HEAVY-TRAFFIC LIMITS FOR AN INFINITE-SERVER FORK-JOIN QUEUEING SYSTEM WITH DEPENDENT AND DISRUPTIVE SERVICES

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Abstract. We study an infinite-server fork-join queueing system with dependent services, which experiences alternating renewal service disruptions. Jobs are forked into a fixed number of parallel tasks upon arrival and processed at the corresponding parallel service stations with multiple servers. Synchronization of a job occurs when its parallel tasks are completed, i.e., non-exchangeable. Service times of the parallel tasks of each job can be correlated, having a general continuous joint distribution function, and moreover, the service vectors of consecutive jobs form a stationary dependent sequence satisfying the strong mixing (α-mixing) condition. The system experiences renewal alternating service disruptions with up and down periods. In each up period, the system operates normally, but in each down period, jobs continue to enter the system, while all the servers will stop working, and services received will be conserved and resume at the beginning of the next up period. We study the impact of both the dependence among service times and these down times upon the service dynamics, the unsynchronized queueing dynamics and the synchronized process, assuming that the down times are asymptotically negligible. We prove FWLLN and FCLT for these processes, where the limit processes in the FCLT possess a stochastic decomposition property and the convergence requires the Skorohod M_1 topology.

1. Introduction

The purpose of this paper is to understand the performance of an infinite-server fork-join queueing system with dependent services which experiences service disruptions with alternating "up" and "down" periods. In the *up* periods, the system operates normally: each job upon arrival is forked into a fixed number of parallel tasks to be processed in the dedicated service stations simultaneously. Specifically, each of the parallel tasks is served at the associated service station with an infinite number of parallel servers. Each job brings in a service vector for its parallel tasks. The service times of the parallel tasks of each job can be correlated. We assume that the service vectors of the consecutive jobs are identically distributed, and moreover, they form a stationary and weakly dependent sequence satisfying the strong mixing (α-mixing) condition. After service completion, each task will join a buffer associated with its service station, referred to as "unsynchronized queue", waiting there for synchronization. Tasks are only synchronized if all tasks tagged with the same job are completed, called "non-exchangeable synchronization" (NES) [3, 24, 25, 43]. After synchronization, jobs will leave the system immediately (zero synchronization time). In the *down* periods, all the servers in each parallel service station stop working. Tasks in service (respectively, waiting for synchronization) will remain in the associated service stations (respectively, unsynchronized queues). The amount of services received will be conserved and services will resume when the down periods end. Each task/job may experience several up and down periods before synchronization. On the other hand, jobs continue to enter the system and are forked into parallel

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tasks. Each of these tasks will be assigned to a free server in the dedicated service station and wait there to receive service when the next up period starts.

In fork-join queueing systems with NES, the response time, as measured by the wait for service, plus service time and the wait for synchronization, is a key performance measure. The model we study captures the two latter measures since jobs do not wait for service due to the infinite number of servers. It is thus important to characterize the impact of the down times, as well as the correlations among parallel service times of each job and services of consecutive jobs, upon the service and synchronization processes. The system dynamics of the fork-join queueing system cannot be analyzed exactly, even if the arrival process is Poisson and the service time distributions are exponential. Our objective is to obtain an approximation of the system dynamics in a heavy-traffic asymptotic regime, where the arrival rate gets large while service times are fixed. The results can be used to approximate the performance of a corresponding fork-join model with finite-server stations in which all stations are in an underloaded (quality-driven) regime. We also scale the disruption/interruption process such that the "up" times have the same order as the service times, while the "down" times are asymptotically negligible compared with the service times (see Assumption 3). We aim to study the impact of these very "short" down times upon the performance of the fork-join queueing system.

