Stochastic integral equations with long-range dependence as scaling limits for interactive Hawkes shot noise processes with power response functions

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Abstract. We study an interactive Hawkes shot noise process, with a power-like shot response function, in which the arrivals for the shot noises processes come from a Hawkes process with an intensity depending on the state of the shot noise process. We prove a functional law of large numbers (FLLN) and a functional central limit theorem (FCLT) for the properly scaled joint Hawkes process and shot noise process. The FLLN limit satisfies a system of nonlinear integral equations, and moreover, the shot noise process limit is given by an integral equation with a power kernel function. The FCLT limit process satisfies a system of stochastic integral equations driven by two independent Brownian motions, with a power-type kernel function. In particular, the FCLT limit for the shot noise process is a stochastic Volterra integral equation, exhibiting both path dependence and long-range dependence.

1. Introduction

Interactive Hawkes shot noise processes are introduced by the authors in [31], where the shot noise process has the "shots" coming from a Hawkes process, while the stochastic intensity of the Hawkes process depends on the state of the shot noise process through the self-exciting function. The new processes generalize both the standard shot noise processes, in which the arrivals of shots (either Poisson or renewal) are usually independent of the state of the shot noise process (see, e.g., [27, 36, 19, 20]), and the standard Hawkes processes, in which the stochastic intensity only depends on the path of the counting process (see, e.g., [2, 16, 28]). They also generalize the so-called "marked" Hawkes processes, in which the marks are included in the self-exciting function and usually assumed to be given exogenously (see, e.g., [8, 9, 23, 17, 30]), while the dependence on the state of the shot noise process in the new processes can be regarded as endogenous "marks". These processes can be potentially used in applications in finance, risk processes, neuroscience, queueing and so on; see, e.g., [38, 37, 25, 32, 14, 11, 7].

In [31], under certain conditions on shot response functions, noises and self-exciting functions, a functional law of large numbers (FLLN) and a functional central limit theorem (FCLT) are established for the joint dynamics of the Hawkes counting process and the shot noise process. The FCLT limit is given by a two-dimensional interactive stochastic differential equations (SDEs) driven by two independent Brownian motions. In particular, the limit for the shot noise process is equivalent in distribution to an Ornstein-Uhlenbeck diffusion process, extending the existing results of Brownian motion limits in [25, 19, 20].

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Shot noise processes with regularly varying or asymptotic power shot response functions have been studied in the literature. In particular, in [24], a scaling limit for the Poisson shot noise process with a regularly varying shot response function is established, where the limit is a fractional Brownian motion with Hurst parameter $H \in (1/2, 1]$, which exhibits long range dependence (see also Chapter 3.4 in [35]). In [25], a self-similar stationary Gaussian limit is also proved for explosive shot noise processes with a Poisson arrival process. More recently, in [33], a generalized fractional Brownian motion is introduced as the scaling limit for power-law shot Poisson noise processes with non-stationary noises. In these works, the arrival processes of shots are assumed to be Poisson. In [19], functional limit theorems are also established for renewal shot-noise process with a response function that is eventually nondecreasing and regularly varying at infinity.

In this paper, we study the interactive Hawkes shot noise processes in the case where the shot response function is asymptotically a power function, which goes beyond the study in [31]. Under suitable conditions, we establish the FLLN and FCLT for the joint dynamics of the Hawkes counting process and the shot noise process. To establish the FLLN, we consider a slightly different scaling in the shot noise process from that used in [31], in order to take into account the asymptotic power response function. The FLLN limit is given by a set of nonlinear integral equations, and the FLLN limit for the shot noise process can be written as an integral equation with a power kernel function (hence exhibiting path-dependence); see Theorem 2.2 and Remark 3.

The FCLT limit is given by a set of stochastic integral equations driven by two independent time-changed Brownian motions, and exhibits both path-dependence and long-range dependence properties (Theorem 2.3). In particular, the limit for the shot noise process in the FCLT can be equivalently (in distribution) written as a stochastic Volterra integral equation with power-law kernel functions (see Proposition 2.4). The limit is not simply a stochastic Volterra integral equation driven by a fractional Brownian motion (as studied in, e.g., [4, 3]), but instead, the driving noise term takes the form $\int_0^t (t-s)^\alpha g(s) dB(s)$ for a measurable deterministic function g, which can be regarded as a generalization of fractional Brownian motion. Although the noise term is a Gaussian process and exhibits long range dependence, it does not have stationary increments or self-similarity property. In the special case of no state dependence, the limit for the shot noise process can be reduced to a fractional Brownian motion as shown in [24] (see Remark 6).

In order to handle the power response function, we must further develop the approach in [31] to prove the convergence of the suitably scaled processes. We recall that the method relying on the renewal equation expressions for standard Hawkes processes from the cluster/immigration-birth representations (see, e.g., [2]) cannot be applied or directly extended to handle the dependence of the self-exciting function upon the state of the shot noise process. The authors have developed an approach by representing the LLN and CLT scaled processes resembling the desired limits together with asymptotically negligible residual terms, together with a localization technique to control the increments of the Hawkes counting process and those of the shot noise process. In the setting with asymptotic power response functions, the estimates associated with these increments (such as maximal inequalities) are much more challenging, and the proofs require novel techniques. One of the main tools we use to establish the FCLT is the criterion for the convergence of stochastic integrals in [26]. However, in order to apply this result, the power-like response function in the shot noise process presents new challenges; in order to check the conditions, we need to overcome several technical hurdles, such as the convergence result of integral representations in Lemma 3.8.

Finally, we remark that the model and results have great potential for applications in finance and risk management. The paper [24] that proved fractional Brownian motion as the scaling limit for Poisson shot noise processes has made a significant impact on the study of financial markets with long range dependence. See [38, 37] for the application of shot noise processes in finance. The interactive Hawkes shot noise processes with power-like response function can be possibly used in similar ways to study financial markets with long range dependence. Hawkes processes have been recently widely used in finance to model the price formation process [2] capturing the self-exciting

and clustering effects from the orders. The new interactive Hawkes shot noise processes can also capture the dependence upon the state of exogenous randomness. This direction of research will be investigated in future work.

1.1. **Organization of the paper.** The paper is organized as follows. In Sections 2.1 and 2.2, we give a detailed description of the model and the associated scaling. In Sections 2.3 and 2.4, we present the FLLN and FCLT results, respectively. The proofs for the main results are given in Section 3, and the proofs for the supporting lemmas are given in Section 4. Some additional proofs are included in the Appendix.

1.2. **Notation.** In this paper, all the processes and random variables are defined on a common probability space $(\Omega, \mathscr{F}, \mathbb{P})$. $\mathbb{N}$ denotes the set of natural numbers, and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. $\mathbb{R}(\mathbb{R}_+)$ denotes the space of real (nonnegative) numbers, and specifically, $\mathbb{R}_+ = [0, \infty)$. For any set $A \subset \mathbb{R}$, let $\mathcal{B}(A)$ be the collection of Borel subsets of A. Let $\mathbb{D} = \mathbb{D}(\mathbb{R}_+, \mathbb{R})$ denote $\mathbb{R}$-valued function space of all càdlàg functions on $\mathbb{R}_+$, and also denote by $\mathbb{C}$ the subspace of continuous functions of $\mathbb{D}$. Let $\mathbb{C}_\uparrow$ be the subspace of $\mathbb{C}$ of nondecreasing functions. $(\mathbb{D}, J_1)$ denotes the space $\mathbb{D}$ equipped with the Skorohod J_1 topology, and $(\mathbb{D}^k, J_1)$ denotes the k-fold product space endowed with the Skorohod J_1 topology, see [5]. We use "u.o.c." as an abbreviation for "uniformly on compact sets". Notations $\to$ and $\Rightarrow$ mean convergence of real numbers and convergence in distribution, respectively. For an event A, we write $\mathbf{1}_A(\cdot)$ being the indicator function. For any real-valued function f, we write $\|f\|_1$ as the L_1 norm, that is, $\|f\|_1 = \int_{\mathbb{R}} |f(x)| dx$. Let $d\nu$ be the Lebesgue-Stieltjes measure induced by the increasing function $\nu \in \mathbb{D}$, that is, $\nu(a, b] = \nu(b) - \nu(a)$, we write $\int_a^b \varphi(y)\, d\nu(y) = \int_{(a,b]} \varphi(y)\, d\nu(y)$ for every measurable function φ.

2. Model and Results

2.1. **The model.** We consider an interactive Hawkes shot noise process described as follows. Let (A, Y) be a pair of Hawkes and shot-noise processes taking values in $\mathbb{N}_0 \times \mathbb{R}$. The shot noise process $Y = \{Y(t), t \geq 0\}$ is defined by

$$Y(t) = \sum_{j \geq 1} \varphi\big(t - \tau_j\big)\, \xi_j\, \mathbf{1}(\tau_j \leq t), \tag{2.1}$$

where $\{\xi_j, j \geq 1\}$ is a sequence of $\mathbb{R}$-valued i.i.d. variables with c.d.f. F, and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a deterministic and measurable functions in $\mathbb{D}$ and referred to as the shot response function, and $\{\tau_j, j \geq 1\}$ is the sequence of event/jump times associated with the Hawkes process A. Let $\{\mathscr{F}_t, t \geq 0\}$ be the natural filtration generated by $\{(\tau_j, \xi_j), j \geq 1\}$, that is,

$$\mathscr{F}_t := \sigma\big\{(\tau_j, \xi_j) : \tau_j \leq t \quad \text{for } j \in \mathbb{N}\big\}.$$

The counting process $A = \{A(t), t \geq 0\}$ is a point process with conditional intensity $\lambda = \{\lambda(t), t \geq 0\}$, that is,

$$\mathbb{P}\big(A \text{ has a jump in } [t, t + dt] \big| \mathscr{F}_{t-}\big) = \lambda(t)\, dt\,,$$

where $\lambda(t)$ is given by

$$\lambda(t) = \lambda_0 + \sum_{j \geq 1} \phi\big(t - \tau_j, Y(\tau_j-)\big)\, \mathbf{1}(\tau_j < t)\,.$$

Here, $\lambda_0 > 0$ is a constant, referred to as the baseline intensity, and $\phi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ is a deterministic and measurable function, referred to as the self-exciting function. Note that the intensity $\{\lambda(t), t \geq 0\}$ depends on the state of the shot noise process through the self-exciting function; hence, the joint dynamics is called the interactive Hawkes shot noise process. This model is first introduced and studied in [31].

In this paper, we particularly consider the case where the shot response function φ satisfies

$$\varphi(x) \sim x^\alpha \quad \text{for some } \alpha > 0 \text{ as } x \to \infty.$$

See Remarks 1 and 5 for further discussions on this condition.

It is clear that the system has two sources of randomness $\{(\tau_j, \xi_j)\}_{j\geq 1}$, and the conditional joint distributions of (τ_k, ξ_k) is written as follows: for every $t > s > 0$ and $k \geq 0$,

$$\mathbb{P}\big(\tau_{k+1} > t, \xi_{k+1} \leq z \big| \mathscr{F}_s\big) = F(z)\, e^{-(\Lambda(t)-\Lambda(s))} \quad \text{on the set } \{A(s) = k\}, \tag{2.2}$$

where Λ is the cumulative intensity process, that is, for $t \geq 0$,

$$\Lambda(t) := \int_0^t \lambda(s)\, ds = \lambda_0 t + \sum_{j\geq 1} \mathbf{1}(\tau_j \leq t) \int_0^{t-\tau_j} \phi\big(u, Y(\tau_j-)\big)\, du,$$

and $\{\xi_j, j \geq 1\}$ is a sequence of unpredictable random noises satisfying (2.2) above. We first have the following well-definedness of the model (A, Y) as shown in Proposition 2.1 in [31], that is, the conditional intensity function (2.2) determines the probability structure of (A, Y) uniquely (see, e.g., [10, Proposition 7.3.IV]). We also say that the process A is non-explosive if $\tau_n < \infty$ and $\sup_n \tau_n = \infty$ with probability one, c.f. [22, page 280], or equivalently, $N(t) < \infty$ for every $t < \infty$.

Proposition 2.1 (Proposition 2.1 [31]). *Suppose $t \to \phi(t, y)$ is integrable for every $y \in \mathbb{R}$. Then the model (2.2) is well-defined. Moreover, $X := A - \Lambda$ is an $\{\mathscr{F}_t, t \geq 0\}$-adapted local martingale.*

2.2. **A sequence of processes $(A^{(n)}, Y^{(n)})$.** We consider a sequence of the joint processes $(A^{(n)}, Y^{(n)})$ indexed by n. Suppose that we have a sequence of functions $\varphi^{(n)}$ and $\phi^{(n)}$, and the variables $\{\xi_j^{(n)}\}_{j\in\mathbb{N}}$. The shot noise process $Y^{(n)}$ is given by

$$Y^{(n)}(t) = \sum_{j\geq 1} \varphi^{(n)}(t - \tau_j^{(n)})\, \xi_j^{(n)}\, \mathbf{1}(\tau_j^{(n)} < t)\,. \tag{2.3}$$

The Hawkes process $A^{(n)}$ has the conditional intensity process

$$\lambda^{(n)}(t) = \lambda_0^{(n)} + \sum_{j\geq 1} \phi^{(n)}\big(t - \tau_j^{(n)}, Y^{(n)}(\tau_j^{(n)}-)\big)\, \mathbf{1}(\tau_j^{(n)} < t)\,, \tag{2.4}$$

with the associated filtration $\{\mathscr{F}_t^{(n)}, t \geq 0\}$ defined by

$$\mathscr{F}_t^{(n)} := \sigma\big\{(\xi_j^{(n)}, \tau_j^{(n)}) : \tau_j^{(n)} \leq t,\ j \in \mathbb{N}\big\}\,. \tag{2.5}$$

We make the following assumptions on the primitives.

Assumption A1. *(i) Suppose $\lambda_0^{(n)} \to \lambda_0$ as $n \to \infty$ and for some function $\varphi \in \mathbb{D}$,*

$$\varphi^{(n)}(x) \equiv \varphi(x) \sim x^\alpha \quad \text{for some } \alpha > 0 \text{ as } x \to \infty. \tag{2.6}$$

(ii) Let $c_n = n^{1+\alpha}$ and for some deterministic and measurable function $\phi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$,

$$\phi^{(n)}(t, c_n\, y) = \phi^{(n)}(t, n^{1+\alpha}\, y) \equiv \phi(t, y).$$

Suppose that $y \to \Phi(y) := \int_0^\infty \phi(t, y)\, dt < 1$ is continuous on $\mathbb{R}$ and ϕ satisfies

$$\sup_{|y|\leq k} \int_t^\infty \phi(u, y)\, du \to 0 \quad \text{as } t \to \infty \text{ for every } k > 0.$$

(iii) Suppose $\{\xi_j^{(n)}, j \geq 1\}$ is a sequence of $\mathbb{R}$-valued i.i.d. random variables with

$$\sup_{n\geq 1} \mathbb{E}\big[|\xi^{(n)}|\big] < \infty, \quad \mu^{(n)} = \mathbb{E}\big[\xi^{(n)}\big] \to \mu \in \mathbb{R} \quad \text{and} \quad \mathbb{E}\big[|\xi^{(n)}|; |\xi^{(n)}| > n\varepsilon\big] \to 0 \quad \forall \varepsilon > 0,$$

where $\xi^{(n)}$ is a generic variable and with the cumulative distribution function (c.d.f.) $F^{(n)}(\cdot)$.

Remark 1. We have assumed that the shot response function φ is asymptotically a power function in (2.6). This is a benchmark case for the more general case that φ satisfies

$$\frac{\varphi(nt)}{\varphi(n)} \to t^\alpha \quad \text{for every } t>0 \text{ as } n\to\infty,$$

that is, φ is regularly varying at ∞ with index $\alpha>0$. The proof for the FLLN for φ can be adapted for the case of being a regularly varying function under some additional technical conditions; however, the proof for the FCLT below would require some new arguments. See further discussions in Remark 5.

2.3. **FLLN.** We define the following LLN-scaled processes:

$$\big(\bar{A}^{(n)}(t),\bar{Y}^{(n)}(t)\big)=\big(n^{-1}A^{(n)}(nt),c_n^{-1}Y^{(n)}(nt)\big),\quad t\ge 0, \tag{2.7}$$

where $c_n=n^{1+\alpha}$ is the scaling parameter defined in Assumption A1-(ii).

Assumption A2. *Suppose that $\Psi(y)=(1-\Phi(y))^{-1}$ is continuous on $\mathbb{R}$ and*

(i) Ψ is locally Lipschitz-continuous, that is, for every $a<b$, there is some constant $c>0$ so that

$$|\Psi(y)-\Psi(z)|\le c\,|y-z|,\quad \forall y,z\in[a,b]\,;$$

(ii) Ψ satisfies the linear growth condition on $\mathbb{R}$, that is, for some $c'>0$,

$$\Psi(y)\le c'\,(1+y),\quad \forall y\in\mathbb{R}.$$

Remark 2. For example, if $\Phi(y)=1-\gamma^{-1}(1+y)^{-\beta}$ for some $\gamma\ge 1,\beta\in(0,1]$, then $\Psi(y)=\gamma(1+y)^\beta$. If $\Phi(y)=(1+\gamma^{-1}e^{\beta y})^{-1}$ for some $\gamma,\beta>0$, then $\Psi(y)=1+\gamma e^{-\beta y}$. In both cases, the local Lipschitz continuity property evidently holds. Note that the local Lipschitz property implies the pathwise uniqueness of local solutions, c.f. [18, Theorem 3.1], while the linear growth condition is a sufficient condition for non-explosion of solutions, c.f. [18, Theorem 2.4].

