paths of X are in \mathbb{C}_2 a.s. \square

Proof of Lemma 6.2. Since $z \in \mathbb{C}$, the convergence of $z^n \rightarrow z$ in (\mathbb{D}, J_1) is equivalent to the convergence under the supremum norm. That is, for any $T > 0$, we have $\|z^n - z\|_T := \sup_{x \in [0, T]} |z^n(x) - z(x)| \rightarrow 0$ as $n \rightarrow \infty$.

It is easy to check that $\phi^r(z), \psi^r(z), \phi^e(z), \psi^e(z) \in \mathbb{C}_{\mathbb{C}}$ (and \mathbb{C}_2), for which it requires the continuity of $F_t(x)$ and $G_t(x)$ in x for each t in Assumptions 2 and 5, respectively. It suffices to prove the convergence in the supremum norm. The proofs for $\psi^r(z), \phi^e(z), \psi^e(z)$ follow from similar arguments. We only prove the convergence $\psi^r(z_n) \rightarrow \psi^r(z)$ in the supremum norm, that is, for any $T, T' > 0$,

$$\sup_{t \in [0, T]} \sup_{y \in [0, T']} |\psi^r(z^n)(t, y) - \psi^r(z)(t, y)| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (9.3)$$

By integration by parts and the fact that $\tilde{F}_{t+y}(s) = F_s(t+y-s) \leq 1$ for all $s, t, y \geq 0$, we have

$$\begin{aligned} & \sup_{t \in [0, T]} \sup_{y \in [0, T']} |\psi^r(z^n)(t, y) - \psi^r(z)(t, y)| \\ &= \sup_{t \in [0, T]} \sup_{y \in [0, T']} \left| \int_0^t (1 - \tilde{F}_{t+y}(s)) dz^n(s) - \int_0^t (1 - \tilde{F}_{t+y}(s)) dz(s) \right| \\ &\leq \sup_{t \in [0, T]} |z^n(t) - z(t)| + |z^n(0) - z(0)| \\ &\quad + \sup_{t \in [0, T]} \sup_{y \in [0, T']} \left| \int_0^t [z^n(s) - z(s)] d\tilde{F}_{t+y}(s) \right| \\ &\leq \|z^n - z\|_T + \|z^n - z\|_T + \|z^n - z\|_T \sup_{t \in [0, T+T']} V_0^T(\tilde{F}_t) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

The convergence follows from the assumption of $z^n \rightarrow z$ and Assumption 3.

We next prove the convergence of $\phi^r(z_n) \rightarrow \phi^r(z)$ in the supremum norm, that is, for any $T, T' > 0$,

$$\sup_{t \in [0, T]} \sup_{y \in [0, T']} |\phi^r(z^n)(t, y) - \phi^r(z)(t, y)| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (9.4)$$

By integration by parts and the fact that $\tilde{G}_t(s) = G_s(t) \leq 1$ for all $s, t \geq 0$, we obtain

$$\begin{aligned}
& \sup_{t \in [0, T]} \sup_{y \in [0, T']} \left| \phi^r(z^n)(t, y) - \phi^r(z)(t, y) \right| \\
&= \sup_{t \in [0, T]} \sup_{y \in [0, T']} \left| \int_0^\infty (1 - \tilde{G}_{t+y}(s)) dz^n(s) - \int_0^\infty (1 - \tilde{G}_{t+y}(s)) dz(s) \right| \\
&= \sup_{t \in [0, T]} \sup_{y \in [0, T']} \left| \int_0^{\bar{y}} (1 - \tilde{G}_{t+y}(s)) dz^n(s) - \int_0^{\bar{y}} (1 - \tilde{G}_{t+y}(s)) dz(s) \right| \\
&\leq |z^n(\bar{y}) - z(\bar{y})| + |z^n(0) - z(0)| \\
&\quad + \sup_{t \in [0, T]} \sup_{y \in [0, T']} \left| \int_0^{\bar{y}} [z^n(s) - z(s)] d\tilde{G}_{t+y}(s) \right| \\
&\leq \|z^n - z\|_{\bar{y}} + \|z^n - z\|_{\bar{y}} + \|z^n - z\|_{\bar{y}} \sup_{t \in [0, T+T']} V_0^{\bar{y}}(\tilde{G}_t).
\end{aligned}$$

All the three terms converges to zero as $n \rightarrow \infty$, under Assumption 6. This completes the proof. \square

We next prove Proposition 8.1. We need the following definition and lemma. Recall that L is the Lipschitz constant of Λ , and note that $\Lambda(t) \equiv 0$ for $t < 0$. Also recall $\tilde{F}_t(u) = F_u(t-u)$ for $t, u \geq 0$ and $F_u(x) \equiv 0$ for $u \geq 0, x < 0$.