We show a functional central limit theorem (FCLT) for the service processes, unsynchronized queueing processes and synchronized processes (Theorem 3.3). The convergence in the FCLT is shown in the Skorohod M_1 topology. There exists a *stochastic decomposition property* for the limit processes, which evidently does not hold for the corresponding prelimit processes. All the limit processes are decomposed into three independent processes, a stochastic integral of the arrival limit process, a functional of a multiparameter generalized Kiefer process driven by the sequence of strong mixing (α-mixing) service vectors, and a jump process arising from the service disruptions. The jump processes in the limit have their jumps occurring simultaneously, being driven by a compound-type jump process, and the jump sizes in each of these limit processes depend on the service time distributions at the corresponding service stations. We also obtain a functional weak law of large numbers (FWLLN) for the aforementioned processes, which shows that the disruptions do not affect the system performance in the fluid scale, under our assumption of asymptotically negligible down times.

We then study the characterization of these limiting processes (Lemmas 3.1–3.3 and Proposition 3.1). Specifically, we obtain the covariance functions of the limiting processes associated with the arrival and service processes and the random environment, when the arrival process is a Lévy process with (deterministic) time change, and when the limit counting process associated with the up-down cycles is Poisson. We study the impact of the service correlations in two aspects: correlation among the service vector of each job, and correlation among service vectors of different jobs. When the service vectors form a first-order discrete vector autoregressive sequence, "DVAR(1)", we find that the correlations from these two aspects can be separated in the covariance functions of the aforementioned processes (Corollary 3.1). The impact of the asymptotically negligible down times is in two folds, affecting both the queueing process and the delay for synchronization. These are shown in the jump terms in the limiting service, unsynchronized queueing and synchronized processes, and their covariance functions are also obtained (Lemma 3.3).

1.1. **Literature Review and Comparisons.** Fork-join networks have been extensively studied in the literature, see, e.g., [20, 28, 29, 40, 3, 19, 9, 42, 43, 31, 30] and references therein. A pure join model with multiple class of jobs is studied in [13], where an asymptotically optimal matching policy is derived to minimize the holding cost in unsynchronized queues. To our best knowledge, this is the very first work on fork-join queueing systems with both dependent and disruptive services.

Our work relates closely to the recent development of multi-server fork-join queueing models in the many-server heavy-traffic regimes. The major challenge in studying multi-server fork-join queueing models with NES is the resequencing of arrival orders after service completion at each service station due to the randomness of service times for parallel tasks [3, 43]. Lu and Pang [24, 25] developed a new approach to resolve the sequencing for these fork-join models, via sequential empirical processes driven by the service vectors of jobs. It is assumed in [24, 25] that the service vectors of jobs in the multi-server fork-join queueing models are i.i.d. and have a continuous joint distribution function. Our first contribution is to generalize that approach to the setting where the service vectors of the consecutive jobs are weakly dependent, forming a stationary sequence and satisfying the strong mixing (α-mixing) condition for random vectors. The proofs require new techniques for the strong mixing conditions.

Queues with dependent service times have been studied in the literature. For example, Pang and Whitt [35] recently studied two-parameter process limits for $G/G/\infty$ queues, where the consecutive service times are assumed to be stationary and weakly dependent, satisfying the ϕ-mixing or S-mixing conditions. See also the extensive references therein. However, in all these studies, correlations are only considered in the unit variate setting. Our work is the first to study the sequential dependence among service vectors associated with each job in queueing systems. As a special case, our work also extends the analysis of the $G/G/\infty$ queues to the case of strong mixing (α-mixing) service times, in terms of the total count processes.

On the other hand, substantial amount work has studied queues in random environments (in particular, service disruptions/interruptions), for example, [8, 16, 17, 18, 32, 33, 36, 26] and references therein. The work closest to ours is [36], where the $G/G/\infty$ queue with weakly dependent and disruptive service times is studied. Two-parameter heavy-traffic limits are proved for the queueing processes in the space $\mathbb{D}([0,\infty),\mathbb{D})$ endowed with Skorohod M_1 topology. The case when the service times have finite support is also discussed in [36]. Our paper generalizes the work [36] to the infinite-server fork-join queueing system with NES, where the disruptions affects all service stations simultaneously. Thus, the effect of the disruptions is not only characterized for the service processes at the service stations, but more importantly, for the synchronization processes.