Theorem 2.2. *Under Assumptions A1 and A2,*

$$\big(\bar{A}^{(n)},\bar{Y}^{(n)}\big)\to\big(\bar{A},\bar{Y}\big)\quad \textit{u.o.c. in probability as } n\to\infty,$$

where $(\bar{A},\bar{Y})$ is the unique non-explosive solution to the nonlinear integral equations

$$\begin{aligned}\bar{A}(t)&=\lambda_0 t+\int_0^t\Phi\big(\bar{Y}(s)\big)\,d\bar{A}(s),\\ \bar{Y}(t)&=\mu\int_0^t(t-s)^\alpha\,d\bar{A}(s).\end{aligned} \tag{2.8}$$

Remark 3. Note that the expressions of $(\bar{A},\bar{Y})$ in (2.8) can be equivalently written as

$$\begin{aligned}\bar{A}(t)&=\lambda_0\int_0^t\Psi\big(\bar{Y}(s)\big)\,ds\,,\\ \bar{Y}(t)&=\lambda_0\mu\int_0^t(t-s)^\alpha\Psi\big(\bar{Y}(s)\big)\,ds.\end{aligned} \tag{2.9}$$

By (2.9), we observe that $\bar{Y}(t)$ is path-dependent, and the kernel function in the integrand is of the particular power function $(t-s)^\alpha$.

In the special case of no state-dependence, that is, $\phi(t,y)=\phi(t)$ without dependence on y (abusing notation), $\Phi(y)$ is simply equal to $\|\phi\|_1<1$. Hence, A is the classical Hawkes process and Y in (2.1) is a special shot-noise process studied in [34], where the noise is independent of the arrival process. In this special case, we have

$$\bar{A}(t)=\frac{\lambda_0}{1-\|\phi\|_1}t\,,$$

and

$$\bar{Y}(t) = \frac{\lambda_0 \mu}{1 - \|\phi\|_1} \int_0^t (t-s)^\alpha \, ds = \frac{\lambda_0 \mu}{1 - \|\phi\|_1} \, \frac{t^{\alpha+1}}{\alpha+1}.$$

The limit $\bar{A}(t)$ becomes the same FLLN limit for the standard Hawkes process (see [1]).

Remark 4. For the example $\Phi(y) = 1 - \gamma^{-1}(1+y)^{-\beta}$ in Remark 2, we have $\Psi(y) = \gamma\,(1+y)^\beta$ and $\bar{Y}$ solves equation

$$\bar{Y}(t) = \lambda_0 \mu \gamma \int_0^t (t-s)^\alpha \big(1 + \bar{Y}(s)\big)^\beta \, ds.$$

If $\beta = 1$, $\bar{Y}$ above can be expressed by Mittag-Leffler function as

$$\bar{Y}(t) = \lambda_0 \mu \sum_{k \geq 1} \frac{\gamma^k \, \Gamma^k(\alpha+1)}{\Gamma(k(\alpha+1)+1)} \, t^{k(\alpha+1)}.$$

2.4. **FCLT.** Define the CLT-scaled processes by

$$(\hat{A}^{(n)}, \hat{Y}^{(n)}) = \sqrt{n}\,(\bar{A}^{(n)} - \bar{A}, \bar{Y}^{(n)} - \bar{Y}), \tag{2.10}$$

where $(\bar{A}, \bar{Y})$ is the fluid limit in (2.8). Note that we have assumed the variable $\xi^{(n)}$ has finite variance, and the long range dependence is caused by the function $\varphi(t) \sim t^\alpha$, thus the scaling parameter for $\hat{Y}^{(n)}$ is still $\sqrt{n}$ (similarly as in [19]), although the scaling c_n is used in $\bar{Y}^{(n)}$.

Assumption A3. *Recalling the notations in Assumption A1, we further assume the following conditions hold.*

(i) There exist $\hat{\lambda}_0$ and $\hat{\mu}$ in $\mathbb{R}$ such that

$$\hat{\lambda}_0^{(n)} = \sqrt{n}\,(\lambda_0^{(n)} - \lambda_0) \to \hat{\lambda}_0 \quad \textit{and} \quad \hat{\mu}^{(n)} = \sqrt{n}\,(\mu^{(n)} - \mu) \to \hat{\mu} \quad \textit{as } n \to \infty.$$

(ii) Suppose Φ is continuously differentiable on $\mathbb{R}$ and for every $k > 0$,

$$\sup_{|y| \leq k} \sqrt{t} \int_t^\infty \phi(u, y)\, du \to 0 \quad \textit{as } t \to \infty.$$

(iii) Suppose that for some $\hat{\kappa} \in \mathbb{R}$,

$$\hat{\varphi}^{(n)}(t) := \sqrt{n}\Big(\frac{\varphi(nt)}{n^\alpha} - t^\alpha\Big) \to \hat{\kappa} \cdot t^{\alpha - 1/2} := \hat{\varphi}(t) \quad \textit{as } n \to \infty. \tag{2.11}$$

(iv) Suppose that as $n \to \infty$,

$$\big(\sigma^{(n)}\big)^2 = \mathit{Var}\big(\xi^{(n)}\big) \to \sigma^2 \quad \textit{and} \quad \mathbb{E}\big[(\xi^{(n)})^2; |\xi^{(n)}| > \sqrt{n}\varepsilon\big] \to 0 \quad \forall \varepsilon > 0.$$

Remark 5. Suppose that φ is regularly varying at ∞ with index $\alpha > 0$ discussed in Remark 1. For the FCLT, we would expect that

$$\hat{\varphi}^{(n)}(t) = \sqrt{n}\Big(\frac{\varphi(nt)}{n^\alpha} - t^\alpha\Big) \to \hat{\varphi}(t) \quad \text{non-trivially on } (0, \infty) \text{ as } n \to \infty$$

for some function $\hat{\varphi}$. It is necessarily that, as proved in [12, Theorem 1], some constant $\hat{\kappa} \in \mathbb{R}$

$$\hat{\varphi}(t) = \hat{\kappa} \cdot t^{\alpha - 1/2} \quad \forall t > 0,$$

and the convergence in (2.11) holds locally uniformly in $(0, \infty)$, c.f. [12, Remarks 4] and [15]. The condition above is also equivalent to

$$\sqrt{t} \cdot (t^{-\alpha}\varphi(t) - 1) \to \hat{\kappa} \quad \text{as } t \to \infty.$$

Theorem 2.3. *Under Assumptions A1, A2 and A3,*

$$\big(\hat{A}^{(n)}, \hat{Y}^{(n)}\big) \Rightarrow \big(\hat{A}, \hat{Y}\big) \quad \text{in } (\mathbb{D}^2, J_1) \text{ as } n \to \infty,$$

where $(\hat{A}, \hat{Y})$ is the unique strong solution to the stochastic integral equations

$$\begin{aligned} \hat{A}(t) &= \hat{\lambda}_0\, t + \int_0^t \Phi'(\bar{Y}(s))\hat{Y}(s)\, d\bar{A}(s) + \int_0^t \Phi(\bar{Y}(s))\, d\hat{A}(s) + \hat{X}(t)\,, \\ \hat{Y}(t) &= \mu\hat{\kappa} \int_0^t (t-s)^{\alpha-1/2}\, d\bar{A}(s) + \int_0^t (t-s)^{\alpha}\, d\big(\hat{Z}(s) + \hat{\mu}\bar{A}(s) + \mu\hat{A}(s)\big), \end{aligned} \tag{2.12}$$

where $\hat{X}$ and $\hat{Z}$ are independent mean-zero time-changed Brownian motions with variance function $\bar{A}$ and $\sigma^2\bar{A}$ with $\bar{A}$ in (2.9), respectively, that is, $\hat{X}(t) \stackrel{\mathrm{d}}{=} B_1(\bar{A}(t))$ and $\hat{Z}(t) \stackrel{\mathrm{d}}{=} \sigma B_2(\bar{A}(t))$ for two independent standard Brownian motions B_1 and B_2.

It is easy to see that $\hat{A}$ in (2.12) can be equivalently expressed as the following SDE:

$$\begin{aligned} d\hat{A}(t) &= \frac{\hat{\lambda}_0\, dt}{1-\Phi(\bar{Y}(t))} + \hat{Y}(t)\frac{\Phi'(\bar{Y}(s))}{1-\Phi(\bar{Y}(s))}\, d\bar{A}(t) + \frac{d\hat{X}(t)}{1-\Phi(\bar{Y}(t))} \\ &= \big(\hat{\lambda}_0\, \Psi(\bar{Y}(t)) + \lambda_0\hat{Y}(t)\Psi'(\bar{Y}(t))\big)\, dt + \Psi\big(\bar{Y}(t)\big)\, d\hat{X}(t), \end{aligned} \tag{2.13}$$

where (2.9) for $\bar{A}$ is applied. Hence, the well-definedness of the limit process $(\hat{A}, \hat{Y})$ follows from that of the process $\hat{Y}$, which is given in the following proposition.

Proposition 2.4. *Under Assumption A2, $\hat{Y}$ has the same law as the unique non-explosive diffusion solving the Volterra integral equation*

$$\begin{aligned} \hat{Y}(t) = &\int_0^t \Big(\lambda_0\mu\hat{\kappa}(t-s)^{\alpha-1/2} + \big(\lambda_0\hat{\mu} + \mu\hat{\lambda}_0\big)(t-s)^{\alpha}\Big)\Psi(\bar{Y}(s))\, ds \\ &+ \lambda_0\mu \int_0^t (t-s)^{\alpha}\Psi'\big(\bar{Y}(s)\big)\hat{Y}(s)\, ds + \int_0^t (t-s)^{\alpha}\mathcal{H}\big(\bar{Y}(s)\big)\, d\hat{B}(s), \end{aligned} \tag{2.14}$$

where $\mathcal{H}(y) = \sqrt{\lambda_0\Psi(y)(\sigma^2 + \mu^2\Psi^2(y))} \in \mathbb{C}$ and $\hat{B}$ is a standard Brownian motion.

Remark 6. The SDE of $\hat{Y}(t)$ in (2.14) indicates both path-dependence and long-range dependence. The first term on the right hand side of (2.14) is deterministic, depending on the fluid limit $\bar{Y}(s)$ in the path-dependence manner. The second term indicates that it is a linear (in $\hat{Y}(s)$) Volterra integral equation with a kernel function being $(t-s)^{\alpha}\Psi'(\bar{Y}(s))$. The third term indicates the long-range dependence in the Brownian noise, with a kernel function being $(t-s)^{\alpha}\mathcal{H}(\bar{Y}(s))$.

Note that in the special case of no state dependence in self-exciting function as discussed in Remark 3, we have $\Psi'\big(\bar{Y}(s)\big) = 0$ and $\mathcal{H}(y) \equiv c_{\mathcal{H}} := \sqrt{\frac{\lambda_0}{1-\|\phi\|_1}\big(\sigma^2 + \frac{\mu^2}{(1-\|\phi\|_1)^2}\big)}$ for each y. Hence, the SDE of $\hat{Y}(t)$ in (2.14) becomes

$$\hat{Y}(t) = \frac{\lambda_0\mu\hat{\kappa}}{1-\|\phi\|_1}\frac{t^{\alpha+1/2}}{\alpha+1/2} + \frac{\lambda_0\hat{\mu} + \mu\hat{\lambda}_0}{1-\|\phi\|_1}\frac{t^{\alpha+1}}{\alpha+1} + c_{\mathcal{H}}\int_0^t (t-s)^{\alpha} d\hat{B}(s). \tag{2.15}$$

Note that the last term is well-defined for all $\alpha > 0$. (The stochastic integral term also appears in [19].) If we further assume that $\hat{\kappa} = 0, \hat{\mu} = 0, \hat{\lambda}_0 = 0$, then the limit $\hat{Y}(t)$ becomes

$$\hat{Y}(t) = c_{\mathcal{H}}\int_0^t (t-s)^{\alpha} d\hat{B}(s)$$

which is a Reiman-Louisville fractional Brownian motion, with Hurst parameter $H = \alpha + 1/2 \in (1/2, 1]$ for $\alpha \in (0, 1/2]$. This resembles the result in [24], while our model has a more general arrival process (Hawkes instead of Poisson).

3. Proofs of the main results

This section is dedicated to the proofs of our main results.

3.1. **Proofs for the well-definedness of the limit processes.** We first consider the limit from the FLLN. Recalling the equivalent expression of $\bar{Y}(t)$ in (2.9), since $\bar{A}$ is an integral with respect to $\bar{Y}$, we only need to focus on the existence and the uniqueness of the solution to the equation for $\bar{Y}$, that is,

$$\bar{Y}(t) = \lambda_0 \mu \int_0^t (t-s)^{\alpha} \Psi\big(\bar{Y}(s)\big)\, ds. \tag{3.1}$$

The proof of the following proposition is given in the appendix.

Proposition 3.1. *Under Assumption A2, (3.1) exists a unique and non-explosive solution on* $\mathbb{R}$.

We next consider the limit process from the FCLT.

Proof of Proposition 2.4. Let $(\hat{A}, \hat{Y})$ be a solution to (2.12).

We start by showing that $\hat{Y}$ satisfies (2.14). Plugging (2.13) into $\hat{Y}$ in (2.12) gives

$$\begin{aligned}
\hat{Y}(t) &= \lambda_0 \mu \hat{\kappa} \int_0^t (t-s)^{\alpha-1/2} \Psi\big(\bar{Y}(s)\big)\, ds + \int_0^t (t-s)^{\alpha}\, d\big(\hat{Z}(s) + \hat{\mu}\bar{A}(s) + \mu \hat{A}(s)\big) \\
&= \lambda_0 \mu \hat{\kappa} \int_0^t (t-s)^{\alpha-1/2} \Psi\big(\bar{Y}(s)\big)\, ds + \big(\lambda_0 \hat{\mu} + \hat{\lambda}_0 \mu\big) \int_0^t (t-s)^{\alpha} \Psi\big(\bar{Y}(s)\big)\, ds \\
&\quad + \int_0^t (t-s)^{\alpha} \left(d\hat{Z}(s) + \mu \Psi(\bar{Y}(s))\, d\hat{X}(s)\right) + \lambda_0 \mu \int_0^t (t-s)^{\alpha} \Psi'(\bar{Y}(s)) \hat{Y}(s)\, ds \\
&=: \hat{Y}_0(t) + \lambda_0 \mu \int_0^t (t-s)^{\alpha} \Psi'(\bar{Y}(s))\, \hat{Y}(s)\, ds.
\end{aligned} \tag{3.2}$$

Recalling that $\hat{X}$ and $\hat{Z}$ in Theorem 2.3 are two independent mean-zero time-changed Brownian motions with variance $\bar{A}$ and $\sigma^2 \bar{A}$, respectively, we can write

$$\hat{B}(t) := \int_0^t \frac{d\hat{Z}(s) + \mu \Psi(\bar{Y}(s))\, d\hat{X}(s)}{\sqrt{\lambda_0 \Psi(\bar{Y}(s))(\sigma^2 + \mu^2 \Psi^2(\bar{Y}(s)))}} = \int_0^t \frac{d\hat{Z}(s) + \mu \Psi(\bar{Y}(s))\, d\hat{X}(s)}{\mathcal{H}(\bar{Y}(s))},$$

is a standard Brownian motion, recalling $\mathcal{H}$ in (2.14). Therefore, we have for the Itô integral part

$$\begin{aligned}
&\int_0^t (t-s)^{\alpha} \left(d\hat{Z}(s) + \mu \Psi(\bar{Y}(s))\, d\hat{X}(s)\right) \\
&= \int_0^t (t-s)^{\alpha} \mathcal{H}\big(\bar{Y}(s)\big)\, d\hat{B}(s) = \int_0^t \Big(\int_0^{t-u} \mathcal{H}(\bar{Y}(u))\, d\hat{B}(u) \Big)\, dv,
\end{aligned}$$

from which $\hat{Y}_0 \in \mathbb{C}$ is a well-defined non-explosive process.