Definition 9.1. For each $0 \leq s < t \leq T$ and $T' > 0$, define a semimetric $d_{s,t}^e$ on $[0, T']$ by

$$d_{s,t}^e(x, y) := \begin{cases} \left(L(y-x) + \int_{t-y}^{t-x} \tilde{F}_t(u) d\Lambda(u) - \int_{s-y}^{s-x} \tilde{F}_s(u) d\Lambda(u) \right)^{1/2}, & \text{if } y-x < t-s, \\ \left(2L(t-s) + \int_0^T (\tilde{F}_t(u) - \tilde{F}_s(u)) d\Lambda(u) \right)^{1/2}, & \text{if } y-x \geq t-s, \end{cases} \quad (9.5)$$

for $0 \leq x \leq y \leq T'$, and by symmetry, for $0 \leq y < x \leq T'$,

$$d_{s,t}^e(x, y) := d_{s,t}^e(y, x). \quad (9.6)$$

It is easy to check that $d_{s,t}^e(x, y)$ defined in (9.5)–(9.6) is indeed a semimetric on $[0, T']$ for any $T' > 0$.

Lemma 9.3. For each $0 \leq s < t \leq T$ and $T' > 0$, the diameter of $[0, T']$ under the semimetric $d_{s,t}^e(x, y)$ is

$$d_{s,t}^e(T') = d_{s,t}^e(0, T') \leq \left(2L(t-s) + \int_0^T (\tilde{F}_t(u) - \tilde{F}_s(u)) d\Lambda(u) \right)^{1/2}, \quad (9.7)$$

and the covering number satisfies (4.22).

Proof. It is easy to check that $d_{s,t}^e(0, T') \geq d_{s,t}^e(x, y)$ for each pair of $(x, y) \in [0, T] \times [0, T']$, which gives the diameter in (9.7). To find the upper bound of the diameter, when $T' \geq t-s$, the upper bound in (9.7) holds with equality by the second line of the definition in (9.5). When $T' < t-s$, by the first line of the definition in (9.5), we obtain

$$\begin{aligned}
d_{s,t}^e(0, T') &= \left(LT' + \int_{t-T'}^t \tilde{F}_t(u) d\Lambda(u) - \int_{s-T'}^s \tilde{F}_s(u) d\Lambda(u) \right)^{1/2} \\
&\leq \left(L(t-s) + \int_{s-T'}^t \tilde{F}_t(u) d\Lambda(u) - \int_{s-T'}^s \tilde{F}_s(u) d\Lambda(u) \right)^{1/2}
\end{aligned}$$

$$\begin{aligned}
 &= \left(L(t-s) + \int_s^t \tilde{F}_t(u) d\Lambda(u) + \int_{s-T'}^s [\tilde{F}_t(u) - \tilde{F}_s(u)] d\Lambda(u) \right)^{1/2} \\
 &\leq \left(L(t-s) + L(t-s) + \int_0^T (\tilde{F}_t(u) - \tilde{F}_s(u)) d\Lambda(u) \right)^{1/2} \\
 &= \left(2L(t-s) + \int_0^T (\tilde{F}_t(u) - \tilde{F}_s(u)) d\Lambda(u) \right)^{1/2},
 \end{aligned}$$

which is the upper bound in (9.7).

To find the upper bound for the covering number, we observe that for each $x \in [0, T']$, the semimetric $d_{s,t}^e(x, y)$, as a function of y , is nonincreasing on $[0, x]$ and nondecreasing on $[x, T']$ (behaves like Euclidean metric). This completes the proof. \square

Proof of Proposition 8.1. We only need to verify that the Gaussian process $\hat{X}_2^e(t, y)$ satisfies (4.21) with $p = 2, 4$. It is evident that $\hat{X}_2^e(t, 0) = 0$ a.s. for each $t \in [0, T]$ (its variance $\text{Var}(\hat{X}_2^e(t, 0)) = 0$ from (3.10)). Then the conclusion follows from (9.7) and (4.23).