1.2. **Organization of the Paper.** The paper is organized as follows. We finish the introduction with notation below. In §2, we provide a detailed model description and the assumptions. The main results are given in §3. The FWLLN and FCLT for the processes tracking the service and waiting dynamics for synchronization are stated in §3.1, and their proofs are provided in §5. We characterize the limiting processes in §3.2, and provide the proof in §4. Some concluding remarks are given in §6.

1.3. **Notation.** The following notations will be used throughout the paper. $\mathbb{R}$ and $\mathbb{R}_+$ ($\mathbb{R}^d$ and $\mathbb{R}^d_+$, respectively) denote sets of real and real non-negative numbers (d-dimensional vectors, respectively, $d \geq 2$). $\mathbb{N}$ denotes the set of natural numbers. For $a, b \in \mathbb{R}$, we denote $a \wedge b := \min\{a,b\}$ and $a \vee b := \max\{a,b\}$. For any $x \in \mathbb{R}_+$, $\lfloor x \rfloor$ ($\lceil x \rceil$, respectively) is used to denote the largest integer no greater than (the smallest integer no less than) x, respectively. We use bold letter to denote a vector, e.g., $\boldsymbol{x} := (x_1, ..., x_d) \in \mathbb{R}^d$, $d \geq 2$. $\mathbf{0}$ and $\boldsymbol{e}$ denote the vectors whose components are all 0 and 1, respectively. $\boldsymbol{e}_i$ is used to denote the vector whose i^{th} component is one and whose other components are infinity. For $x \in \mathbb{R}$ and $\boldsymbol{e}, \boldsymbol{e}_i \in \mathbb{R}^d$, $d \geq 2$, we define $x\boldsymbol{e} := (x, ..., x) \in \mathbb{R}^d$ and let $x\boldsymbol{e}_i$ be the vector whose i^{th} entry is x and whose other components are infinity, $1 \leq i \leq d$. For $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$, $d \geq 2$, we denote $\boldsymbol{x} \leq \boldsymbol{y}$ and $\boldsymbol{x} < \boldsymbol{y}$ in the componentwise sense, and let $\boldsymbol{x} \wedge \boldsymbol{y} = (x_1 \wedge y_1, ..., x_d \wedge y_d)$. Denote $(\mathbb{R}^2)^\infty := \{((a_1, b_1), \ldots, (a_j, b_j), \ldots) : a_j, b_j \in \mathbb{R}, j \in \mathbb{N}\}$ with the metric ρ defined by $\rho(\gamma_1, \gamma_2) := \sum_{i=1}^\infty \frac{1}{2^i} \min\{\max\{|a_i - c_i|, |b_i - d_i|\}, 1\}$, for any $\gamma_1 := ((a_1, b_1), (a_2, b_2), ...)$ and $\gamma_2 := ((c_1, d_1), (c_2, d_2), ...)$ in $(\mathbb{R}^2)^\infty$ (the well-definedness of the metric can be easily checked as Example 1.2 in [5]). We use $\mathbf{1}(A)$ to denote the indicator function of a set A. For any univariate distribution function G, we denote $G^c := 1 - G$. For any two deterministic real-valued functions f and g, we write $f(x) = O(g(x))$ if $\limsup_{x\to\infty} |f(x)/g(x)| < \infty$ and $f(x) = o(g(x))$ if $\limsup_{x\to\infty} |f(x)/g(x)| = 0$. For a sequence of random variables $\{Z_n : n \geq 1\}$ and a sequence of constants $\{b_n : n \geq 1\}$, we write

$Z_n = o_p(b_n)$ if Z_n/b_n converges to zero in probability as $n \to \infty$, i.e., $\lim_{n\to\infty} P(|Z_n/b_n| > \epsilon) = 0$ for any $\epsilon > 0$.