We show next that, given $\hat{Y}_0$, $\hat{Y}$ is a pathwise, unique non-explosive solution to (3.2) by applying the Banach contraction principle. Given $\bar{Y}$ from FLLN, $\hat{y}_0 \in \mathbb{C}$, $\Psi' \in \mathbb{C}$ and $T > 0$, let

$$\sup_{t \le T} \lambda_0 \, |\mu\, \Psi'|(\bar{Y}(t)) = c_0 < \infty,$$

define $\delta_0 = (2c_0 \Gamma(\alpha+1))^{\frac{1}{\alpha+1}}$ and denote $\mathcal{F}_\delta : \mathbb{C} \to \mathbb{C}$ by

$$\mathcal{F}_\delta(x)(t) = e^{-\delta_0 t} \Big(\hat{y}_0(t) + \lambda_0 \mu \int_0^t (t-s)^{\alpha} \Psi'(\bar{Y}(s))\, e^{\delta_0 s} x(s)\, ds \Big).$$

We obtain for any $x, y \in \mathbb{C}$ and $t < T$,

$$\begin{aligned}\big|\mathcal{F}_\delta(x) - \mathcal{F}_\delta(y)\big|(t) &\le \lambda_0|\mu| \int_0^t e^{\delta_0(s-t)}(t-s)^\alpha \big|\Psi'(\bar{Y}(s))\big|\,|x-y|(s)\,ds \\ &\le c_0 \int_0^t e^{\delta(s-t)}(t-s)^\alpha\,|x-y|(s)\,ds \\ &\le \sup_{s\le t}|x-y|(s)\Big(c_0\int_0^\infty e^{-\delta_0 u}u^\alpha\,du\Big) = \frac{1}{2}\sup_{s\le t}|x-y|(s).\end{aligned}$$

for our choice of δ_0. Thus, one can find that $\mathcal{F}_\delta$ is a contraction mapping on $\mathbb{C}$, and $\hat{Y}$ is the fixed point of equation $\mathcal{F}_\delta(x) = x$ given $\hat{y}_0 = \hat{Y}_0$. One may also check that it is independent of the choice of δ_0. This proves the assertion. □

3.2. **Proof of FLLN.** We define the process $X^{(n)} = \{X^{(n)}(t) : t \ge 0\}$:

$$X^{(n)}(t) := A^{(n)}(t) - \Lambda^{(n)}(t).$$

It is evident that $X^{(n)}$ is a local martingale with respect to the filtration $\{\mathscr{F}_t^{(n)}, t \ge 0\}$ in (2.5) by Proposition 2.1. We also define the associated LLN-scaled process for $X^{(n)}$:

$$\bar{X}^{(n)}(t) := n^{-1}X^{(n)}(nt) = \bar{A}^{(n)}(t) - \bar{\Lambda}^{(n)}(t)\,. \tag{3.3}$$

Let

$$\bar{\mathscr{F}}_t^{(n)} = \mathscr{F}_{nt}^{(n)}$$

and

$$\bar{\tau}_j^{(n)} = n^{-1}\tau_j^{(n)}.$$

Under Assumption A1-(i), by (2.3), recalling $c_n = n^{-\alpha-1}$, we can write $\bar{Y}^{(n)}$ as

$$\bar{Y}^{(n)}(t) = c_n\,Y^{(n)}(nt) = \int_0^t \bar{\varphi}^{(n)}(t-s)\,d\bar{Z}^{(n)}(s), \tag{3.4}$$

where

$$\bar{\varphi}^{(n)}(t) := n^{-\alpha}\varphi(nt), \tag{3.5}$$

and

$$\bar{Z}^{(n)}(t) := \bar{W}^{(n)}(\bar{A}^{(n)}(t)) \quad \text{with} \quad \bar{W}^{(n)}(t) := \frac{1}{n}\sum_{j\ge1}\xi_j^{(n)}\mathbf{1}(j \le nt). \tag{3.6}$$

Next, we can express $\bar{A}^{(n)}(t)$ as

$$\begin{aligned}\bar{A}^{(n)}(t) &= \bar{X}^{(n)}(t) + \bar{\Lambda}^{(n)}(t) = \bar{X}^{(n)}(t) + \int_0^t \lambda^{(n)}(nu)\,du \\ &= \bar{X}^{(n)}(t) + \lambda_0^{(n)}\,t + \int_0^t \Big(\int_0^{n(t-u)} \phi(v, \bar{Y}^{(n)}(u-))\,dv\Big)\,d\bar{A}^{(n)}(u) \\ &= \bar{X}^{(n)}(t) + \lambda_0^{(n)}\,t + \int_0^t \Phi\big(\bar{Y}^{(n)}(u-)\big)\,d\bar{A}^{(n)}(u) - \bar{\varepsilon}^{(n)}(t),\end{aligned} \tag{3.7}$$

where

$$\bar{\varepsilon}^{(n)}(t) = \int_0^t \Big(\int_{n(t-s)}^\infty \phi\big(u, \bar{Y}^{(n)}(s-)\big)\,du\Big)\,d\bar{A}^{(n)}(s). \tag{3.8}$$

We will use a localization technique in the proofs, so that martingale properties can be applied. For an arbitrary fixed $T > 0$, let $k_0 > 0$ be a constant so that the solution $\bar{Y}$ to (2.8) satisfies

$$\sup_{t\le T} |\bar{Y}|(t) < k_0, \tag{3.9}$$

and define for such k_0

$$\bar{\tau}^{(n)}_{+,k_0} := \inf\big\{t > 0, |\bar{Y}^{(n)}|(t-) \vee |\bar{Y}^{(n)}|(t) \ge k_0\big\}, \tag{3.10}$$

with the convention $\inf \emptyset = \infty$, and denote by

$$\beta_{k_0} := \sup_{|y|\le k_0} \Phi(y) \quad \text{and} \quad \bar{J}_{k_0}(t) := \sup_{|y|\le k_0} \int_t^\infty \phi(u,y)\, du \quad \text{for } t \ge 0. \tag{3.11}$$

For notational brevity we drop the subscript k_0 in $\bar{\tau}^{(n)}_{+,k_0}$, β_{k_0} and $\bar{J}_{k_0}$, and c_0 denotes a constant that may vary from line to line. Under Assumption A1-(ii), the following facts are frequently used

$$\Phi(y) \le \bar{J}(0) = \beta < 1 \text{ for every } |y| \le k_0 \text{ and } \bar{J} \text{ is decreasing with } \bar{J}(\infty) = 0.$$

In addition, we define the $\bar{\tau}^{(n)}_+$-truncated version of $\bar{\varepsilon}^{(n)}$ and $\bar{Y}^{(n)}$ by

$$\bar{\varepsilon}^{(n)}_{k_0}(t) := \int_0^t \left(\int_{n(t-s)}^\infty \phi\big(u, \bar{Y}^{(n)}(s-)\big)\, du \right) d\bar{A}^{(n)}(s \wedge \bar{\tau}^{(n)}_{+,k_0}), \tag{3.12}$$

$$\bar{Y}^{(n)}_{k_0}(t) := \int_0^t \bar{\varphi}^{(n)}(t-s)\, d\bar{Z}^{(n)}\big(s \wedge \bar{\tau}^{(n)}_{+,k_0}\big), \tag{3.13}$$

comparing with (3.8) and (3.4), respectively. It is clear that

$$\big(\bar{\varepsilon}^{(n)}(t), \bar{Y}^{(n)}(t)\big) = \big(\bar{\varepsilon}^{(n)}_{k_0}(t), \bar{Y}^{(n)}_{k_0}(t)\big) \qquad \forall t < \bar{\tau}^{(n)}_+. \tag{3.14}$$

We next present several lemmas, which will be used extensively in the proof.

Lemma 3.1. *Under Assumption A1-(ii), $\bar{X}^{(n)}(\cdot \wedge \bar{\tau}^{(n)}_+)$ is an $\{\bar{\mathscr{F}}^{(n)}_t\}_{t\ge0}$-adapted martingale, and*

$$\mathbb{E}\big[\bar{A}^{(n)}(t \wedge \bar{\tau}^{(n)}_+)\big] \le \frac{\lambda^{(n)}_0}{1-\beta} \cdot t \qquad \forall t > 0. \tag{3.15}$$

There is a constant $c_0 > 0$, such that, for every $t > 0$,

$$\mathbb{E}\Big[\sup_{s\le t} (\bar{X}^{(n)})^2(s \wedge \bar{\tau}^{(n)}_+)\Big] \le \frac{c_0}{n} \cdot t.$$

Hence, $\bar{X}^{(n)}(\cdot \wedge \bar{\tau}^{(n)}_+)$ converges to 0 u.o.c. in $L^2(\mathbb{P})$.

Recalling the the following modulus of continuity of a function x on $[0,T]$ from [5, equation (7.1)]:

$$w_\delta(x,T) := \sup_{0\le u\le v\le T, u-v\le\delta} \big|x(u) - x(v)\big|, \quad \forall \delta > 0.$$

Lemma 3.2. *Under Assumption A1-(ii), for every $T > 0$, there is $c_0 > 0$ such that*

$$\limsup_{n\to\infty} \mathbb{E}\Big[w_\delta\big(\bar{A}^{(n)}(\cdot \wedge \bar{\tau}^{(n)}_+), T\big)\Big] \le c_0 \cdot \delta, \tag{3.16}$$

for every $\delta > 0$, and

$$\limsup_{n\to\infty} \mathbb{E}\Big[\sup_{t\le T} \bar{\varepsilon}^{(n)}_{k_0}(t)\Big] = 0.$$

Lemma 3.3. *Under Assumption A1-(iii), for $\bar{W}^{(n)}$ in (3.6), we have*

$$\bar{W}^{(n)} \to \mu\mathfrak{e} \quad \textit{u.o.c. in probability as } n \to \infty,$$

where $\mathfrak{e}(t) = t$ denotes the identity function. If, in addition, Assumption A3-(iv) holds, then

$$\hat{W}^{(n)} := \sqrt{n}\,(\bar{W}^{(n)} - \mu^{(n)}\mathfrak{e}) \Rightarrow \hat{W} \quad \textit{in } (\mathbb{D}, J_1) \textit{ as } n \to \infty,$$

where $\hat{W}$ is a mean-zero Brownian motions with variance σ^2.

Lemma 3.4. *Let $z_n, a_n \in \mathbb{D}$ and a_n be an increasing function. Suppose $z_n \to z$ and $a_n \to a$ u.o.c. for some $z \in \mathbb{D}$ and $a \in \mathbb{C}$. Then,*

$$\int_0^t z_n(t-s)\,da_n(s) \to \int_0^t z(t-s)\,da(s) \quad \textit{u.o.c. as } n \to \infty.$$

Lemma 3.5. *Recalling $\bar{\varphi}^{(n)}$ in (3.5), under Assumption A1, we have*

$$\int_0^t \left(\bar{\varphi}^{(n)}(t-s) - (t-s)^\alpha\right) d\bar{Z}^{(n)}(s \wedge \bar{\tau}_+^{(n)}) \to 0 \quad \textit{u.o.c. in probability as } n \to \infty.$$

Lemmas 3.1 and 3.2 can be proved following the same procedures as those in [31] (see equation (5.1), Lemmas 5.1, 5.2 & 5.3 therein). Lemma 3.3 is essentially the weak convergence for triangular array, which holds under more general conditions by localization, c.f. [13, Theorem 2.2.11]. Hence the proofs for these lemmas are omitted for brevity. Lemma 3.4 is related to the continuity of measures in the sense of convolution, which extends the results [34, Lemma 6.1] and [29, Lemma 5.2], see also [26, Example 5.3]. Lemma 3.5 looks similar to Lemma 3.4, where we do have the convergence for $\bar{\varphi}^{(n)}$ u.o.c. as shown in Remark 1; however, $\bar{Z}^{(n)}$ fails to have monotone paths. The proofs of Lemmas 3.4 and 3.5 are given in Section 4.1.

Now, we are ready to prove our FLLN. Basically, we show that the joint process $(\bar{A}^{(n)}, \bar{Y}^{(n)})$ is relatively compact in $(\mathbb{D}^2, J_1)$ with a continuous limit, and every limit satisfies (2.8). Then, the uniqueness of solution in Proposition 3.1 can be used to finish the proof.

Proof of Theorem 2.2. By the fact that $\Delta\bar{A}^{(n)}$ is uniformly bounded, and by (3.15) in Lemma 3.1 and (3.16) in Lemma 3.2, it is evident that $\bar{A}^{(n)}(\cdot \wedge \bar{\tau}_+^{(n)})$ is a $\mathbb{C}$-tight family in $(\mathbb{D}, J_1)$. Hence, $(\bar{A}^{(n)}(\cdot \wedge \bar{\tau}_+^{(n)}), T \wedge \bar{\tau}_+^{(n)})$ is relatively compact in $(\mathbb{D}, J_1) \times ([0,T], \mathcal{B}([0,T]))$.

Let $(\bar{A}_{k_0}, T_{k_0}) \in \mathbb{C} \times [0,T]$ be a weak limit point over some convergent subsequence, say $\{n_k, k \geq 1\}$,

$$\left(\bar{A}^{(n)}(\cdot \wedge \bar{\tau}_+^{(n)}), T \wedge \bar{\tau}_+^{(n)}\right)\Big|_{n=n_k} \Rightarrow \left(\bar{A}_{k_0}, T_{k_0}\right) \quad \text{in } (\mathbb{D}, J_1) \times ([0,T], \mathcal{B}([0,T])).$$

Notice that we only have convergence in distribution at the moment.

Applying Lemma 3.3 to $\bar{Z}^{(n)}$ in (3.6), one can see that over the same subsequence,

$$\left(\left(\bar{A}^{(n)}, \bar{Z}^{(n)}\right)(\cdot \wedge \bar{\tau}_+^{(n)}), T \wedge \bar{\tau}_+^{(n)}\right)\Big|_{n=n_k} \Rightarrow \left(\bar{A}_{k_0}, \bar{Z}_{k_0}, T_{k_0}\right) \quad \text{in } (\mathbb{D}^2, J_1) \times ([0,T], \mathcal{B}([0,T])), \tag{3.17}$$

where $\bar{Z}_{k_0} = \mu\bar{A}_{k_0} \in \mathbb{C}$. We can rewrite $\bar{Y}_{k_0}^{(n)}$ in (3.13) as

$$\begin{aligned}
\bar{Y}_{k_0}^{(n)}(t) &= \int_0^t \left(\bar{\varphi}^{(n)}(t-s) - (t-s)^\alpha\right) d\bar{Z}^{(n)}(s \wedge \bar{\tau}_+^{(n)}) + \int_0^t (t-s)^\alpha\, d\bar{Z}^{(n)}(s \wedge \bar{\tau}_+^{(n)}) \\
&= \int_0^t \left(\bar{\varphi}^{(n)}(t-s) - (t-s)^\alpha\right) d\bar{Z}^{(n)}(s \wedge \bar{\tau}_+^{(n)}) + \int_0^t \bar{Z}^{(n)}\big((t-s) \wedge \bar{\tau}_+^{(n)}\big)\, ds^\alpha,
\end{aligned}$$

where in the n^{th}-system, $\bar{Z}^{(n)}$ in (3.6) is a random partial sum so that integration by parts is applicable to the second term above, and s^α defines a continuous measure on $\mathbb{R}_+$ for $\alpha > 0$.

Applying Lemma 3.5 to the first term above, Lemma 3.4 and (3.17) to the second term, we will obtain

$$\begin{aligned}&\Big(\big(\bar{A}^{(n)},\bar{Z}^{(n)}\big)(\cdot\wedge\bar{\tau}_+^{(n)}),\bar{Y}_{k_0}^{(n)}(\cdot),T\wedge\bar{\tau}_+^{(n)}\Big)\Big|_{n=n_k}\\ &\Rightarrow\big(\bar{A}_{k_0},\bar{Z}_{k_0},\bar{Y}_{k_0},T_{k_0}\big)\quad\text{in }(\mathbb{D}^3,J_1)\times([0,T],\mathcal{B}([0,T])\text{ as }n\to\infty,\end{aligned}\tag{3.18}$$

where

$$\bar{Y}_{k_0}(t)=\int_0^t\bar{Z}_{k_0}(t-s)\,ds^\alpha=\mu\int_0^t\bar{A}_{k_0}(t-s)\,ds^\alpha\in\mathbb{C},\quad\forall t\ge 0.\tag{3.19}$$

Now, we return to $\bar{A}^{(n)}(\cdot\wedge\bar{\tau}_+^{(n)})$ in (3.7). For $t\le T$, we can write it as

$$\begin{aligned}\bar{A}^{(n)}(t\wedge\bar{\tau}_+^{(n)})&=\lambda_0^{(n)}\big(t\wedge(T\wedge\bar{\tau}_+^{(n)})\big)+\bar{X}^{(n)}\big(t\wedge\bar{\tau}_+^{(n)}\big)\\&\quad+\int_0^t\Phi\big(\bar{Y}_{k_0}^{(n)}(s-)\big)\,d\bar{A}^{(n)}(s\wedge\bar{\tau}_+^{(n)})-\bar{\varepsilon}^{(n)}(t\wedge\bar{\tau}_+^{(n)})\end{aligned}\tag{3.20}$$

where the fact (3.14) is applied. Moreover, we have from (3.12) and (3.8) that

$$\bar{\varepsilon}^{(n)}(t\wedge\bar{\tau}_+^{(n)})-\bar{\varepsilon}_{k_0}^{(n)}(t)=\frac{\mathbf{1}(\bar{\tau}_+^{(n)}\le t)}{n}\int_0^{n(t-\bar{\tau}_+^{(n)})}\phi\big(u,\bar{Y}^{(n)}(\bar{\tau}_+^{(n)}-)\big)\,du\le\frac{1}{n}\to 0\quad\text{u.o.c.}$$

Applying the limit for $\bar{X}^{(n)}$ in Lemma 3.1, the limit for $\bar{\varepsilon}_{k_0}^{(n)}$ in Lemma 3.2 and the joint convergence in (3.18), one can employ [26, Theorem 2.7] to (3.20) to find that $\bar{A}_{k_0}$ solves the equation

$$\bar{A}_{k_0}(t)=\lambda_0\,(t\wedge T_{k_0})+\int_0^t\Phi\big(\bar{Y}_{k_0}(s)\big)\,d\bar{A}_{k_0}(s)\quad\forall t\in[0,T].$$