We now verify the condition (4.21) for $\hat{X}_2^e(t, y)$ with $p = 2$. By (3.10), we obtain

$$\begin{aligned}
 &E \left[\left| (\hat{X}_2^e(t, y) - \hat{X}_2^e(s, y)) - (\hat{X}_2^e(t, x) - \hat{X}_2^e(s, x)) \right|^2 \right] \\
 &= \int_{t-y}^{t-x} \tilde{F}_t(u) \tilde{F}_t^c(u) d\Lambda(u) + \int_{s-y}^{s-x} \tilde{F}_s(u) \tilde{F}_s^c(u) d\Lambda(u) \\
 &\quad - 2 \int_{t-y}^{(t-y) \vee (s-x)} \tilde{F}_s(u) \tilde{F}_t^c(u) d\Lambda(u). \tag{9.8}
 \end{aligned}$$

When $t - y > s - x$, that is, $y - x < t - s$, the right hand side of (9.8) becomes

$$\begin{aligned}
 &\int_{t-y}^{t-x} \tilde{F}_t(u) \tilde{F}_t^c(u) d\Lambda(u) + \int_{s-y}^{s-x} \tilde{F}_s(u) \tilde{F}_s^c(u) d\Lambda(u) - 0 \\
 &\leq \int_{t-y}^{t-x} \tilde{F}_t(u) d\Lambda(u) + \int_{s-y}^{s-x} \tilde{F}_s^c(u) d\Lambda(u) \\
 &= \Lambda(s-x) - \Lambda(s-y) + \int_{t-y}^{t-x} \tilde{F}_t(u) d\Lambda(u) - \int_{s-y}^{s-x} \tilde{F}_s(u) d\Lambda(u) \\
 &\leq L(y-x) + \int_{t-y}^{t-x} \tilde{F}_t(u) d\Lambda(u) - \int_{s-y}^{s-x} \tilde{F}_s(u) d\Lambda(u). \tag{9.9}
 \end{aligned}$$

When $t - y \leq s - x$, that is, $y - x \geq t - s$, the right hand side of (9.8) becomes

$$\begin{aligned}
 &\int_{t-y}^{t-x} \tilde{F}_t(u) \tilde{F}_t^c(u) d\Lambda(u) + \int_{s-y}^{s-x} \tilde{F}_s(u) \tilde{F}_s^c(u) d\Lambda(u) - 2 \int_{t-y}^{s-x} \tilde{F}_s(u) \tilde{F}_t^c(u) d\Lambda(u) \\
 &= \int_{t-y}^{t-x} \tilde{F}_t(u) d\Lambda(u) - \int_{t-y}^{t-x} (\tilde{F}_t(u))^2 d\Lambda(u) \\
 &\quad + \int_{s-y}^{s-x} \tilde{F}_s(u) d\Lambda(u) - \int_{s-y}^{s-x} (\tilde{F}_s(u))^2 d\Lambda(u) - 2 \int_{t-y}^{s-x} \tilde{F}_s(u) \tilde{F}_t^c(u) d\Lambda(u) \\
 &= \int_{t-y}^{t-x} \tilde{F}_t(u) d\Lambda(u) - \int_{s-y}^{s-x} \tilde{F}_s(u) d\Lambda(u)
 \end{aligned}$$

$$\begin{aligned}
& +2 \int_{s-y}^{s-x} \tilde{F}_s(u) d\Lambda(u) - \int_{t-y}^{t-x} (\tilde{F}_t(u))^2 d\Lambda(u) - \int_{s-y}^{s-x} (\tilde{F}_s(u))^2 d\Lambda(u) \\
& -2 \int_{t-y}^{s-x} \tilde{F}_s(u) \tilde{F}_t^c(u) d\Lambda(u) \\
= & \int_{t-y}^{t-x} \tilde{F}_t(u) d\Lambda(u) - \int_{s-y}^{s-x} \tilde{F}_s(u) d\Lambda(u) + \int_{s-y}^{t-y} (2\tilde{F}_s(u) - (\tilde{F}_s(u))^2) d\Lambda(u) \\
& - \int_{t-y}^{s-x} [\tilde{F}_t(u) - \tilde{F}_s(u)]^2 d\Lambda(u) - \int_{s-x}^{t-x} (\tilde{F}_t(u))^2 d\Lambda(u) \\
\leq & \int_{t-y}^{t-x} \tilde{F}_t(u) d\Lambda(u) - \int_{s-y}^{s-x} \tilde{F}_s(u) d\Lambda(u) + \Lambda(t-y) - \Lambda(s-y) \\
& - \int_{t-y}^{s-x} [\tilde{F}_t(u) - \tilde{F}_s(u)]^2 d\Lambda(u) - \int_{s-x}^{t-x} (\tilde{F}_t(u))^2 d\Lambda(u) \\
\leq & L(t-s) + \int_{t-y}^{t-x} \tilde{F}_t(u) d\Lambda(u) - \int_{s-y}^{s-x} \tilde{F}_s(u) d\Lambda(u). \tag{9.10}
\end{aligned}$$