All random variables and processes are defined on a common probability space $(\Omega, \mathcal{F}, P)$. For any two complete separable metric spaces $\mathcal{S}_1$ and $\mathcal{S}_2$, we denote $\mathcal{S}_1 \times \mathcal{S}_2$ as their product space, endowed with the maximum metric ρ defined by $m((x_1, x_2), (y_1, y_2)) := \max\{m_1(x_1, y_1), m_2(x_2, y_2)\}$, for $(x_1, x_2), (y_1, y_2) \in \mathcal{S}_1 \times \mathcal{S}_2$, where m_1 and m_2 are metrics for $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively (see, e.g., §11.4 of [41]). $\mathcal{S}^k$ is used to represent k-fold product space of any complete and separable metric space $\mathcal{S}$ for $k \in \mathbb{N}$. For a complete separable metric space $\mathcal{S}$, $\mathbb{D}([0,\infty), \mathcal{S})$ denotes the space of all $\mathcal{S}$-valued càdlàg functions on $[0,\infty)$, and is endowed with the Skorohod J_1 topology (see, e.g., [12]). Let $\mathbb{D} \equiv \mathbb{D}([0,\infty), \mathbb{R})$. $(\mathbb{D}, J_1)$ denotes the space $\mathbb{D}$ endowed with J_1 topology while $(\mathbb{D}, M_1)$ denotes the space $\mathbb{D}$ endowed with M_1 topology. We refer to §3.3 and §12 of [41] for the definition and convergence criteria in $(\mathbb{D}, M_1)$. Let $(\mathbb{D}^k, J_1) \equiv (\mathbb{D}, J_1) \times \cdots \times (\mathbb{D}, J_1)$ ($(\mathbb{D}^k, M_1) \equiv (\mathbb{D}, M_1) \times \cdots \times (\mathbb{D}, M_1)$, respectively) be the k-fold product of $(\mathbb{D}, J_1)$ ($(\mathbb{D}, M_1)$, respectively) with the maximum metric as the product topology. Let $\mathbb{C} \equiv \mathbb{C}([0,\infty), \mathbb{R})$ be the space of all continuous real-valued functions on $[0,\infty)$ and denote $\mathbb{C}_\uparrow$ as the space of nondecreasing continuous functions in $\mathbb{C}$. Let $\mathbb{C}^k$ be the k-fold product of $\mathbb{C}$. Denote $\|\cdot\|$ as the uniform norm, i.e., for any real-valued function x and $T > 0$, $\|x\|_T = \sup_{0\le t\le T} |x(t)|$. $\mathbb{D}(\mathbb{T}, \mathbb{R})$ denotes the space of all "continuous from above with limits from below" real-valued functions in the sense of [4, 27, 39, 14] on $\mathbb{T} \subseteq \mathbb{R}^k_+$ for $k \ge 2$; see [39, 4, 27] for $\mathbb{T} = [0,1]^k$ and [14] for $\mathbb{T} = [0,\infty)^k$. Denote $\mathbb{D}_k \equiv \mathbb{D}([0,\infty)^k, \mathbb{R})$ for $k \ge 2$. Weak convergence of probability measures μ_n to μ will be denoted as $\mu_n \Rightarrow \mu$. We also use $\Rightarrow$ to denote the convergence in distribution without abuse of notation. For a sequence of processes $\{\mathcal{X}^n : n \ge 1\}$ and a process $\mathcal{X}$, $\mathcal{X}^n \xrightarrow{d_f} \mathcal{X}$ denotes the convergence in finite-dimensional distributions of $\mathcal{X}^n$ to $\mathcal{X}$. We use the abbreviations *a.s.* and *a.e.* to represent *almost surely* and *almost everywhere*.