Therefore, by the identity above and (3.19), $(\bar{A}_{k_0},\bar{Y}_{k_0})$ solves the same set of nonlinear integral equations as (2.8) on $[0,T_{k_0})$. By the uniqueness of solution in Proposition 3.1, it holds that

$$\big(\bar{A}_{k_0},\bar{Y}_{k_0}\big)(t)=\big(\bar{A},\bar{Y}\big)(t)\quad\forall t\in[0,T_{k_0}),\tag{3.21}$$

which is now a deterministic function on possible random interval $[0,T_{k_0})$. Finally, by the continuous mapping theorem, as a weak limit of $T\wedge\bar{\tau}_+^{(n)}$, we should have on the set $\{T_{k_0}<T\}$ that

$$k_0\le\sup_{t\le T_{k_0}}\big|\bar{Y}_{k_0}\big|(t-)\vee\big|\bar{Y}_{k_0}\big|(t)=\sup_{t\le T_{k_0}}\big|\bar{Y}\big|(t)\le\sup_{t\le T}\big|\bar{Y}\big|(t)$$

where (3.21) is applied. However, this cannot happen by for our choice of (T,k_0) in (3.9). Therefore,

$$\mathbb{P}\big(\bar{\tau}_{+,k_0}^{(n)}\le T\big)\to 0.\tag{3.22}$$

Putting back to (3.18) proves the limit without stopping. □

3.3. **Proof of FCLT.** We first obtain the following representations for the CLT-scaled processes $(\hat{A}^{(n)},\hat{Y}^{(n)})$. By (3.4) and (2.8), recalling $\bar{Z}=\mu\bar{A}$, we obtain

$$\begin{aligned}\hat{Y}^{(n)}(t)&=\sqrt{n}\bigg(\int_0^t\bar{\varphi}^{(n)}(t-s)\,d\bar{Z}^{(n)}(s)-\mu\int_0^t(t-s)^\alpha\,d\bar{A}(s)\bigg)\\&=\int_0^t\hat{\varphi}^{(n)}(t-s)\,d\bar{Z}^{(n)}(s)+\sqrt{n}\int_0^t(t-s)^\alpha d\big(\bar{Z}^{(n)}(s)-\bar{A}^{(n)}(s)\big),\end{aligned}$$

recalling $\hat{\varphi}^{(n)}$ in (2.11). By integration by parts to the second term above, it further gives

$$\begin{aligned}\hat{Y}^{(n)}(t)&=\int_0^t\hat{\varphi}^{(n)}(t-s)\,d\bar{Z}^{(n)}(s)+\int_0^t\big(\hat{Z}^{(n)}(t-s)+\hat{\mu}^{(n)}\bar{A}^{(n)}(t-s)\big)\,ds^\alpha\\&\quad+\mu\int_0^t\hat{A}^{(n)}(t-s)\,ds^\alpha=:\hat{\Xi}^{(n)}(t)+\mathcal{F}(\hat{A}^{(n)})(t),\end{aligned}\tag{3.23}$$

where $\hat{\mu}^{(n)}$ is given in Assumption A3, and

$$\hat{Z}^{(n)}(t) := \sqrt{n}\big(\bar{Z}^{(n)}(t) - \mu^{(n)}\bar{A}^{(n)}(t)\big) = \hat{W}^{(n)}\big(\bar{A}^{(n)}(t)\big), \tag{3.24}$$

recalling $\bar{Z}^{(n)}$ in (3.6) and $\hat{W}^{(n)}$ defined in Lemma 3.3. Here, $\mathcal{F} : \mathbb{D} \to \mathbb{C}$ is defined by

$$\mathcal{F}(x)(t) := \mu \int_0^t x(t-s)\, ds^{\alpha} \quad \forall x \in \mathbb{D}. \tag{3.25}$$

Next, by (2.8) and (3.7), we obtain

$$\begin{aligned} \hat{A}^{(n)}(t) = \hat{X}^{(n)}(t) + \hat{\lambda}_0^{(n)}\, t + \int_0^t \Phi(\bar{Y}^{(n)}(s-))\, d\hat{A}^{(n)}(s) \\ + \sqrt{n} \cdot \int_0^t \big(\Phi(\bar{Y}^{(n)}(s)) - \Phi(\bar{Y}(s))\big)\, d\bar{A}(s) - \hat{\varepsilon}^{(n)}(t), \end{aligned} \tag{3.26}$$

where $\hat{\lambda}_0^{(n)}$ is given in Assumption A3, and

$$\hat{X}^{(n)}(t) := \sqrt{n} \cdot \bar{X}^{(n)}(t), \quad \hat{\varepsilon}^{(n)}(t) = \sqrt{n} \cdot \bar{\varepsilon}^{(n)}(t)\,.$$

We will employ [26, Theorem 5.4] to establish the FCLT. However, the integral term of $d\hat{A}^{(n)}$ on the right-hand side of the governing equation (3.26) precludes the direct application of the theorem. Hence, we propose the following alternative representation for $\hat{A}^{(n)}$, to which we can apply [26, Theorem 5.4] directly.

For that purpose, we first observe that

$$\begin{aligned} &\int_0^t \Psi\big(\bar{Y}(s)\big)\, d\bar{A}^{(n)}(s) = \int_0^t \Psi\big(\bar{Y}(s)\big)\, d\bar{X}^{(n)}(s) + \int_0^t \Psi\big(\bar{Y}(s)\big)\lambda^{(n)}(ns)\, ds \\ &= \bar{X}_{\Psi}^{(n)}(t) + \lambda_0^{(n)} \int_0^t \Psi\big(\bar{Y}(s)\big)\, ds + \int_0^t \Psi\big(\bar{Y}(u)\big)\, du \int_0^u n\phi\big(n(u-v), \bar{Y}^{(n)}(v-)\big)\, d\bar{A}^{(n)}(v) \\ &= \bar{X}_{\Psi}^{(n)}(t) + \frac{\lambda_0^{(n)}}{\lambda_0}\, \bar{A}(t) + \int_0^t \Psi\big(\bar{Y}(s)\big)\Phi\big(\bar{Y}^{(n)}(s-)\big)\, d\bar{A}^{(n)}(s) - \bar{\varepsilon}_{\Psi}^{(n)}(t), \end{aligned}$$

where $\bar{X}_{\Psi}^{(n)}(t) := \int_0^t \Psi\big(\bar{Y}(s)\big)\, d\bar{X}^{(n)}(s)$, the expression of $\bar{A}$ in (2.9) is used to get the second equality, and

$$\bar{\varepsilon}_{\Psi}^{(n)}(t) = \int_0^t d\bar{A}^{(n)}(s)\Big(\Psi\big(\bar{Y}(s)\big)\Phi\big(\bar{Y}^{(n)}(s-)\big) - \int_0^{t-s} \Psi\big(\bar{Y}(u+s)\big) n\phi\big(nu, \bar{Y}^{(n)}(s-)\big)\, du\Big), \tag{3.27}$$

which follows from a change of variables. Hence, from $\Psi(y)(1-\Phi(y)) = 1$, we obtain that

$$\begin{aligned} \hat{A}^{(n)}(t) &= \sqrt{n}\Big(\int_0^t \Psi(\bar{Y}(s))\big(1 - \Phi(\bar{Y}(s))\big)\, d\bar{A}^{(n)}(s) - \bar{A}(t)\Big) \\ &= \hat{X}_{\Psi}^{(n)}(t) + \frac{\hat{\lambda}_0^{(n)}}{\lambda_0}\bar{A}(t) + \int_0^t \hat{Y}^{(n)}(s-)\, d\bar{I}^{(n)}(s) - \hat{\varepsilon}_{\Psi}^{(n)}(t), \end{aligned} \tag{3.28}$$

where with the understanding that $\frac{\Phi(z)-\Phi(y)}{z-y}\Big|_{z=y} = \Phi'(y)$

$$\hat{X}_{\Psi}^{(n)}(t) = \sqrt{n}\int_0^t \Psi\big(\bar{Y}(s)\big)\, d\bar{X}^{(n)}(s) \quad \text{and} \quad \hat{\varepsilon}_{\Psi}^{(n)}(t) = \sqrt{n} \cdot \bar{\varepsilon}_{\Psi}^{(n)}(t), \tag{3.29}$$

$$\bar{I}^{(n)}(t) = \int_0^t \Psi(\bar{Y}(s)) \frac{\Phi(\bar{Y}^{(n)}(s-)) - \Phi(\bar{Y}(s))}{\bar{Y}^{(n)}(s-) - \bar{Y}(s)}\, d\bar{A}^{(n)}(s). \tag{3.30}$$

Plugging the expression of $\hat{Y}^{(n)}$ in (3.23) into (3.28), we obtain

$$\hat{A}^{(n)}(t) = \hat{U}^{(n)}(t) + \int_0^t \mathcal{F}(\hat{A}^{(n)})(s)\, d\bar{I}^{(n)}(s) - \hat{\varepsilon}_{\Psi}^{(n)}(t), \tag{3.31}$$

recalling that $\mathcal{F}(\hat{A}^{(n)}) \in \mathbb{C}$ by definition (3.25) and $\hat{\Xi}^{(n)}$ in (3.23), where

$$\hat{U}^{(n)}(t) = \hat{X}_{\Psi}^{(n)}(t) + \frac{\hat{\lambda}_0^{(n)}}{\lambda_0}\bar{A}(t) + \int_0^t \hat{\Xi}^{(n)}(s-)\, d\bar{I}^{(n)}(s). \tag{3.32}$$

To prove FCLT, we need the following lemmas whose proofs are given in Section 4.2.

Lemma 3.6. *Under Assumptions A1, A2 and A3-(iv),*

$$\big(\hat{X}^{(n)}, \hat{Z}^{(n)}\big) \Rightarrow \big(\hat{X}, \hat{Z}\big) \quad \text{in } (\mathbb{D}^3, J_1) \text{ as } n \to \infty.$$

where $\hat{X}$, $\hat{Z}$ are mutually independent time-changed Brownian motions as given in Theorem 2.3.

Lemma 3.7. *Under Assumptions A1, A2, A3-(i) and (ii),*

$$\hat{\varepsilon}^{(n)}(t) \to 0 \quad \text{and} \quad \hat{\varepsilon}_{\Psi}^{(n)}(t) \to 0 \quad \text{u.o.c. in probability as } n \to \infty.$$

Lemma 3.6 provides the joint convergence of the main sources of randomness for FCLT, which the FLLN under Assumptions A1 and A2 and the second moment condition in Assumption A3-(iv) gives a Brownian limit. Lemma 3.7 shows that $\hat{\varepsilon}^{(n)}$ and $\hat{\varepsilon}_{\Psi}^{(n)}$ in (3.26) and (3.29) are indeed negligible for n large enough, where the limit for $\hat{\varepsilon}^{(n)}$ can be proved by following the procedures similar to [31, Lemma 5.5] and Assumption A3-(ii) is necessary.

Recall Assumption A3-(iii) and Remark 5 that

$$\sqrt{t}\cdot(t^{-\alpha}\varphi(t) - 1) \to \hat{\kappa} \quad \text{as } t \to \infty \text{ and as } n \to \infty$$

and $\hat{\varphi}^{(n)}(t) = \sqrt{n}\big(\bar{\varphi}^{(n)}(t) - t^{\alpha}\big)$. The following Lemma 3.8 is similar to Lemmas 3.5 and 3.4. However, for the case of small $\alpha \in (0, 1/2)$, we only have uniform convergence for $\hat{\varphi}^{(n)}$ on $(0, \infty)$ as discussed in Remark 5, and 0 is a singular point in this case which causes significant technical challenges. Lemma 3.9 is a verification for condition C5.4(ii) in [26] for the functional $\mathcal{F}$ in (3.25), where we use the notations in that paper. Also notice that, for the case $\alpha \in (0, 1)$, $\mathcal{F}$ in (3.25) is not included in the Example 5.4-(b) in [26].

Lemma 3.8. *Under Assumptions A1, A2 and A3,*

$$\int_0^t \hat{\varphi}^{(n)}(t-s)\, d\bar{Z}^{(n)}(s) \to \int_0^t \hat{\varphi}(t-s)\, d\bar{Z}(s) \quad \text{u.o.c. in probability as } n \to \infty.$$

where $\bar{Z} = \mu\bar{A}$ and $\hat{\varphi}(t) = \hat{\kappa}\cdot t^{\alpha-1/2}$.

Lemma 3.9. *Let $\mathbb{C}_{\uparrow}$ be the collection of nondecreasing continuous functions onto $\mathbb{R}_+$, that is, $\theta(0) = 0$ and $\theta(\infty) = \infty$ for every $\theta \in \mathbb{C}_{\uparrow}$. Define $\mathcal{G} : \mathbb{D} \times \mathbb{C}_{\uparrow} \to \mathbb{C}$ by*

$$\mathcal{G}(z, \theta)(t) = \mu\alpha \int_0^t (\theta(t) - \theta(s))^{\alpha-1} z(s)\, d\theta(s).$$

Then $\{(z_n, \theta_n)\} \subset \mathbb{D} \times \mathbb{C}_{\uparrow}$ with $z_n \to z$ and $\theta_n \to \theta$ u.o.c., implies $\mathcal{G}(z_n, \theta_n) \to \mathcal{G}(z, \theta)$ u.o.c.

Now, we are ready to prove Theorem 2.3, where [26, Theorem 5.4] concerning the convergence of stochastic integrals is applied to (3.31), for which the joint convergence of the component processes is required.

Proof of Theorem 2.3. Recalling the facts that $(\bar{A}^{(n)}, \bar{Y}^{(n)}) \to (\bar{A}, \bar{Y})$ u.o.c. in probability from FLLN, $(\bar{A}, \bar{Y})$ is a deterministic and continuous solution to (2.8), and the joint convergence $(\hat{X}^{(n)}, \hat{Z}^{(n)}) \Rightarrow (\hat{X}, \hat{Z})$ in $(\mathbb{D}^2, J_1)$ holds as shown in Lemma 3.6, we immediately have

$$\big(\bar{A}^{(n)}, \bar{Y}^{(n)}, \hat{X}^{(n)}, \hat{Z}^{(n)}\big) \Rightarrow \big(\bar{A}, \bar{Y}, \hat{X}, \hat{Z}\big) \quad \text{in } (\mathbb{D}^4, J_1) \text{ as } n \to \infty.$$

Applying [26, Theorem 2.7] to $\bar{I}^{(n)}$ in (3.30) with the observation that

$$\frac{\Phi(\bar{Y}^{(n)}(s-)) - \Phi(\bar{Y}(s))}{\bar{Y}^{(n)}(s-) - \bar{Y}(s)} \to \Phi'(\bar{Y}(s)) \quad \text{u.o.c. in probability as } n \to \infty$$

from the limit above, we have

$$\big(\bar{A}^{(n)}, \bar{Y}^{(n)}, \bar{I}^{(n)}, \hat{X}^{(n)}, \hat{Z}^{(n)}\big) \Rightarrow \big(\bar{A}, \bar{Y}, \bar{I}, \hat{X}, \hat{Z}\big) \quad \text{in } (\mathbb{D}^5, J_1) \text{ as } n \to \infty.$$

where by the facts $\Psi = (1-\Phi)^{-1}$ and the identity for $\bar{A}$ in (2.9)

$$\bar{I}(t) = \lambda_0 \int_0^t \Psi'(\bar{Y}(s))\, ds.$$

Applying Lemma 3.8 and the limit above to $\hat{\Xi}^{(n)}$ in (3.23) gives

$$\big(\bar{A}^{(n)}, \bar{Y}^{(n)}, \bar{I}^{(n)}, \hat{X}^{(n)}, \hat{Z}^{(n)}, \hat{\Xi}^{(n)}\big) \Rightarrow \big(\bar{A}, \bar{Y}, \bar{I}, \hat{X}, \hat{Z}, \hat{\Xi}\big) \quad \text{in } (\mathbb{D}^6, J_1) \text{ as } n \to \infty.$$

where

$$\hat{\Xi}(t) = \int_0^t \hat{\varphi}(t-s)\, d\bar{Z}(s) + \int_0^t \big(\hat{Z}(t-s) + \hat{\mu}\bar{A}(t-s)\big)\, ds^{\alpha}.$$

Plugging the joint limit above into (3.32) for $\hat{U}^{(n)}$, and applying [26, Theorem 2.7] gives

$$\big(\bar{A}^{(n)}, \bar{Y}^{(n)}, \bar{I}^{(n)}, \hat{X}^{(n)}, \hat{Z}^{(n)}, \hat{\Xi}^{(n)}, \hat{U}^{(n)}\big) \Rightarrow \big(\bar{A}, \bar{Y}, \bar{I}, \hat{X}, \hat{Z}, \hat{\Xi}, \hat{U}\big) \tag{3.33}$$

in $(\mathbb{D}^7, J_1)$ as $n \to \infty$, where

$$\hat{U}(t) = \frac{\hat{\lambda}_0}{\lambda_0}\bar{A}(t) + \int_0^t \Psi\big(\bar{Y}(s)\big)\, d\hat{X}(s) + \int_0^t \hat{\Xi}(s)\, d\bar{I}(s).$$

Now, we apply [26, Theorem 5.4] to (3.31):

(i) One can find that the integrator $\bar{I}^{(n)}$ in (3.30) is a semi-martingale which has uniformly bounded variation on every compact set by the local Lipschitz continuity property of $\Psi = (1-\Phi)^{-1}$ in Assumption A2-(i) and (3.15), thus C.2.2(i) in [26] is verified;

(ii) The functional on $\mathbb{D} \times \mathbb{C}_{\uparrow}$ associated to $\mathcal{F}$ in (3.24), in the condition C5.4(ii) in [26], is given by $\mathcal{G}$ as one can check, whose continuity is proved in Lemma 3.9, thus C.5.4. in [26] is verified.