Note that (9.9) and (9.10) show some symmetry. However, (9.10) does not fit our purpose here. The reason is that if we define $d_{s,t}^e(x, y)$ as the square root of (9.10) when $y-x \geq t-s$, it can be shown that it is a semimetric, but the diameter under this semimetric seems impossible to calculate and thus it is hard to obtain a convenient covering number. That is also the motivation for the definition of $d_{s,t}^e$ in (9.5). Thus, we need to derive the following upper bound on (9.10). (9.10) can be written as

$$\begin{aligned}
& L(t-s) + \int_{t-y}^{s-x} [\tilde{F}_t(u) - \tilde{F}_s(u)] d\Lambda(u) + \int_{s-x}^{t-x} \tilde{F}_t(u) d\Lambda(u) - \int_{s-y}^{t-y} \tilde{F}_s(u) d\Lambda(u) \\
\leq & L(t-s) + \int_0^T [\tilde{F}_t(u) - \tilde{F}_s(u)] d\Lambda(u) + \Lambda(t-x) - \Lambda(s-x) \\
\leq & 2L(t-s) + \int_0^T [\tilde{F}_t(u) - \tilde{F}_s(u)] d\Lambda(u). \tag{9.11}
\end{aligned}$$

Combining the two cases above, we see that $\hat{X}_2^e(t, y)$ satisfies (4.21) with $p = 2$ and $C = 1$.

To verify (4.21) with $p = 4$, we note that $(\hat{X}_2^e(t, y) - \hat{X}_2^e(s, y)) - (\hat{X}_2^e(t, x) - \hat{X}_2^e(s, x))$ is a normal random variable. Recall that the kurtosis for a normal random variable is 3. So we obtain

$$\begin{aligned}
& E \left[|(\hat{X}_2^e(t, y) - \hat{X}_2^e(s, y)) - (\hat{X}_2^e(t, x) - \hat{X}_2^e(s, x))|^4 \right] \\
= & 3 \left(E \left[|(\hat{X}_2^e(t, y) - \hat{X}_2^e(s, y)) - (\hat{X}_2^e(t, x) - \hat{X}_2^e(s, x))|^2 \right] \right)^2 \\
\leq & 3 (d_{s,t}^e(x, y))^4. \tag{9.12}
\end{aligned}$$

Thus, we see that $\hat{X}_2^e(t, y)$ satisfies (4.21) with $p = 4$ and $C = 3$. This completes the proof. \square

Proof of Proposition 8.2. We apply Proposition 5.1, (4.19) in Theorem 4.3 and the fact that $\hat{X}_2^T(t, y)$ has the same distribution with $\hat{V}(t, t+y)$. We only consider the case when $p = 2$

since a similar argument follows for $p = 4$.

$$\begin{aligned}
 & E \left[\sup_{y \in [0, T']} |\hat{X}_2^r(t, y) - \hat{X}_2^r(s, y)|^2 \right] \\
 = & E \left[\sup_{y \in [0, T']} |\hat{V}(t, t+y) - \hat{V}(s, s+y)|^2 \right] \\
 \leq & 2E \left[\sup_{y \in [0, T']} |\hat{V}(t, t+y) - \hat{V}(s, t+y)|^2 \right] + 2E \left[\sup_{y \in [0, T']} |\hat{V}(s, t+y) - \hat{V}(s, s+y)|^2 \right] \\
 \leq & 2E \left[\sup_{y \in [0, T+T']} |\hat{V}(t, y) - \hat{V}(s, y)|^2 \right] + 2E \left[\sup_{|x-y| \leq t-s} |\hat{V}(s, y) - \hat{V}(s, x)|^2 \right]. \quad (9.13)
 \end{aligned}$$

By Proposition 5.1, the first term is upper bounded by

$$K_1(t - s + (\Lambda(t) - \Lambda(s))), \quad (9.14)$$

for some constant $K_1 > 0$.