2. Model and Assumptions

In this section, we present the model and assumptions in detail. In the fork-join queueing model, there is a single class of arrivals and each job upon arrival is forked into K parallel tasks, which are processed in the corresponding parallel service stations with an infinite number of servers. After service completion, each task will join a waiting buffer for synchronization, called *unsynchronized queue*, associated with each service station. When all tasks of the same job are finished, they will be synchronized and leave the system immediately. The system is operating in an alternating renewal environment with *up-down* cycles (down periods as the service disruptive process). In the up periods, the system operates normally. In the down periods, all servers will stop functioning and resume at the beginning of the next up period. New jobs continue to enter the system and their tasks will be assigned to free servers in the service stations and wait there for service to start at the beginning of the next up period. Tasks in service (respectively, waiting for synchronization) will be kept in their stations (respectively, unsynchronized queues), and the amount of services received will be conserved, and their services will be resumed when the down period ends.

Let $A = \{A(t) : t \ge 0\}$ be the arrival process of the system with τ_i representing the arrival time of the i^{th} job, $i \in \mathbb{N}$. Let $\{\boldsymbol{\eta}^i : i \ge 1\}$ be a sequence of identically distributed service vectors of the parallel tasks. The joint distribution of the service time vector for the i^{th} job $\boldsymbol{\eta}^i$ is $F(\boldsymbol{x}) := F(x_1, ..., x_K)$ for $x_1, ..., x_K \ge 0$ and $F(\boldsymbol{x}) = 0$ if $x_k \le 0$ for some $1 \le k \le K$. The service times of the parallel tasks of the same job can be correlated. The case of perfectly positively correlated parallel services (the components of the service vector are equal, $\eta^i_1 = \eta^i_2 = \cdots = \eta^i_K$ for each i) is excluded since that will lead to empty waiting buffers for synchronization. Moreover, we assume that the sequence $\{\boldsymbol{\eta}^i : i \ge 1\}$ is stationary and weakly dependent, satisfying the strong mixing (α-mixing) condition.

Assumption 1. *The sequence of service vectors $\{\boldsymbol{\eta}^i : i \ge 1\}$ is weakly dependent and constitute a one-sided stationary sequence with each $\boldsymbol{\eta}^i$ having a continuous distribution function $F(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^K_+$.*

The sequence $\{\boldsymbol{\eta}^i : i \geq 1\}$ *satisfies the* α*-mixing condition, that is,* $\alpha_n = O(n^{-a})$*, for* $a > 1$*, where* $\alpha_i := \sup\{|P(A \cap B) - P(A)P(B)| : A \in \mathcal{F}_j, B \in \mathcal{G}_{j+i}, j \geq 1\}$*, with* $\mathcal{F}_j := \sigma\{\boldsymbol{\eta}^i : 1 \leq i \leq j\}$ *and* $\mathcal{G}_j := \sigma\{\boldsymbol{\eta}^i : i \geq j\}$*.*

The joint distribution of any two service times η_j^i and η_k^i is $F_{j,k}(x_j, x_k) := P(\eta_j^i \leq x_j, \eta_k^i \leq x_k)$ for $x_j, x_k \geq 0$, $j, k = 1, ..., K$. Note $F_{j,k}(\cdot,\cdot) = F_k(\cdot)$ when $j = k$ for $j, k = 1, ..., K$. We denote $F^c_{j,k}(x_j, x_k) := P(\eta_j^i > x_j, \eta_k^i > x_k) = 1 - F_j(x_j) - F_k(x_k) + F_{j,k}(x_j, x_k)$ for $x_j, x_k \geq 0$, $j, k = 1, ..., K$. Note $F^c_{j,k}(\cdot,\cdot) = F^c_k(\cdot)$ when $j = k$ for $j, k = 1, ..., K$. Let $\eta^i_{\mathrm{m}} := \max\{\eta^i_1, ..., \eta^i_K\}$ be the maximum of the components in the service vector $\boldsymbol{\eta}^i$, and $F_{\mathrm{m}}(x) := P(\eta^i_{\mathrm{m}} \leq x) = F(x, ..., x)$ for $x \geq 0$. (Throughout the paper, we use subscript "m" to index quantities and processes associated with the maximum.)