Therefore, applying [26, Theorem 5.4] to (3.31), with the joint limits in (3.33) and Lemma 3.7 for $\hat{\varepsilon}_{\Psi}^{(n)}$, one can find that

$$\big(\bar{A}^{(n)}, \bar{Y}^{(n)}, \bar{I}^{(n)}, \hat{X}^{(n)}, \hat{Z}^{(n)}, \hat{\Xi}^{(n)}, \hat{U}^{(n)}, \hat{A}^{(n)}\big) \Rightarrow \big(\bar{A}, \bar{Y}, \bar{I}, \hat{X}, \hat{Z}, \hat{\Xi}, \hat{U}, \hat{A}\big)$$

in $(\mathbb{D}^8, J_1)$ as $n \to \infty$. Further plugging into (3.23) and applying Lemma 3.4 leads to

$$\big(\bar{A}^{(n)}, \bar{Y}^{(n)}, \bar{I}^{(n)}, \hat{X}^{(n)}, \hat{Z}^{(n)}, \hat{\Xi}^{(n)}, \hat{U}^{(n)}, \hat{A}^{(n)}, \hat{Y}^{(n)}\big) \Rightarrow \big(\bar{A}, \bar{Y}, \bar{I}, \hat{X}, \hat{Z}, \hat{\Xi}, \hat{U}, \hat{A}, \hat{Y}\big)$$

in $(\mathbb{D}^9, J_1)$ as $n \to \infty$, where

$$\hat{A}(t) = \hat{U}(t) + \int_0^t \mathcal{F}(\hat{A})(s)\, d\bar{I}(s) \quad \text{and} \quad \hat{Y}(t) = \hat{\Xi}(t) + \mathcal{F}(\hat{A})(t).$$

To see the SDE for $(\hat{A}, \hat{Y})$ in (2.12), substituting the joint limit above into (3.23) and (3.26), and applying Lemma 3.7 for $\hat{\varepsilon}^{(n)}$, Lemma 3.4 and 3.8, and [26, Theorem 2.7], we obtain

$$\hat{Y}(t) = \hat{\kappa} \int_0^t (t-s)^{\alpha-1/2}\, d\bar{Z}(s) + \int_0^t \big(\hat{Z} + \hat{\mu}\bar{A} + \mu\hat{A}\big)(t-s)\, ds^{\alpha},$$
$$\hat{A}(t) = \hat{\lambda}_0\, t + \hat{X}(t) + \int_0^t \Phi\big(\bar{Y}(s)\big)\, d\hat{A}(s) + \int_0^t \hat{Y}(s)\Phi'(\bar{Y}(s))\, d\bar{A}(s).$$

Finally, applying Itô formula gives (2.12) and finishes the proof. □

4. Proofs for the supporting lemmas

4.1. **Lemmas for FLLN.** We provide the detailed proofs for Lemmas 3.4 and 3.5, which are particularly associated with the LLN-scaled process $\bar{Y}^{(n)}$ with the power response function φ under Assumption A1-(i).

Proof of Lemma 3.4. Let $z^{\varepsilon} \in \mathbb{D}$ be a piecewise constant function approximating z, that is,

$$z^{\varepsilon}(t) = \sum_{j\geq 1} z_j \mathbf{1}\big(t \in [t_{j-1}, t_j)\big) \quad \text{and} \quad \sup_{t\geq 0} |z^{\varepsilon}(t) - z(t)| \leq \varepsilon, \tag{4.1}$$

with $t_0 = 0$. See [26, Lemma 6.1] for more technical results. First, we obtain

$$\begin{aligned}
&\Big| \int_0^t z_n(t-s)\, da_n(s) - \int_0^t z(t-s)\, da(s) \Big| \\
&\leq \int_0^t \big|z_n - z\big|(t-s)\, da_n(s) + \Big| \int_0^t z(t-s)\, d(a_n - a)(s) \Big| \\
&\leq \sup_{s\leq t} |z - z_n|(s) \cdot a_n(t) + \int_0^t \big|z - z^{\varepsilon}\big|(t-s)\, d(a_n + a)(s) + \Big| \int_0^t z^{\varepsilon}(t-s)\, d(a_n - a)(s) \Big| \\
&\leq \sup_{s\leq t} |z - z_n|(s) \cdot a_n(t) + \varepsilon \cdot \big(a_n(t) + a(t)\big) + \Big| \int_0^t z^{\varepsilon}(t-s)\, d(a_n - a)(s) \Big|.
\end{aligned} \tag{4.2}$$

Next, by the definition of z^{ε} in (4.1), we also have

$$\begin{aligned}
\int_0^t z^{\varepsilon}(t-s)\, d(a_n - a)(s) &= \sum_{j\geq 1} z_j \cdot \int_0^t \mathbf{1}\big(t-s \in [t_{j-1}, t_j)\big)\, d(a_n - a)(s) \\
&= \sum_{j\geq 1} z_j \cdot \Big(\big(a_n(t-t_{j-1}) - a(t-t_{j-1})\big) - \big(a_n(t-t_j) - a(t-t_j)\big) \Big),
\end{aligned}$$

with the understanding that $a(u) = 0 = a_n(u)$ for $u < 0$. Plugging into (4.2) gives

$$\begin{aligned}
&\sup_{t\leq T} \Big| \int_0^t z_n(t-s)\, da_n(s) - \int_0^t z(t-s)\, da(s) \Big| \\
&\leq \sup_{s\leq T} |z - z_n|(s) \cdot a_n(T) + \varepsilon \cdot \big(a_n(T) + a(T)\big) + \sup_{t\leq T} \big(|z(t)| + \varepsilon\big) \cdot \sup_{t\leq T} |a(t) - a_n(t)| \cdot 2\, N_{\varepsilon},
\end{aligned}$$

where N_{ε} is the finite number for arbitrary fixed $\varepsilon > 0$ so that $T \in (t_{N_{\varepsilon}-1}, t_{N_{\varepsilon}}]$ and independent to n. Letting $n \to \infty$ and then $\varepsilon \to 0$ proves the lemma. □

Proof of Lemma 3.5. Recalling $\bar{Z}^{(n)}$ in (3.6), we have

$$\Big| \int_0^t \big(\bar{\varphi}^{(n)}(t-s) - (t-s)^{\alpha}\big)\, d\bar{Z}^{(n)}(s \wedge \bar{\tau}_+^{(n)}) \Big| \leq \sup_{s\leq t} \big|\bar{\varphi}^{(n)}(s) - s^{\alpha}\big| \cdot \check{Z}_{k_0}^{(n)}(t), \tag{4.3}$$

where

$$\check{Z}_{k_0}^{(n)}(t) = \frac{1}{n}\sum_{j\geq 1} \big|\xi_j^{(n)}\big| \mathbf{1}\big(\bar{\tau}_j^{(n)} \leq t \wedge \bar{\tau}_+^{(n)}\big). \tag{4.4}$$

By the unpredictable assumption on $\xi^{(n)}$, Assumption A1-(iii) and (3.15), one can see that

$$\mathbb{E}\big[\check{Z}_{k_0}^{(n)}(t)\big] = \mathbb{E}\big[|\xi^{(n)}|\big] \cdot \mathbb{E}\big[\bar{A}^{(n)}(t \wedge \bar{\tau}_+^{(n)})\big] \leq \frac{\lambda_0^{(n)}\, \mathbb{E}\big[|\xi^{(n)}|\big]}{1-\beta} \cdot t. \tag{4.5}$$

Applying the uniform convergence theorem for φ satisfying (2.6), c.f. [6, Theorem 1.2.1], we have

$$\big|\bar{\varphi}^{(n)}(s) - s^\alpha\big| \leq \frac{\varphi(n)}{n^\alpha} \cdot \Big|\frac{\varphi(ns)}{\varphi(n)} - s^\alpha\Big| + s^\alpha \cdot \Big|\frac{\varphi(n)}{n^\alpha} - 1\Big| \to 0 \quad \text{u.o.c. as } n \to \infty.$$

Substituting the two facts above into (4.3) proves the lemma. □

4.2. **Lemmas for FCLT.** The proof of Lemma 3.6 follows exactly the same argument as that of Lemma 5.4 in [31] by making use of [21, Theorem 3.22 in Chapter VIII]. The proof of $\hat{\varepsilon}^{(n)}$ in Lemma 3.7 also follow the same argument as that of Lemma 5.5 in [31]. Hence the proofs for these results are omitted for brevity. In the following, we begin with the proof of $\hat{\varepsilon}_\Psi^{(n)}$ in Lemma 3.7. Given that the proof of Lemma 3.8 is lengthy and requires a step-by-step development, we shall focus on it in the next subsection. The proof of Lemma 3.9, meanwhile, will be provided immediately after that of Lemma 3.7.

For every fixed $T > 0$, the lemmas are proved uniformly on $[0, T]$ in the following. Let $k_0 > 0$ be the constant from (3.9) so that (3.22) holds. By the choice of T and $\bar{\tau}_+^{(n)}$ in (3.9) and (3.10), we have from (3.11) for Ψ in Assumption A2 that

$$\Psi(\bar{Y}^{(n)}(t)) \leq (1-\beta)^{-1} \quad \text{and} \quad \Psi\big(\bar{Y}(t)\big) \leq (1-\beta)^{-1} \quad \forall t \leq T \wedge \bar{\tau}_+^{(n)}. \tag{4.6}$$

Proof of Lemma 3.7 – $\hat{\varepsilon}_\Psi^{(n)}$. Recalling that $\bar{Y}$ in (2.8) is non-decreasing and Ψ is assumed Lipschitz-continuous on $[0, k_0]$ in Assumption A2-(i), it is straightforward that $\forall 0 < s < t < T$,

$$\begin{aligned}
&\Big|\Psi(\bar{Y}(s))\Phi(y) - \int_0^{t-s} \Psi(\bar{Y}(u+s)) n\phi(nu, y)\, du\Big| \\
&\leq \Psi(\bar{Y}(s)) \int_{t-s}^\infty n\phi(nu, y)\, du + \int_0^{t-s} \big|\Psi(\bar{Y}(u+s)) - \Psi(\bar{Y}(s))\big| n\phi(nu, y)\, du \\
&\leq c_0 \cdot \bigg(\int_{t-s}^\infty n\phi(nu, y)\, du + \int_0^{t-s} \big(\bar{Y}(u+s) - \bar{Y}(s)\big)\, n\phi(nu, y)\, du \bigg),
\end{aligned} \tag{4.7}$$

where (4.6) is applied. Let $d\bar{Y}$ be the Stieltjes measure induced by $\bar{Y}$. We further have

$$\begin{aligned}
\text{LHS of (4.7)} &= c_0 \cdot \bigg(\int_{t-s}^\infty n\phi(nu, y)\, du + \int_0^{t-s} \Big(\int_s^{s+u} d\bar{Y}(v) \Big) n\phi(nu, y)\, du \bigg) \\
&\leq c_0 \cdot \bigg(\int_{t-s}^\infty n\phi(nu, y)\, du + \int_s^t d\bar{Y}(v) \int_{v-s}^\infty n\phi(nu, y)\, du \bigg).
\end{aligned}$$

Plugging the inequality above into $\bar{\varepsilon}_\Psi^{(n)}$ in (3.27), and recalling $\bar{\varepsilon}^{(n)}$ in (3.8), one can find that

$$\bar{\varepsilon}_\Psi^{(n)}(t) \leq c_0 \cdot \Big(\bar{\varepsilon}^{(n)}(t) + \int_0^t \bar{\varepsilon}^{(n)}(v)\, d\bar{Y}(v) \Big) \quad \forall t \leq T \text{ on the set } \{T < \bar{\tau}_+^{(n)}\}.$$

For the last, by the fact $(\hat{\varepsilon}^{(n)}, \hat{\varepsilon}_\Psi^{(n)}) = \sqrt{n}(\bar{\varepsilon}^{(n)}, \bar{\varepsilon}_\Psi^{(n)})$, the convergence of $\hat{\varepsilon}^{(n)}$ and (3.22) for our choice of k_0, we obtain the convergence of $\hat{\varepsilon}_\Psi^{(n)}$. This finish the proof of the lemma. □

Proof of Lemma 3.9. Let $z \in \mathbb{D}$ and $\theta \in \mathbb{C}_\uparrow$ and suppose $z_n \to z$ and $\theta_n \to \theta$ u.o.c. on $\mathbb{R}_+$. The piecewise constant function approximating in (4.1) is applied to z, that is, for every $\varepsilon > 0$, let $z^\varepsilon \in \mathbb{D}$ be a function so that (4.1) holds.

For every $t > 0$, we have

$$\begin{aligned}\big|\mathcal{G}(z_n,\theta_n)(t)-\mathcal{G}(z,\theta)(t)\big| &\le \mu\alpha \int_0^t \big(\theta_n(t)-\theta_n(s)\big)^{\alpha-1}\big|z_n(s)-z(s)\big|\,d\theta_n(s)\\ &\quad+\mu\alpha \int_0^t \big(\theta_n(t)-\theta_n(s)\big)^{\alpha-1}\big|z^\varepsilon(s)-z(s)\big|\,d\theta_n(s)\\ &\quad+\mu\alpha \int_0^t \big(\theta(t)-\theta(s)\big)^{\alpha-1}\big|z^\varepsilon(s)-z(s)\big|\,d\theta(s)\\ &\quad+\mu\alpha \Big|\int_0^t \big(\theta_n(t)-\theta_n(s)\big)^{\alpha-1}z^\varepsilon(s)\,d\theta_n(s)-\int_0^t \big(\theta(t)-\theta(s)\big)^{\alpha-1}z^\varepsilon(s)\,d\theta(s)\Big|\\ &=: g_1(t)+g_2(t)+g_3(t)+g_4(t).\end{aligned} \tag{4.8}$$

It is straightforward to find in (4.8) that

$$\begin{aligned} g_1(t) &\le \mu \sup_{s\le t}\big|z_n(s)-z(s)\big|\cdot\theta_n^\alpha(t)\to 0 \quad \text{u.o.c. as } n\to\infty,\\ g_2(t) &\le \mu\,\varepsilon\cdot\theta_n^\alpha(t)\le c_0\cdot\varepsilon \quad \text{uniformly in } n \text{ and } t\in[0,T],\\ g_3(t) &\le \mu\,\varepsilon\cdot\theta^\alpha(t)\le c_0\cdot\varepsilon \quad \text{uniformly in } n \text{ and } t\in[0,T].\end{aligned} \tag{4.9}$$

We next check the last term g_4 in (4.8). For every $t\in[t_{v-1},t_v)$ from (4.1), we have

$$\begin{aligned}&\alpha\int_0^t \big(\theta(t)-\theta(s)\big)^{\alpha-1}z^\varepsilon(s)\,d\theta(s)\\ &=\sum_{j=1}^{v-1} z_j\cdot\alpha\int_{t_{j-1}}^{t_j}\big(\theta(t)-\theta(s)\big)^{\alpha-1}\,d\theta(s)+z_v\cdot\alpha\int_{t_{v-1}}^{t}\big(\theta(t)-\theta(s)\big)^{\alpha-1}\,d\theta(s)\\ &=\sum_{1\le j\le v-1} z_j\cdot\Big(\big(\theta(t)-\theta(t_{j-1})\big)^\alpha-\big(\theta(t)-\theta(t_j)\big)^\alpha\Big)+z_v\cdot\big(\theta(t)-\theta(t_{v-1})\big)^\alpha.\end{aligned}$$

Plugging into $g_4(t)$ in (4.8) gives

$$g_4(t)\le \mu\cdot\sup_{s\le t}\big(|z(s)|+\varepsilon\big)\cdot(2v-1)\cdot\sup_{1\le j<v}\big|(\theta(t)-\theta(t_j))^\alpha-(\theta_n(t)-\theta_n(t_j))^\alpha\big|. \tag{4.10}$$

Observing that, we have for all $b>a>0$

$$b^\gamma-a^\gamma\le\begin{cases}(b-a)^\gamma & \text{if}\quad \gamma\in(0,1],\\ (b-a)\cdot\gamma\,b^\gamma & \text{if}\quad \gamma>1,\end{cases} \tag{4.11}$$

from which we have if $\alpha\in(0,1]$,

$$\big|(\theta(t)-\theta(t_j))^\alpha-(\theta_n(t)-\theta_n(t_j))^\alpha\big|\le 2^\alpha\cdot\sup_{s\le t}\big|\theta(s)-\theta_n(s)\big|^\alpha\to 0 \quad \text{u.o.c.}$$

and if $\alpha>1$,

$$\begin{aligned}&\big|(\theta(t)-\theta(t_j))^\alpha-(\theta_n(t)-\theta_n(t_j))^\alpha\big|\\ &\le\big|(\theta(t)-\theta(t_j))-(\theta_n(t)-\theta_n(t_j))\big|\cdot\gamma\cdot\big|(\theta(t)-\theta(t_j))^\alpha+(\theta_n(t)-\theta_n(t_j))^\alpha\big|\\ &\le 2\gamma\cdot\sup_{s\le t}|\theta(s)-\theta_n(s)|\cdot\big(\theta^\alpha(t)+\theta_n^\alpha(t)\big)\to 0 \quad \text{u.o.c.}\end{aligned}$$

Notice that v in (4.10) comes from (4.1) and depends only on ε. Thus, plugging the limits above and (4.9) into (4.8), letting $n\to\infty$ and then $\varepsilon\to 0$ proves the lemma. □

4.2.1. *Proof of Lemma 3.8.* Recalling the fact (3.22) for our choice of T and k_0, it is sufficient to focus on the $\bar\tau_+^{(n)}$-truncated process

$$\bar Z^{(n)}_{\varphi,k_0}(t) := \int_0^t \hat\varphi^{(n)}(t-s)\, d\bar Z^{(n)}(s\wedge\bar\tau_+^{(n)}).$$

and check that as $n\to\infty$,

$$\bar Z^{(n)}_{\varphi,k_0}(t) - \int_0^t \hat\varphi(t-s)\, d\bar Z(s\wedge\bar\tau_+^{(n)}) \to 0 \quad \text{uniformly on } [0,T] \text{ in probability.} \tag{4.12}$$

For arbitrary $\varepsilon>0$, by Assumption A3-(iii) on φ, let $k>0$ be a constant so that

$$\begin{aligned} \big|\hat\varphi^{(n)}(t)-\hat\varphi(t)\big| &= \Big|\sqrt{n}\,\frac{\varphi(nt)-(nt)^\alpha}{n^\alpha} - \hat\kappa\, t^{\alpha-1/2}\Big| \\ &= \big|\sqrt{s}\,\big(s^{-\alpha}\varphi(s)-1\big)-\hat\kappa\big|\cdot t^{\alpha-1/2} \le \varepsilon\cdot t^{\alpha-1/2}, \quad \forall s=nt>k, \end{aligned} \tag{4.13}$$

and let c_k be a constant defined by

$$n^{\alpha-1/2}\cdot|\hat\varphi^{(n)}(t)| = \big|\varphi(nt)-(nt)^\alpha\big| \le \sup_{s\le k}\varphi(s)+k^\alpha =: c_k, \quad \text{for } nt\le k. \tag{4.14}$$

With k introduced above, we rewrite (4.12) as, with $s=t-\frac{k}{n}$ for every $t>0$,

$$\begin{aligned} &\bar Z^{(n)}_{\varphi,k_0}(t) - \int_0^t \hat\varphi(t-u)\, d\bar Z(u\wedge\bar\tau_+^{(n)}) \\ =& \int_{t-\frac{k}{n}}^t \hat\varphi^{(n)}(t-u)\, d\bar Z^{(n)}(u\wedge\bar\tau_+^{(n)}) + \int_0^s \big(\hat\varphi^{(n)}-\hat\varphi\big)\big(\frac{k}{n}+s-u\big)\, d\bar Z^{(n)}(u\wedge\bar\tau_+^{(n)}) \\ &+ \int_0^s \hat\varphi\big(\frac{k}{n}+s-u\big)\, d\big(\bar Z^{(n)}-\mu\bar A\big)(u\wedge\bar\tau_+^{(n)}) - \int_{t-\frac{k}{n}}^t \hat\varphi(t-u)\, d\bar Z(u\wedge\bar\tau_+^{(n)}) \\ =& \bar\varepsilon^{(n)}_{\varphi,1}(t) + \bar\varepsilon^{(n)}_{\varphi,2}(s) + \frac{1}{\sqrt{n}}\int_0^s \hat\varphi\big(\frac{k}{n}+s-u\big)\, d\big(\hat Z^{(n)}+\mu^{(n)}\hat A^{(n)}+\hat\mu^{(n)}\bar A\big)(u\wedge\bar\tau_+^{(n)}) - \bar\varepsilon^{(n)}_{\varphi,6}(t) \\ =& \bar\varepsilon^{(n)}_{\varphi,1}(t) + \bar\varepsilon^{(n)}_{\varphi,2}(s) + \big(\bar\varepsilon^{(n)}_{\varphi,3}(s)+\bar\varepsilon^{(n)}_{\varphi,4}(s)+\bar\varepsilon^{(n)}_{\varphi,5}(s)\big) - \bar\varepsilon^{(n)}_{\varphi,6}(t) \end{aligned} \tag{4.15}$$

recalling $\hat Z^{(n)}$ in (3.24), $\hat A^{(n)}$ in (2.10) and $\hat\mu^{(n)}$ in Assumption A3, where we understand that $\bar Z^{(n)}(u)=\bar Z(u)=\hat\varphi^{(n)}(u)=\hat\varphi(u)=0$ for $u<0$ and $\bar\varepsilon^{(n)}_{\varphi,j}(s)=0$ for $s<0$ and $j=2,3,4,5$.