Now we consider the second term in (9.13). Recall the definition of the semimetric $d_{s,t}(x, y)$ in Definition 5.1. We first observe that under the additional assumption on $F_u(x - u)$, we have

$$\begin{aligned}
 d_{0,s}(y, x) &= \left(s \wedge (y - x) + \int_0^s [F_u(y - u) - F_u(x - u)] d\Lambda(u) \right)^{1/2} \\
 &\leq (y - x + L_T(y - x))^{1/2} \\
 &\leq (L_T + 1)^{1/2}(y - x)^{1/2}.
 \end{aligned}$$

Therefore, the second term in (9.13) is bounded by

$$2E \left[\sup_{d_{0,s}(y,x) \leq ((L_T+1)(t-s))^{1/2}} |\hat{V}(s, y) - \hat{V}(s, x)|^2 \right]. \quad (9.15)$$

It is straightforward to verify that

$$E \left[|\hat{V}(s, y) - \hat{V}(s, x)|^2 \right] \leq C_V (d_{0,s}(y, x))^2$$

for some $C_V > 0$. By (4.19) in Theorem 4.3, we have that for any $\zeta, \delta > 0$,

$$2E \left[\sup_{d_{0,s}(x,y) \leq \delta} |\hat{V}(s, y) - \hat{V}(s, x)|^2 \right] \leq K_2 \left(\int_0^\zeta (N(\epsilon/2, d_{0,s}))^{1/2} d\epsilon + \delta N(\zeta/2, d_{0,s}) \right)^2, \quad (9.16)$$

for some $K_2 > 0$.

Taking $\delta = ((L_T + 1)(t - s))^{1/2}$ and $\zeta = \delta^{1/2}$ in (9.16), we obtain

$$\begin{aligned}
 & 2E \left[\sup_{d_{0,s}(x,y) \leq ((L_T+1)(t-s))^{1/2}} |\hat{V}(s, y) - \hat{V}(s, x)|^2 \right] \\
 \leq & K_2 \left(\int_0^{\delta^{1/2}} (N(\epsilon/2, d_{0,s}))^{1/2} d\epsilon + \delta \cdot \frac{d_{0,s}(0, T') + \delta^{1/2}}{\delta^{1/2}} \right)^2
 \end{aligned}$$

$$\begin{aligned}
&\leq K_2 \left(\int_0^{\delta^{1/2}} \left(\frac{d_{0,s}(0, T') + \delta^{1/2}}{\epsilon} \right)^{1/2} d\epsilon + \delta \cdot \frac{d_{0,s}(0, T') + \delta^{1/2}}{\delta^{1/2}} \right)^2 \\
&\leq K_2 \left((d_{0,s}(0, T') + \delta^{1/2})^{1/2} \cdot 2\delta^{1/4} + (d_{0,s}(0, T') + \delta^{1/2}) \cdot \delta^{1/2} \right)^2 \\
&\leq K_2 \left(8(d_{0,s}(0, T') + \delta^{1/2}) \cdot \delta^{1/2} + 2(d_{0,s}(0, T') + \delta^{1/2})^2 \cdot \delta \right) \\
&\leq \hat{K}_2 ((t-s)^{1/4} + (t-s)^{1/2}). \tag{9.17}
\end{aligned}$$

for some large enough \hat{K}_2 since $d_{0,s}(0, T') + \delta^{1/2}$ is upper bounded by $T' + \Lambda(T)$ and $\delta^{1/2} \leq ((L_T + 1)(t-s))^{1/4}$.

Combining this upper bound with (9.14), we obtain that

$$\begin{aligned}
&E \left[\sup_{y \in [0, T']} |\hat{X}_2^r(t, y) - \hat{X}_2^r(s, y)|^2 \right] \\
&\leq K_1(t-s + (\Lambda(t) - \Lambda(s))) + \hat{K}_2 \left((t-s)^{1/4} + (t-s)^{1/2} \right) \\
&\leq K_r \left((t-s)^{1/4} + (t-s)^{1/2} + (t-s) + \Lambda(t) - \Lambda(s) \right) \tag{9.18}
\end{aligned}$$

for some constant $K_r > 0$. This completes the proof. \square

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