Let $\{(u_i, d_i) : i \geq 1\}$ be a sequence of i.i.d. random vectors with u_i and d_i representing the up and down times in the i^{th} *up-down* cycle of the underlying renewal process for all parallel service stations. We assume that the arrival and service processes and renewal disruptive process are mutually independent.

Recall that each job is forked into K tasks and each task is processed in a parallel station, for which we call the service station serving task k of each job as "service station k". Let $X_k := \{X_k(t) : t \geq 0\}$ be the process counting the number of tasks in service at the service station k, and $Y_k := \{Y_k(t) : t \geq 0\}$ be the process counting the number of tasks in the waiting buffer for synchronization (unsynchronized queue) after service completion at service station k, $k = 1, ..., K$. Let $S := \{S(t) : t \geq 0\}$ be the process counting the number of synchronized jobs and $D_k := \{D_k(t) : t \geq 0\}$ be the process counting the number of tasks that have completed service at station k, $k = 1, ..., K$. Denote $\boldsymbol{X} := (X_1, ..., X_K)$, $\boldsymbol{Y} := (Y_1, ..., Y_K)$ and $\boldsymbol{D} := (D_1, ..., D_K)$. We assume that the system starts empty at the beginning of an up period.

2.1. **A Sequence of Systems.** The exact distributions of $(\boldsymbol{X}, \boldsymbol{Y}, S)$ cannot be obtained directly, even when the arrival process is Poisson and the service times are exponential. Thus, our objective is to establish heavy-traffic limits for the fluid and diffusion scaled processes of $(\boldsymbol{X}, \boldsymbol{Y}, S)$ jointly. We consider a sequence of such systems indexed by n and use superscript n for the processes $A, \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{D}, S$, the sequence of $\{(u_i, d_i) : i \geq 1\}$, and the arrival times $\{\tau_i : i \geq 1\}$, but we let the service times $\{\boldsymbol{\eta}^i : i \geq 1\}$ and their distribution functions be independent of n. We will let the arrival rate grow large for the system to be in heavy traffic and make the following assumption on the arrival process A^n.

Assumption 2 (FCLT for arrivals)**.** *There exist: (i) a nondecreasing deterministic real-valued continuous function* $\bar{a}$ *with a density function* $\lambda \in \mathbb{D}$ *a.e. and* $\bar{a}(0) = 0$ *and (ii) a stochastic process* $\hat{A}$ *with sample paths in* $\mathbb{D}$*, such that*

$$\hat{A}^n := n^{-\frac{1}{2}}(A^n - n\bar{a}) \Rightarrow \hat{A} \quad \text{in} \quad (\mathbb{D}, M_1) \quad \text{as} \quad n \to \infty. \tag{2.1}$$

This implies that an FWLLN holds for A^n:

$$\bar{A}^n := n^{-1}A^n \Rightarrow \bar{a} \quad \text{in} \quad (\mathbb{D}, M_1) \quad \text{as} \quad n \to \infty. \tag{2.2}$$

Note that since the limit $\bar{a}$ is continuous, the convergence is equivalent to be in any Skorohod topology [41].

We consider a scaling regime for the underlying random environment, where the down times are asymptotically negligible. Specifically, we make the following assumption.

Assumption 3 (Scaling of the Service Disruptive Process)**.** *The sequence of up and down times* $\{(u^n_i, d^n_i) : i \geq 1\}$ *satisfies*

$$\{(u^n_i, n^{1/2}d^n_i) : i \geq 1\} \Rightarrow \{(u_i, d_i) : i \geq 1\} \quad \text{in} \quad \left(\mathbb{R}^2\right)^\infty \quad \text{as} \quad n \to \infty, \tag{2.3}$$

where $u_i, d_i > 0$ *a.s. for each* $i \geq 1$*.*

2.2. **System Dynamics.** We provide a representation of the processes $(\boldsymbol{X}^n, \boldsymbol{Y}^n, S)$. We first introduce some processes associated with the underlying renewal random environment. Let the sequence $\{T_i^n : i \geq 1\}$ be the renewal times, defined by

$$T_i^n := \sum_{j=1}^{i} (u_j^n + d_j^n), \quad \text{for} \quad i \geq 1 \quad \text{and} \quad T_0^n := 0. \tag{2.4}$$