In addition to $\check Z^{(n)}_{k_0}$ in (4.4), we further define

$$\hat{\check Z}^{(n)}_{k_0}(t) := \frac{1}{\sqrt{n}}\sum_{j\ge1}\Big(|\xi_j^{(n)}| - \mathbb{E}\big[|\xi^{(n)}|\big]\Big)\mathbf{1}\big(\bar\tau_j^{(n)}\le t\wedge\bar\tau_+^{(n)}\big). \tag{4.16}$$

By the unpredictability of the noises $\{\xi_j^{(n)}\}$, similar to $\hat Z^{(n)}$ in (3.24), one can check that $\hat{\check Z}^{(n)}_{k_0}$ is a local-martingale. One can apply the Burkholder-Davis-Gundy (BDG) inequality, c.f. [22, Theorem 20.12], to find that, for some constant c_2 from the inequality,

$$\mathbb{E}\Big[\sup_{s\le t}(\hat{\check Z}^{(n)}_{k_0})^2(s)\Big] \le c_2\cdot\mathrm{Var}\big(|\xi^{(n)}|\big)\cdot\mathbb{E}\big[\bar A^{(n)}(t\wedge\bar\tau_+^{(n)})\big] \le \frac{c_2\,\lambda_0^{(n)}\,\mathbb{E}\big[(\xi^{(n)})^2\big]}{1-\beta}\cdot t, \tag{4.17}$$

where (3.15) in Lemma 3.1 is applied. Thus, $\hat{\check Z}^{(n)}_{k_0}$ is locally stochastically bounded.

Recalling Assumption A3-(iii), for the case $\alpha\in(0,1/2)$, 0 is a singular point for $\hat\varphi$ and the term $\int_{t-k/n}^t \hat\varphi(t-u)\, d\bar Z^{(n)}(u\wedge\bar\tau_+^{(n)})$ could be ∞-valued at some countable points. Thus, $\hat{\check Z}^{(n)}_{k_0}$ is introduced to control the error term $\bar\varepsilon^{(n)}_{\varphi,1}$ in (4.15). In the following, Lemma 3.8 is proven separately for the

two cases, $\alpha \geq 1/2$ and $\alpha \in (0, 1/2)$. The next lemma applies to both cases, while the proofs for $\bar{\varepsilon}^{(n)}_{\varphi,2}$, $\bar{\varepsilon}^{(n)}_{\varphi,3}$, $\bar{\varepsilon}^{(n)}_{\varphi,4}$ and $\bar{\varepsilon}^{(n)}_{\varphi,5}$ require different arguments in the two cases.

Lemma 4.1. *Under Assumptions A1, A2 and A3, for every $\alpha > 0$, we have*

$$\bar{\varepsilon}^{(n)}_{\varphi,1}(t) \to 0 \quad \textit{and} \quad \bar{\varepsilon}^{(n)}_{\varphi,6}(t) \to 0 \quad \textit{uniformly on } [0,T] \textit{ in probability as } n \to \infty.$$

Proof. By the identity $\bar{Z} = \mu \bar{A}$ and (2.9), we have for $\bar{\varepsilon}^{(n)}_{\varphi,6}$ in (4.15),

$$\begin{aligned} \big|\bar{\varepsilon}^{(n)}_{\varphi,6}(t)\big| &= \lambda_0 |\mu\,\hat{\kappa}| \int_{t-\frac{k}{n}}^{t} (t-u)^{\alpha-1/2} \Psi(\bar{Y}(u)) \mathbf{1}(u \leq \bar{\tau}^{(n)}_+)\, du \\ &\leq \frac{\lambda_0 |\mu\,\hat{\kappa}|}{1-\beta} \int_0^{k/n} u^{\alpha-1/2}\, du = \frac{2\lambda_0 |\mu\,\hat{\kappa}|}{(1-\beta)(1+2\alpha)} \big(\frac{k}{n}\big)^{\alpha+1/2}, \end{aligned}$$

where (4.6) is applied. Letting $n \to \infty$ proves the limit for $\hat{\varepsilon}^{(n)}_{\varphi,6}$.

On the other hand, recalling $\check{Z}^{(n)}_{k_0}$ in (4.4) and $\hat{\check{Z}}^{(n)}_{k_0}$ in (4.16), we have by applying (4.14) to $\bar{\varepsilon}^{(n)}_{\varphi,1}$ in (4.15) that

$$\begin{aligned} & n^{\alpha}\big(\bar{\varepsilon}^{(n)}_{\varphi,1}(t)\big) \leq c_k \cdot \sqrt{n}\left(\check{Z}^{(n)}_{k_0}(t) - \check{Z}^{(n)}_{k_0}(t-\frac{k}{n})\right) \\ &= c_k \cdot \Big(\big(\hat{\check{Z}}^{(n)}_{k_0}(t) - \hat{\check{Z}}^{(n)}_{k_0}(t-\frac{k}{n})\big) + \mathbb{E}\big[|\xi^{(n)}|\big] \cdot \sqrt{n}\big(\bar{A}^{(n)}(t\wedge\bar{\tau}^{(n)}_+) - \bar{A}^{(n)}((t-\frac{k}{n})\wedge\bar{\tau}^{(n)}_+)\big)\Big). \end{aligned} \tag{4.18}$$

Since $\alpha > 0$, to prove the convergence for $\bar{\varepsilon}^{(n)}_{\varphi,1}$, we only need to show the stochastic boundedness on $[0,T]$ for the processes on the right above. Noticing that $\hat{\check{Z}}^{(n)}_{k_0}$ above is already proved to be locally stochastically bounded from (4.17), it is thus sufficient to claim that

$$\begin{aligned} & (1-\beta)\cdot\sqrt{n}\left(\bar{A}^{(n)}(t\wedge\bar{\tau}^{(n)}_+) - \bar{A}^{(n)}\big((t-\frac{k}{n})\wedge\bar{\tau}^{(n)}_+\big)\right) \\ &\leq \lambda^{(n)}_0 \cdot \frac{k}{\sqrt{n}} + 2\cdot\sup_{s\leq t}\big|\hat{X}^{(n)}\big|(s\wedge\bar{\tau}^{(n)}_+) + \sup_{s\leq t}\hat{\varepsilon}^{(n)}(s). \end{aligned} \tag{4.19}$$

Then, the convergence of $\hat{X}^{(n)}$ in Lemma 3.6 and $\hat{\varepsilon}^{(n)}$ in Lemma 3.7 gives the locally stochastically bounded on the right hand side of (4.19).

To prove (4.19), one has from (3.7) that for every $\bar{\tau}^{(n)}_+ \geq t' \geq s' > 0$,

$$\begin{aligned} \bar{A}^{(n)}(t') - \bar{A}^{(n)}(s') &= \big(\bar{X}^{(n)}(t') - \bar{X}^{(n)}(s')\big) + \lambda^{(n)}_0(t'-s') \\ &\quad + \int_{s'}^{t'} \Phi(\bar{Y}^{(n)}(u-))\, d\bar{A}^{(n)}(u) + \big(\bar{\varepsilon}^{(n)}(s') - \bar{\varepsilon}^{(n)}(t')\big) \\ &\leq \big(\bar{X}^{(n)}(t') - \bar{X}^{(n)}(s')\big) + \lambda^{(n)}_0(t'-s') \\ &\quad + \beta\cdot\big(\bar{A}^{(n)}(t') - \bar{A}^{(n)}(s')\big) + \bar{\varepsilon}^{(n)}(s')\mathbf{1}(s'<t') \end{aligned}$$

where (3.11) is applied to $\Phi(\bar{Y}^{(n)}(u-))$ with $u < t' \leq \bar{\tau}^{(n)}_+$. In particular,

$$\begin{aligned} & (1-\beta)\cdot\big(\bar{A}^{(n)}(t\wedge\bar{\tau}^{(n)}_+) - \bar{A}^{(n)}(s\wedge\bar{\tau}^{(n)}_+)\big) \\ &\leq \lambda^{(n)}_0(t-s) + \big(\bar{X}^{(n)}(t\wedge\bar{\tau}^{(n)}_+) - \bar{X}^{(n)}(s\wedge\bar{\tau}^{(n)}_+)\big) + \bar{\varepsilon}^{(n)}(s), \end{aligned}$$

for every $t > s > 0$ by taking $(t', s') = (t\wedge\bar{\tau}^{(n)}_+, s\wedge\bar{\tau}^{(n)}_+)$, where the fact $\bar{\varepsilon}^{(n)} > 0$ is also applied. The inequality above can also be found from the proof of Lemma 5.2 in [31]. Taking $s = t - \frac{k}{n}$ and multiplying on both sides above by $\sqrt{n}$ gives (4.19). This finishes the proof. □

Proof of Lemma 3.8 – $\alpha \geq 1/2$. Applying (4.13) to $\bar{\varepsilon}^{(n)}_{\varphi,2}$ in (4.15), one can find that

$$\big|\bar{\varepsilon}^{(n)}_{\varphi,2}(t)\big| \leq \varepsilon \int_0^t \big(\frac{k}{n} + t - u\big)^{\alpha-1/2}\, d\check{Z}^{(n)}_{k_0}(u) \leq \varepsilon\,(1+t)^{\alpha-1/2}\,\check{Z}^{(n)}_{k_0}(t) \quad \forall t>0, n \geq k, \tag{4.20}$$

where the fact $\alpha \geq 1/2$ is applied.

Recalling that $\hat{\varphi}(t) = \hat{\kappa}\, t^{\alpha-1/2}$, we can consider the remaining three terms in (4.15) jointly:

$$\bar{\varepsilon}^{(n)}_{\varphi,3+4+5}(t) = \bar{\varepsilon}^{(n)}_{\varphi,3}(t) + \bar{\varepsilon}^{(n)}_{\varphi,4}(t) + \bar{\varepsilon}^{(n)}_{\varphi,5}(t) = \int_0^t \hat{\varphi}\big(\frac{k}{n} + t - u\big)\, d\big(\bar{Z}^{(n)} - \bar{Z}\big)(u \wedge \bar{\tau}^{(n)}_+).$$

If $\alpha = 1/2$, we have $\hat{\varphi}(t) \equiv \hat{\kappa}$ for $t > 0$. Thus, applying Lemma 3.3 to $\bar{Z}^{(n)}$ in (3.6) and the FLLN, we obtain

$$\bar{\varepsilon}^{(n)}_{\varphi,3+4+5}(t) = \hat{\kappa} \cdot \big(\bar{Z}^{(n)} - \bar{Z}\big)(t \wedge \bar{\tau}^{(n)}_+) \to 0 \quad \text{u.o.c. as } n \to \infty. \tag{4.21}$$

If $\alpha > 1/2$, applying Fubini's theorem, we obtain

$$\begin{aligned}
\bar{\varepsilon}^{(n)}_{\varphi,3+4+5}(t) &= \hat{\kappa}(\alpha - \frac{1}{2}) \int_0^t \Big(\int_0^{t-s+k/n} u^{\alpha-3/2}\, du \Big)\, d\big(\bar{Z}^{(n)} - \bar{Z}\big)(s \wedge \bar{\tau}^{(n)}_+) \\
&= \hat{\kappa}(\alpha - \frac{1}{2}) \int_0^{t+k/n} u^{\alpha-3/2}\, \big(\bar{Z}^{(n)} - \bar{Z}\big)(t \wedge (t + \frac{k}{n} - u) \wedge \bar{\tau}^{(n)}_+)\, du.
\end{aligned}$$

It follows that for $n \geq k$,

$$\begin{aligned}
\big|\bar{\varepsilon}^{(n)}_{\varphi,3+4+5}(t)\big| &\leq |\hat{\kappa}| \cdot \sup_{s\leq t} \big|\bar{Z}^{(n)} - \bar{Z}\big|(s \wedge \bar{\tau}^{(n)}_+) \cdot (\alpha - \frac{1}{2}) \int_0^{t+1} u^{\alpha-3/2}\, du \\
&= |\hat{\kappa}| \cdot (1+t)^{\alpha-1/2} \cdot \sup_{s\leq t} \big|\bar{Z}^{(n)} - \bar{Z}\big|(s \wedge \bar{\tau}^{(n)}_+) \to 0 \quad \text{u.o.c. as } n \to \infty.
\end{aligned} \tag{4.22}$$

Finally, plugging Lemma 4.1, (4.20), (4.21) and (4.22) into (4.15), and making use of the stochastically boundedness of $\check{Z}^{(n)}_{k_0}$ on $[0,T]$ from (4.5) to (4.20), (4.12) is proved by letting $n \to \infty$ and then $\varepsilon \to 0$ for the case $\alpha \geq 1/2$. □

We next consider the case of $\alpha \in (0, 1/2)$. Recall that Lemma 4.1 for $\bar{\varepsilon}^{(n)}_{\varphi,1}$ and $\bar{\varepsilon}^{(n)}_{\varphi,6}$ is still applicable. Hence, we focus on the proof of $\bar{\varepsilon}^{(n)}_{\varphi,j}$ with $j = 2,3,4,5$ below.

To prove the limit for $\bar{\varepsilon}^{(n)}_{\varphi,2}$ in (4.15) for the case $\alpha \in (0,1/2)$, we need following lemma.

Lemma 4.2. *Under Assumptions A1, A2 and A3, for every $k > 0$ and $\gamma \in (0,1)$, we have*

$$\begin{aligned}
\bar{K}^{(n)}(t) &:= \int_0^t \big(\frac{k}{n} + t - u\big)^{-\gamma}\, d\bar{A}^{(n)}(u \wedge \bar{\tau}^{(n)}_+) \\
&\leq \frac{2\,n^\gamma}{(1-\beta)k^\gamma} \Big(\sup_{s\leq t} \big|\bar{X}^{(n)}\big|(s \wedge \bar{\tau}^{(n)}_+) + \sup_{s\leq t} \big|\bar{\varepsilon}^{(n)}_{k_0}\big|(s) \Big) + \frac{\lambda^{(n)}_0\, t^{1-\gamma}}{(1-\gamma)(1-\beta)},
\end{aligned}$$

where $\bar{X}^{(n)}$ is defined in (3.3) and $\bar{\varepsilon}^{(n)}_{k_0}$ is defined in (3.12).