Let $N^n := \{N^n(t) : t \geq 0\}$ be the associated renewal counting process, defined by

$$N^n(t) := \max\{i \geq 0 : T_i^n \leq t\}, \quad t \geq 0. \tag{2.5}$$

Let $\chi^n := \{\chi^n(t) : t \geq 0\}$ be the service-availability process, defined by

$$\chi^n(t) := \begin{cases} 1, & T_i^n \leq t \leq T_i^n + u_{i+1}^n, \quad \text{for} \quad i \geq 0, \\ 0, & T_i^n + u_{i+1}^n < t < T_{i+1}^n, \quad \text{for} \quad i \geq 0. \end{cases} \tag{2.6}$$

The cumulative up-time process $\xi^n := \{\xi^n(t) : t \geq 0\}$ is defined by

$$\xi^n(t) := \int_0^t \chi^n(s) ds, \quad t \geq 0. \tag{2.7}$$

The cumulative down-time process $\zeta^n := \{\zeta^n(t) : t \geq 0\}$ is defined by $\zeta^n(t) := t - \xi^n(t)$ for each $t \geq 0$.

Now, for each $t \geq 0$ and $k = 1, ..., K$, we can write

$$X_k^n(t) = \sum_{i=1}^{A^n(t)} \mathbf{1}(\eta_k^i > \xi^n(t) - \xi^n(\tau_i^n)), \tag{2.8}$$

$$S^n(t) = \sum_{i=1}^{A^n(t)} \mathbf{1}(\eta_j^i \leq \xi^n(t) - \xi^n(\tau_i^n), \quad \forall j = 1, ..., K), \tag{2.9}$$

$$\begin{aligned} Y_k^n(t) &= D_k^n(t) - S^n(t) \\ &= \sum_{i=1}^{A^n(t)} \mathbf{1}(\eta_k^i \leq \xi^n(t) - \xi^n(\tau_i^n) \text{ and } \eta_{k'}^i > \xi^n(t) - \xi^n(\tau_i^n) \quad \text{for some } k' \neq k). \end{aligned} \tag{2.10}$$

The following balanced equations hold for each $t \geq 0$ and $k = 1, ..., K$,

$$D_k^n(t) = A^n(t) - X_k^n(t), \tag{2.11}$$

$$Y_k^n(t) = D_k^n(t) - S^n(t). \tag{2.12}$$

Define the diffusion-scaled processes

$$\hat{\xi}^n := n^{1/2}(\xi^n - e) \quad \text{and} \quad \hat{\zeta}^n := n^{1/2}\zeta^n = -\hat{\xi}^n, \tag{2.13}$$

where $e(t) := t$ is the identity function, and ξ^n and ζ^n are the cumulative up and down times. We then obtain the following limits for the processes associated with service disruptions.

Lemma 2.1. *Under Assumption 3,*

$$\left(\xi^n, \zeta^n, N^n, \hat{\xi}^n, \hat{\zeta}^n\right) \Rightarrow \left(e, 0, \hat{N}, -\hat{J}, \hat{J}\right) \quad \text{in} \quad (\mathbb{D}^3, J_1) \times (\mathbb{D}^2, M_1) \quad \text{as} \quad n \to \infty, \tag{2.14}$$

where the limit process $\hat{J} := \{\hat{J}(t) : t \geq 0\}$ is defined by

$$\hat{J}(t) := \sum_{i=1}^{\hat{N}(t)} d_i, \quad t \geq 0, \tag{2.15}$$

where $\hat{N} := \{\hat{N}(t) : t \geq 0\}$ is the associated renewal counting process in the limit, defined by

$$\hat{N}(t) := \max\{i \geq 0 : \hat{T}_i \leq t\}, \quad t \geq 0, \tag{2.16}$$

and $\{\hat{T}_i : i \geq 0\}$ *is the limiting sequence of cycle renewal times, defined by*

$$\hat{T}_i := \sum_{j=1}^{i} u_j, \quad i \geq 1, \quad \textit{and} \quad \hat{T}_0 := 0. \tag{2.17}$$