Proof. By the identity $\bar{A}^{(n)} = \bar{X}^{(n)} + \bar{\Lambda}^{(n)}$ and $\lambda^{(n)}$ in (2.4), we can rewrite

$$\begin{aligned}
\bar{K}^{(n)}(t) &= \int_0^t \big(\frac{k}{n} + t - u\big)^{-\gamma} d\bar{X}^{(n)}(u \wedge \bar{\tau}_+^{(n)}) + \lambda_0^{(n)} \int_0^t \big(\frac{k}{n} + t - u\big)^{-\gamma} \mathbf{1}(u \le \bar{\tau}_+^{(n)})\, du \\
&\quad + \int_0^t \big(\frac{k}{n} + t - u\big)^{-\gamma} \Big(\int_0^u n\phi\big(n(u-v), \bar{Y}^{(n)}(v-)\big)\, d\bar{A}^{(n)}(v)\Big) \mathbf{1}(u \le \bar{\tau}_+^{(n)})\, du \\
&\le \int_0^t \big(\frac{k}{n} + t - u\big)^{-\gamma} d\bar{X}^{(n)}(u \wedge \bar{\tau}_+^{(n)}) + \lambda_0^{(n)} \int_0^t \big(\frac{k}{n} + t - u\big)^{-\gamma} du \\
&\quad + \int_0^t \big(\frac{k}{n} + t - u\big)^{-\gamma} \Big(\int_0^u n\phi\big(n(u-v), \bar{Y}^{(n)}(v-)\big)\, d\bar{A}^{(n)}(v \wedge \bar{\tau}_+^{(n)})\Big)\, du \\
&=: \bar{K}_1^{(n)}(t) + \bar{K}_2^{(n)}(t) + \bar{K}_3^{(n)}(t).
\end{aligned} \tag{4.23}$$

Applying Fubini's theorem to $\bar{K}_1^{(n)}$ above, we have

$$\begin{aligned}
\bar{K}_1^{(n)}(t) &= \gamma \int_0^t \Big(\int_{t-u}^\infty \big(\frac{k}{n} + v\big)^{-\gamma-1} dv\Big) d\bar{X}^{(n)}(u \wedge \bar{\tau}_+^{(n)}) \\
&= \gamma \int_0^\infty \big(\frac{k}{n} + v\big)^{-\gamma-1} \big(\bar{X}^{(n)}(t \wedge \bar{\tau}_+^{(n)}) - \bar{X}^{(n)}((t-v) \wedge \bar{\tau}_+^{(n)})\big)\, dv,
\end{aligned} \tag{4.24}$$

where we understand that $\bar{X}^{(n)}(u) = 0$ for $u < 0$. This implies

$$|\bar{K}_1^{(n)}|(t) \le 2\gamma \sup_{s \le t} |\bar{X}^{(n)}|(s \wedge \bar{\tau}_+^{(n)}) \int_0^\infty \big(\frac{k}{n} + u\big)^{-\gamma-1} du = 2 \sup_{s \le t} \big|\bar{X}^{(n)}\big|(s \wedge \bar{\tau}_+^{(n)}) \cdot \big(\frac{k}{n}\big)^{-\gamma}. \tag{4.25}$$

Changing the order of integration in $\bar{K}_3^{(n)}$ above gives

$$\bar{K}_3^{(n)}(t) = \int_0^t d\bar{A}^{(n)}(v \wedge \bar{\tau}_+^{(n)}) \Big(\int_0^{t-v} \big(\frac{k}{n} + t - v - u\big)^{-\gamma} n\phi\big(nu, \bar{Y}^{(n)}(v-)\big)\, du\Big). \tag{4.26}$$

By integration by parts, one can check that $\forall s > 0, |y| \le k_0$

$$\begin{aligned}
&\int_0^s \big(\frac{k}{n} + s - u\big)^{-\gamma} n\phi(nu, y)\, du \\
&= \big(\frac{k}{n} + s\big)^{-\gamma} \Phi(y) - \big(\frac{k}{n}\big)^{-\gamma} \int_{ns}^\infty \phi(u, y)\, du + \gamma \int_0^s \big(\frac{k}{n} + v\big)^{-\gamma-1} \Big(\int_{n(s-v)}^\infty \phi(u, y)\, du\Big)\, dv \\
&\le \beta \big(\frac{k}{n} + s\big)^{-\gamma} + \gamma \int_0^s \big(\frac{k}{n} + v\big)^{-\gamma-1} \int_{n(s-v)}^\infty \phi(u, y)\, du dv
\end{aligned}$$

where the bound (3.11) is applied to $\Phi(y)$ above. Plugging the inequality above into (4.26) gives

$$\bar{K}_3^{(n)}(t) \le \gamma \int_0^t \bar{\varepsilon}_{k_0}^{(n)}(t-v) \big(\frac{k}{n} + v\big)^{-\gamma-1} dv + \beta \cdot \bar{K}^{(n)}(t), \tag{4.27}$$

recalling $\bar{\varepsilon}_{k_0}^{(n)}$ in (3.12) and $\bar{K}^{(n)}$ in the Lemma.

For the last, plugging (4.27) into (4.23), eliminating $\beta\, \bar{K}^{(n)}$ on both sides gives

$$\begin{aligned}
(1-\beta)\, \bar{K}^{(n)}(t) &\le \big|\bar{K}_1^{(n)}(t)\big| + \gamma \int_0^t \bar{\varepsilon}_{k_0}^{(n)}(t-u) \big(\frac{k}{n} + u\big)^{-\gamma-1} dv + \bar{K}_2^{(n)}(t) \\
&\le \big|\bar{K}_1^{(n)}(t)\big| + \sup_{s \le t} \big|\bar{\varepsilon}_{k_0}^{(n)}\big|(s) \cdot \gamma \int_0^\infty \big(\frac{k}{n} + u\big)^{-\gamma-1} dv + \lambda_0^{(n)} \int_0^t \big(\frac{k}{n} + u\big)^{-\gamma} dv \\
&\le \big|\bar{K}_1^{(n)}(t)\big| + \sup_{s \le t} \big|\bar{\varepsilon}_{k_0}^{(n)}\big|(s) \cdot \big(\frac{k}{n}\big)^{-\gamma} + \frac{\lambda_0^{(n)}}{1-\gamma} \Big(\big(\frac{k}{n} + t\big)^{1-\gamma} - \big(\frac{k}{n}\big)^{1-\gamma}\Big).
\end{aligned}$$

The lemma is proved by applying (4.25) and (4.11) with $1-\gamma \in (0,1)$. □

Lemma 4.3. *Under Assumptions A1, A2 and A3, for every $\alpha \in (0,1/2)$, we have*

$$\bar{\varepsilon}^{(n)}_{\varphi,2}(t) \to 0 \quad \textit{uniformly on } [0,T] \textit{ in probability as } n \to \infty.$$

Proof. Applying (4.13) to $\bar{\varepsilon}^{(n)}_{\varphi,2}$ in (4.15), recalling $\check{Z}^{(n)}_{k_0}$ in (4.4) and $\hat{\check{Z}}^{(n)}_{k_0}$ in (4.16), we have

$$\begin{aligned} \big|\bar{\varepsilon}^{(n)}_{\varphi,2}\big|(t) &\le \varepsilon \int_0^t \big(\frac{k}{n}+t-u\big)^{\alpha-1/2}\, d\check{Z}^{(n)}_{k_0}(u) \\ &= \varepsilon \int_0^t \big(\frac{k}{n}+t-u\big)^{\alpha-1/2}\, d\Big(\frac{1}{\sqrt{n}}\hat{\check{Z}}^{(n)}_{k_0}(u) + \mathbb{E}\big[|\xi^{(n)}|\big]\,\bar{A}^{(n)}(u\wedge\bar{\tau}^{(n)}_+)\Big) \\ &=: \bar{\varepsilon}^{(n)}_{\varphi,21}(t) + \bar{\varepsilon}^{(n)}_{\varphi,22}(t). \end{aligned} \tag{4.28}$$

Next, by the same procedure as the proof of $\bar{K}^{(n)}_1$ in (4.24), one can find that

$$\bar{\varepsilon}^{(n)}_{\varphi,21}(t) = \frac{\varepsilon}{\sqrt{n}}\cdot\big(\frac{1}{2}-\alpha\big)\cdot\int_0^\infty \Big(\hat{\check{Z}}^{(n)}_{k_0}(t) - \hat{\check{Z}}^{(n)}_{k_0}(t-u)\Big)\big(\frac{k}{n}+u\big)^{\alpha-3/2}\,du.$$

Further applying the maximal inequality for $\hat{\check{Z}}^{(n)}_{k_0}$ in (4.17), we have

$$\begin{aligned} \big|\bar{\varepsilon}^{(n)}_{\varphi,21}\big|(t) &\le \frac{2\,\varepsilon}{\sqrt{n}}\cdot\sup_{s\le t}\big|\hat{\check{Z}}^{(n)}_{k_0}\big|(s)\cdot\big(\frac{1}{2}-\alpha\big)\int_0^\infty\big(\frac{k}{n}+u\big)^{\alpha-3/2}\,du \\ &= \varepsilon\cdot\frac{2\,k^{\alpha-1/2}}{n^\alpha}\cdot\sup_{s\le t}\big|\hat{\check{Z}}^{(n)}_{k_0}\big|(s)\to 0 \quad \text{u.o.c. as } n\to\infty. \end{aligned}$$

For $\bar{\varepsilon}^{(n)}_{\varphi,22}$ in (4.28), we can apply Lemma 4.2 with $\gamma = \frac{1}{2}-\alpha$ to obtain

$$\begin{aligned} \bar{\varepsilon}^{(n)}_{\varphi,22}(t) &= \varepsilon\cdot\mathbb{E}\big[|\xi^{(n)}|\big]\cdot\int_0^t\big(\frac{k}{n}+t-u\big)^{\alpha-1/2}\,d\bar{A}^{(n)}(u\wedge\bar{\tau}^{(n)}_+) \\ &\le \varepsilon\cdot c_0\cdot\Big(n^{\frac{1}{2}-\alpha}\Big(\sup_{s\le t}\big|\bar{X}^{(n)}\big|(s\wedge\bar{\tau}^{(n)}_+) + \sup_{s\le t}\big|\bar{\varepsilon}^{(n)}_{k_0}\big|(s)\Big) + t^{\frac{1}{2}+\alpha}\Big) \end{aligned}$$

for some constant $c_0>0$ depends on k and independent to n.

For the last, applying the tightness for $\hat{X}^{(n)} = \sqrt{n}\bar{X}^{(n)}$ from Lemma 3.6, the limit for $\hat{\varepsilon}^{(n)} = \sqrt{n}\bar{\varepsilon}^{(n)}$ from Lemma 3.7 as well as the fact (3.22), letting $n\to\infty$ and then $\varepsilon\to 0$ proves the convergence of $\bar{\varepsilon}^{(n)}_{\varphi,22}$ to 0. This proves the convergence of $\bar{\varepsilon}^{(n)}_{\varphi,2}$ as claimed. □

Lemma 4.4. *Under Assumptions A1, A2 and A3, for every $\alpha \in (0,1/2)$, we have*

$$\bar{\varepsilon}^{(n)}_{\varphi,3}(t)\to 0 \quad \textit{and} \quad \bar{\varepsilon}^{(n)}_{\varphi,5}(t)\to 0 \quad \textit{uniformly on } [0,T] \textit{ in probability as } n\to\infty.$$

Proof. Applying Fubini's theorem to $\bar{\varepsilon}^{(n)}_{\varphi,3}$ in (4.15), similar to (4.24), we have

$$\begin{aligned} \bar{\varepsilon}^{(n)}_{\varphi,3}(t) &= \big(\frac{1}{2}-\alpha\big)\frac{\hat{\kappa}}{\sqrt{n}}\int_0^t\Big(\int_{t-u}^\infty\big(\frac{k}{n}+v\big)^{\alpha-3/2}\,dv\Big)\,d\hat{Z}^{(n)}(u\wedge\bar{\tau}^{(n)}_+) \\ &= \big(\frac{1}{2}-\alpha\big)\frac{\hat{\kappa}}{\sqrt{n}}\int_0^\infty\big(\frac{k}{n}+v\big)^{\alpha-3/2}\Big(\hat{Z}^{(n)}(t\wedge\bar{\tau}^{(n)}_+) - \hat{Z}^{(n)}\big((t-v)\wedge\bar{\tau}^{(n)}_+\big)\Big)\,dv, \end{aligned}$$

where we understand that $\hat{Z}^{(n)}(u)=0$ for $u<0$. It follows that

$$\begin{aligned} \big|\bar{\varepsilon}^{(n)}_{\varphi,3}\big|(t) &\le 2\cdot\frac{|\hat{\kappa}|}{\sqrt{n}}\cdot\sup_{s\le t}\big|\hat{Z}^{(n)}(s\wedge\bar{\tau}^{(n)}_+)\big|\cdot\big(\frac{1}{2}-\alpha\big)\int_0^\infty\big(\frac{k}{n}+v\big)^{\alpha-3/2}\,dv \\ &\le 2\cdot|\hat{\kappa}|\cdot\sup_{s\le t}\big|\hat{Z}^{(n)}(s\wedge\bar{\tau}^{(n)}_+)\big|\cdot\frac{k^{\alpha-1/2}}{n^\alpha} \quad \text{u.o.c. in probability as } n\to\infty, \end{aligned}$$

by the tightness of $\hat{Z}^{(n)}$ from Lemma 3.6.

For $\bar{\varepsilon}^{(n)}_{\varphi,5}$ in (4.15) with $\alpha < 1/2$, we have directly from (2.9) that for every $t \le T$,

$$\begin{aligned}\big|\bar{\varepsilon}^{(n)}_{\varphi,5}\big|(t) &= \frac{\lambda_0|\hat{\mu}^{(n)}\,\hat{\kappa}|}{\sqrt{n}} \int_0^t \big(\frac{k}{n} + t - u\big)^{\alpha-1/2} \Psi\big(\bar{Y}(u)\big)\, \mathbf{1}(u \le \bar{\tau}^{(n)}_+)\, du \\ &\le \frac{2\lambda_0|\hat{\mu}^{(n)}\,\hat{\kappa}|}{(1-\beta)(2\alpha+1)\sqrt{n}} \Big(\big(\frac{k}{n} + t\big)^{\alpha+\frac{1}{2}} - \big(\frac{k}{n}\big)^{\alpha+\frac{1}{2}} \Big) \le \frac{2\lambda_0|\hat{\mu}^{(n)}\,\hat{\kappa}| \cdot t^{\alpha+1/2}}{(1-\beta)(2\alpha+1)\sqrt{n}},\end{aligned}$$

where (4.6) and (4.11) are applied for $\gamma = \alpha + \frac{1}{2} \in (0,1)$. This finishes the proof. □

To prove the limit for $\bar{\varepsilon}^{(n)}_{\varphi,4}$ in (4.15), we need the following result similar to Gronwall's inequality.

Lemma 4.5. *Suppose $\alpha > 0$, and let $A \in \mathbb{D}$ be an increasing function. Assume $U, V \in \mathbb{C}$ with $U, V \ge 0$, and that V satisfies*

$$0 \le V(t) \le U(t) + \int_0^t (t-u)^\alpha V(u)\, dA(u), \quad \forall t > 0. \tag{4.29}$$

Then we have

$$V(t) \le U(t) + \Big(\int_0^t (t-u)^\alpha U(u)\, dA(u) \Big) \exp\Big(\int_0^t (t-u)^\alpha\, dA(u) \Big), \quad \forall t > 0.$$

Proof. Let I_1 and I_2 be the continuous and increasing functions on $\mathbb{R}_+$ defined by

$$I_1(t) = \int_0^t (t-u)^\alpha U(u)\, dA(u), \quad I_2(t) = \int_0^t (t-u)^\alpha V(u)\, dA(u).$$

Without loss of generality, we assume $V \not\equiv 0$, then $I_2(\infty) = \infty$. We further define I_3 by

$$I_2(t)\, dI_3(t) = d\Big(\int_0^t (t-u)^\alpha I_2(u)\, dA(u) \Big), \tag{4.30}$$

where we understand that $I_3(t) = 0$ whenever $I_2(t) = 0$. Making use of (4.29), one can see that

$$I_2(t)\, dI_3(t) + dI_1(t) - dI_2(t) = d\Big(\int_0^t (t-u)^\alpha \big(I_2(u) + U(u) - V(u)\big)\, dA(u) \Big) \ge 0,$$

which gives

$$d\big(\exp(-I_3(t))\, I_2(t) \big) = e^{-I_3(t)} \big(dI_2(t) - I_2(t)\, dI_3(t) \big) \le e^{-I_3(t)}\, dI_1(t).$$

By the facts that $I_1(0) = I_2(0) = 0$ and $I_3 \ge 0$, we have

$$I_2(t) \le I_1(t)\, \exp(I_3(t)) \quad \forall t > 0.$$

To prove the lemma, by (4.29), it is sufficient to show that

$$I_3(t) \le \int_0^t (t-u)^\alpha\, dA(u) := I_4(t) \quad \forall t > 0. \tag{4.31}$$

Since $I_2(\infty) = 0$ and I_2 is increasing on $\mathbb{R}_+$, letting $d\nu_2$ be Stieltjes measure induced by $-1/I_2$ on $\mathbb{R}_+$, that is, $\nu_2(x, \infty) = 1/I_2(x)$, and applying Fubini's theorem, we obtain

$$\begin{aligned} I_3(t) &= \int_0^t \frac{1}{I_2(u)}\, I_2(u)\, dI_3(u) = \int_0^t \Big(\int_u^\infty d\nu_2(s) \Big)\, I_2(u) dI_3(u) \\ &= \int_0^\infty \Big(\int_0^{t\wedge s} I_2(u)\, dI_3(u) \Big)\, d\nu_2(s). \end{aligned} \tag{4.32}$$

On the other hand, recalling (4.30) and the fact that I_2 is increasing and $I_2(0)=0$, we further have

$$\begin{aligned}\int_0^t I_2(u)\,dI_3(u) &= \int_0^t (t-u)^\alpha\Big(\int_0^u dI_2(v)\Big)\,dA(u) = \int_0^t dI_2(v)\Big(\int_v^t (t-u)^\alpha\,dA(u)\Big)\\ &= \int_0^t dI_2(v)\Big(\int_0^t (t-u)^\alpha\,dA(u) - \int_0^v (t-u)^\alpha\,dA(u)\Big)\\ &\le \int_0^t dI_2(v)\Big(\int_0^t (t-u)^\alpha\,dA(u) - \int_0^v (v-u)^\alpha\,dA(u)\Big)\\ &= \int_0^t \big(I_4(t)-I_4(v)\big)\,dI_2(v) = \int_0^t\int_v^t dI_4(u)dI_2(v) = \int_0^t I_2(u)\,dI_4(u),\end{aligned}$$

where I_4 is given in (4.31). Plugging the inequality above into (4.32) gives

$$I_3(t) \le \int_0^\infty d\nu_2(s)\Big(\int_0^{t\wedge s} I_2(u)\,dI_4(u)\Big) = \int_0^t \frac{1}{I_2(u)}\,I_2(u)\,dI_4(u) = I_4(t),$$

where in the second identity the calculations in (4.32) are applied with I_3 replaced by I_4. This proves (4.31) and finishes the proof. □

Corollary 4.1. $\sup_{0\le s\le T}|\hat{A}^{(n)}|(s)$ *is stochastically bounded.*

Proof. We make use of (3.31) for the proof. Recalling $\mathcal{F}$ defined in (3.25), we have

$$\begin{aligned}\mathcal{F}(\hat{A}^{(n)})(t) &= \mathcal{F}\big(\hat{U}^{(n)}-\hat{\varepsilon}^{(n)}_\Psi\big)(t) + \mu\int_0^t\Big(\int_0^{t-s}\mathcal{F}(\hat{A}^{(n)})(u)\,d\bar{I}^{(n)}(u)\Big)\,ds^\alpha\\ &= \mathcal{F}\big(\hat{U}^{(n)}-\hat{\varepsilon}^{(n)}_\Psi\big)(t) + \mu\int_0^t (t-u)^\alpha\mathcal{F}(\hat{A}^{(n)})(u)\,d\bar{I}^{(n)}(u).\end{aligned}$$

Recalling $\bar{I}^{(n)}$ in (3.30), the Lipschitz-continuity of Ψ on $[0,k_0]$ under Assumption A2-(i), the fact $\Psi(x)=(1-\Phi(x))^{-1}$ and the bounds in (4.6), one can find for some constant $c_0>0$ that

$$|\mathcal{F}(\hat{A}^{(n)})|(t) \le \mathcal{F}\big(|\hat{U}^{(n)}|+|\hat{\varepsilon}^{(n)}_\Psi|\big)(t) + c_0\int_0^t (t-s)^\alpha|\mathcal{F}(\hat{A}^{(n)})|(s)\,d\bar{A}^{(n)}(s).$$

Noticing that $\mathcal{F}(x)\in\mathbb{C}$ for every $x\in\mathbb{D}$, and applying Lemma 4.5, we obtain

$$\begin{aligned}|\mathcal{F}(\hat{A}^{(n)})|(t) \le{}& \mathcal{F}\big(|\hat{U}^{(n)}|+|\hat{\varepsilon}^{(n)}_\Psi|\big)(t)\\ &+ c_0\Big(\int_0^t (t-s)^\alpha\mathcal{F}\big(|\hat{U}^{(n)}|+|\hat{\varepsilon}^{(n)}_\Psi|\big)(s)\,d\bar{A}^{(n)}(s)\Big)\exp\Big(c_0\int_0^t (t-s)^\alpha\,d\bar{A}^{(n)}(s)\Big).\end{aligned}$$

Plugging the inequality above back into (3.31), recalling (3.15) and the convergence of $\hat{\varepsilon}^{(n)}_\Psi\to 0$ in Lemma 3.7, the stochastic boundedness for $\hat{A}^{(n)}$ on $[0,T]$ is proved by showing the locally stochastically boundedness for $\hat{U}^{(n)}$ in (3.32).

To that end, applying Doob's maximal inequality to $\hat{X}^{(n)}_\Psi$ in (3.29), one can find

$$\begin{aligned}&\mathbb{E}\Big[\sup_{t\le T}\big(\hat{X}^{(n)}_\Psi\big)^2(t\wedge\bar{\tau}^{(n)}_+)\Big] \le 4\,\mathbb{E}\Big[\big(\hat{X}^{(n)}_\Psi\big)^2(T\wedge\bar{\tau}^{(n)}_+)\Big]\\ &= 4\,\mathbb{E}\Big[\int_0^T \Psi^2(\bar{Y}(s))\,d\bar{A}^{(n)}(s\wedge\bar{\tau}^{(n)}_+)\Big] \le \frac{4}{(1-\beta)^2}\,\mathbb{E}\big[\bar{A}^{(n)}(T\wedge\bar{\tau}^{(n)}_+)\big] \le \frac{4\,\lambda_0^{(n)}\,T}{(1-\beta)^3},\end{aligned} \tag{4.33}$$

where(4.6) and (3.15) are applied. Recalling the identity (4.15) for $\bar{Z}^{(n)}_{\varphi,k_0}$, we have

$$\bar{Z}^{(n)}_{\varphi,k_0}(t) = \bar{\varepsilon}^{(n)}_{\varphi,1}(t) + \bar{\varepsilon}^{(n)}_{\varphi,2}(s) + \bar{\varepsilon}^{(n)}_{\varphi,3}(s) + \mu^{(n)}\hat{\kappa}\int_0^s (\frac{k}{n}+s-u)^{\alpha-1/2}\,d\bar{A}^{(n)}(u\wedge\bar{\tau}^{(n)}_+).$$

Applying Lemma 4.1 for $\bar{\varepsilon}^{(n)}_{\varphi,1}$, Lemma 4.3 for $\bar{\varepsilon}^{(n)}_{\varphi,2}$, Lemma 4.4 for $\bar{\varepsilon}^{(n)}_{\varphi,3}$ and Lemma 4.2 with $\gamma = 1/2-\alpha \in (0,1/2)$, one can find that $\bar{Z}^{(n)}_{\varphi,k_0}$ is locally stochastically bounded. Together with the weak convergence for $\hat{Z}^{(n)}$ in Lemma 3.6 and (3.15), this implies the locally stochastically boundedness of $\hat{\Xi}^{(n)}$ in (3.23).

For the last, recalling $\bar{I}^{(n)}$ in (3.30) and the Lipschitz-continuity of Ψ on $[0,k_0]$ from Assumption A2, we obtain the locally stochastically boundedness for $\hat{U}^{(n)}$. This proves the locally stochastically boundedness for $\hat{A}^{(n)}$. □

Lemma 4.6. *Under Assumptions A1, A2 and A3, for every $\alpha \in (0,1/2)$, we have*

$$\bar{\varepsilon}^{(n)}_{\varphi,4}(t) \to 0 \quad \textit{uniformly on } [0,T] \textit{ in probability as } n \to \infty.$$

Proof. Recalling $\bar{\varepsilon}^{(n)}_{\varphi,4}$ in (4.15), by Fubini's theorem, we have

$$\begin{aligned}\bar{\varepsilon}^{(n)}_{\varphi,4}(t) &= \frac{\hat{\kappa}\mu^{(n)}}{\sqrt{n}} \int_0^t \big(\frac{k}{n} + t - s\big)^{\alpha-1/2}\, d\hat{A}^{(n)}(s \wedge \bar{\tau}^{(n)}_+) \\ &= \frac{\hat{\kappa}\mu^{(n)}}{\sqrt{n}} \big(\frac{1}{2} - \alpha\big) \int_0^\infty \big(\frac{k}{n} + u\big)^{\alpha-3/2} \big(\hat{A}^{(n)}(t \wedge \bar{\tau}^{(n)}_+) - \hat{A}^{(n)}((t-u) \wedge \bar{\tau}^{(n)}_+)\big)\, du.\end{aligned}$$

Then we obtain

$$\begin{aligned}\big|\bar{\varepsilon}^{(n)}_{\varphi,4}\big|(t) &\le \frac{|\hat{\kappa}\mu^{(n)}|(1-2\alpha)}{\sqrt{n}} \cdot \sup_{s\le t} \big|\hat{A}^{(n)}\big|(s \wedge \bar{\tau}^{(n)}_+) \cdot \int_0^\infty \big(\frac{k}{n} + u\big)^{\alpha-3/2}\, du \\ &= 2|\hat{\kappa}\mu^{(n)}| \cdot \sup_{s\le t} \big|\hat{A}^{(n)}\big|(s \wedge \bar{\tau}^{(n)}_+) \cdot \frac{k^{\alpha-1/2}}{n^\alpha}.\end{aligned}$$

Therefore, the limit for $\bar{\varepsilon}^{(n)}_{\varphi,4}$ is proved by applying the boundedness of $\hat{A}^{(n)}$ from Lemma 4.1. □

Proof of Lemma 3.8 – $\alpha \in (0,1/2)$**.** The lemma is proved by applying Lemmas 4.1, 4.3, 4.4 and 4.6. □

Appendix A. Appendix

Proof of Proposition 3.1. Suppose first that $\Psi : \mathbb{R} \to \mathbb{R}_+$ has an upper bound, say c_0.

For every $n \in \mathbb{N}$, define by induction

$$t^{(n)}_k = \frac{k}{2^n}, \quad \bar{y}^{(n)}_0 = 0 \quad \text{and} \quad \bar{y}^{(n)}_k = \lambda_0\mu \sum_{1\le j\le k} \int_{t^{(n)}_{j-1}}^{t^{(n)}_j} (t^{(n)}_k - u)^\alpha \Psi\big(\bar{y}^{(n)}_{j-1}\big)\, du, \tag{A.1}$$

and let $\{\bar{y}^{(n)}(t), t \ge 0\}$ be the linear interpolation of $\bar{y}^{(n)}_k$ with $\bar{y}^{(n)}(t^{(n)}_k) = \bar{y}^{(n)}_k$.

It is true from (A.1) that, for every fixed n, $\{\bar{y}^{(n)}_k, k \ge 0\}$ is monotone in k and

$$\begin{aligned}|\bar{y}^{(n)}_{k+1} - \bar{y}^{(n)}_k| &\le \lambda_0\mu\, c_0 \cdot \Big(\int_0^{t^{(n)}_{k+1}} (t^{(n)}_{k+1} - u)^\alpha\, du - \int_0^{t^{(n)}_k} (t^{(n)}_k - u)^\alpha\, du \Big) \\ &= \frac{\lambda_0\mu\, c_0}{\alpha+1} \cdot \big((t^{(n)}_{k+1})^{\alpha+1} - (t^{(n)}_k)^{\alpha+1}\big),\end{aligned}$$

from which one can check that $\{\bar{y}^{(n)}(t), t \ge 0\}_{n\ge1}$ is relatively compact in $\mathbb{C}$.

Let $\bar{y} \in \mathbb{C}$ be a limit point. For every $n \in \mathbb{N}$ and $t > 0$, we have $t \in [t_k^{(n)}, t_{k+1}^{(n)})$ for some $k \geq 0$. One can check directly that

$$\begin{aligned}
&\Big|\bar{y}(t) - \lambda_0\mu \int_0^t (t-u)^\alpha \Psi(\bar{y}(u))\, du\Big| \\
&\leq \big|\bar{y}(t) - \bar{y}^{(n)}(t_k^{(n)})\big| + \lambda_0\mu\Big(\int_0^t (t-u)^\alpha \Psi(\bar{y}(u))\, du - \int_0^{t_k^{(n)}} (t_k^{(n)} - u)^\alpha \Psi(\bar{y}(u))\, du\Big) \\
&\quad + \Big|\bar{y}^{(n)}(t_k^{(n)}) - \lambda_0\mu \int_0^{t_k^{(n)}} (t_k^{(n)} - u)^\alpha \Psi(\bar{y}(u))\, du\Big| = I_1^{(n)}(t) + I_2^{(n)}(t) + I_3^{(n)}(t).
\end{aligned}$$

By the continuity of $\bar{y}$, over some subsequence of $\{n\}_{n\geq 1}$, we have

$$\begin{aligned}
I_1^{(n)}(t) &\leq \big|\bar{y}(t) - \bar{y}(t_k^{(n)})\big| + \big|\bar{y} - \bar{y}^{(n)}\big|(t_k^{(n)}) \to 0 \quad \text{u.o.c. in probability,} \\
I_2^{(n)}(t) &\leq \lambda_0\mu\, c_0 \cdot \Big(\int_0^t (t-u)^\alpha\, du - \int_0^{t_k^{(n)}} (t_k^{(n)} - u)^\alpha\, du\Big) \\
&= \frac{\lambda_0\mu\, c_0}{\alpha+1} \cdot \big(t^{\alpha+1} - (t_k^{(n)})^{\alpha+1}\big) \to 0 \quad \text{u.o.c. in probability.}
\end{aligned}$$

Moreover, by the identity for $\bar{y}^{(n)}(t_k^{(n)})$ in (A.1), we also have

$$\begin{aligned}
I_3^{(n)}(t) &\leq \lambda_0\mu \sum_{1\leq j\leq k} \int_{t_{j-1}^{(n)}}^{t_j^{(n)}} (t_k^{(n)} - u)^\alpha \big|\Psi(\bar{y}(u)) - \Psi(\bar{y}^{(n)}(t_{j-1}^{(n)})\big|\, du \\
&\leq \lambda_0\mu \sup_{\substack{0\leq v,u\leq t \\ 0\leq u-v\leq 2^{-n}}} \big|\Psi(\bar{y}(u)) - \Psi(\bar{y}^{(n)}(v))\big| \cdot \int_0^{t_k^{(n)}} (t_k^{(n)} - u)^\alpha\, du \\
&\leq \frac{\lambda_0\mu}{\alpha+1} \cdot \sup_{\substack{0\leq v,u\leq t \\ 0\leq u-v\leq 2^{-n}}} \big|\Psi(\bar{y}(u)) - \Psi(\bar{y}^{(n)}(v))\big| \cdot t^{\alpha+1} \to 0 \quad \text{u.o.c. in probability,}
\end{aligned}$$

by the continuity of Ψ and the convergence of $\bar{y}^{(n)}$. Therefore, we conclude that $\bar{y}$ solves (3.1).

Now, suppose that Ψ is additionally global Lipschitz continuous, and we show that the pathwise uniqueness of solutions holds. Let $\bar{z}$ be another solution to (3.1). Then, for some constant $k_0 > 0$,

$$\begin{aligned}
\big|\bar{y} - \bar{z}\big|(t) &\leq \lambda_0|\mu| \int_0^t (t-s)^\alpha \big|\Psi(y(s)) - \Psi(\bar{z}(s))\big|\, ds \\
&\leq k_0 \int_0^t (t-s)^\alpha \big|\bar{y} - \bar{z}\big|(s)\, ds = \int_0^t \big|\bar{y} - \bar{z}\big|(t-s)\, d\nu_c(s)\,,
\end{aligned}$$

where $\nu_k(ds) = k_0\, s^\alpha\, ds$. It also implies that for every $n \in \mathbb{N}$,

$$\big|\bar{y} - \bar{z}\big|(t) \leq \int_0^t \big|\bar{y} - \bar{z}\big|(t-s)\, d\nu_k^{*n}(s) \leq \sup_{u\leq t} \big|\bar{y} - \bar{z}\big|(u) \cdot \nu_k^{*n}(t)\,,$$

where ν_k^{*n} denotes the n^{th}-self-convolution of ν_k. Thus, we obtain

$$\big|\bar{y} - \bar{z}\big|(t) \leq \sup_{u\leq t} \big|\bar{y} - \bar{z}\big|(u) \cdot \frac{\nu_k^n(t)}{n!} = \sup_{u\leq t} \big|\bar{y} - \bar{z}\big|(u) \cdot \frac{(k_0 t^{\sigma+1})^n}{(\alpha+1)^n n!} \to 0 \quad \text{as } n \to \infty.$$

This proves the uniqueness.

For the last, suppose that Ψ is locally Lipschitz and satisfies the linear growth condition in Assumption A2. For every $k > 0$, it is true that $\Psi(\cdot \wedge k)$ is bounded and global Lipschitz continuous

on $\mathbb{R}$. Let $\bar{y}_k$ be the unique solution to

$$\bar{y}_k(t) = \lambda_0 \mu \int_0^t (t-s)^\alpha \Psi\big(\bar{y}_k(s) \wedge k\big)\, ds.$$

Let $\bar{\tau}_k = \inf\{t > 0, \bar{y}_k(t) > k\}$. It is true that

$$\bar{y}_k(t) = \lambda_0 \mu \int_0^t (t-s)^\alpha \Psi\big(\bar{y}_k(s)\big)\, ds \quad \forall t \le \bar{\tau}_k \text{ and } \quad \bar{y}_k(t) = \bar{y}_{k'}(t) \quad \forall t < \bar{\tau}_k \wedge \bar{\tau}_{k'},$$

from which one can define uniquely on $[0, \tau_\infty)$ with $\bar{\tau}_\infty = \lim_{k\to\infty} \bar{\tau}_k$ such that

$$\bar{y}(t) = \bar{y}_k(t) \quad \forall t < \bar{\tau}_k \quad \text{and} \quad \bar{\tau}_k = \inf\{t > 0, \bar{y}(t) > k\}.$$

On the other hand, by the linear growth condition, for some constant $c_0 > 0$ independent to k, we have

$$\begin{aligned} |\bar{y}|(t \wedge \bar{\tau}_k) = |\bar{y}_k|(t \wedge \bar{\tau}_k) &= \lambda_0 |\mu| \int_0^{t\wedge\bar{\tau}_k} \big((t \wedge \bar{\tau}_k) - u\big)^\alpha \Psi\big(\bar{y}(u)\big)\, du \\ &\le \lambda_0 |\mu| \int_0^t (t-u)^\alpha \Psi(\bar{y}(u \wedge \bar{\tau}_k))\, du \le c_0 \int_0^t (t-u)^\alpha \big(1 + |\bar{y}|(u \wedge \bar{\tau}_k)\big)\, du\,, \end{aligned} \tag{A.2}$$

for every $t > 0$. Let ψ_c be the solution to

$$\psi_c(t) = c_0 t^\alpha + c_0 \int_0^t \psi_c(t-u) u^\alpha\, du.$$

One can check directly from (A.2) that

$$|\bar{y}|(t \wedge \bar{\tau}_k) \le \int_0^t \psi_c(u)\, du.$$

Thus, letting $k \to \infty$ proves $\bar{\tau}_\infty = \infty$ and the existence of a unique global solution. □

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