Proof. The convergence of (ξ^n, ζ^n) in (2.14) follows directly from Assumption 3. The convergence of $(N^n, \hat{\xi}^n, \hat{\zeta}^n)$ in (2.14) can be found in §5.4 of [33]. The joint convergence $(\xi^n, \zeta^n, N^n, \hat{\xi}^n, \hat{\zeta}^n)$ is obtained from Theorem 11.4.5 of [41], by noting that the limit of (ξ^n, ζ^n) is deterministic. □

3. Main Results

3.1. **Limit Theorems.** We first define multiparameter sequential empirical processes $\hat{U}^n := \{\hat{U}^n(t, \boldsymbol{x}) : t \geq 0,\ \boldsymbol{x} \in [0,1]^K\}$ driven by a sequence of random vectors with uniform marginals:

$$\hat{U}^n(t, \boldsymbol{x}) := \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} \tilde{\gamma}_i(\boldsymbol{x}), \quad t \geq 0, \quad \boldsymbol{x} \in [0,1]^K,$$

where for each $i \in \mathbb{N}$, $\tilde{\gamma}_i(\boldsymbol{x}) := \mathbf{1}(\mathbf{U}^i \leq \boldsymbol{x}) - H(\boldsymbol{x})$, $\boldsymbol{x} \in [0,1]^K$, and $\{\mathbf{U}^i : i \geq 1\}$ constitutes a one-sided stationary sequence, and each $\mathbf{U}^i$ is a vector of nonnegative random variables with continuous joint distribution function $H(\cdot)$ and uniform marginals over $[0,1]$. We assume that the sequence $\{\mathbf{U}^i : i \geq 1\}$ satisfies the strong mixing (α-mixing) condition, as in Assumption 1 for $\{\boldsymbol{\eta}^i : i \geq 1\}$, that is, $\alpha_n^{\mathbf{U}} = O(n^{-a})$, for $a > 1$, where $\alpha_i^{\mathbf{U}} := \sup\{|P(A \cap B) - P(A)P(B)| : A \in \mathcal{F}_j^{\mathbf{U}}, B \in \mathcal{G}_{j+i}^{\mathbf{U}}, j \geq 1\}$, with $\mathcal{F}_j^{\mathbf{U}} := \sigma\{\mathbf{U}^i : 1 \leq i \leq j\}$ and $\mathcal{G}_j^{\mathbf{U}} := \sigma\{\mathbf{U}^i : i \geq j\}$. Here we state an FCLT for the multiparameter sequential empirical processes (see Theorem 1 in [7]).

Theorem 3.1. (*An FCLT for Strong Mixing* (α*-mixing*) *Uniformly Distributed Random Vectors*) *The processes* $\hat{U}^n(t, \boldsymbol{x})$ *converge in distribution,*

$$\hat{U}^n(t, \boldsymbol{x}) \Rightarrow \hat{U}(t, \boldsymbol{x}) \quad \textit{in} \quad (\mathbb{D}([0,1]^{K+1}, \mathbb{R}), J_1) \quad \textit{as} \quad n \to \infty, \tag{3.1}$$

where $\hat{U} := \{\hat{U}(t, \boldsymbol{x}) : t \geq 0, \boldsymbol{x} \in \mathbb{R}_+^K\}$ *is a continuous Gaussian random field with mean* 0 *and covariance function*

$$Cov(\hat{U}(t, \boldsymbol{x}), \hat{U}(s, \boldsymbol{y})) = (t \wedge s)\Gamma_U(\boldsymbol{x}, \boldsymbol{y}),$$
$$\Gamma_U(\boldsymbol{x}, \boldsymbol{y}) := (t \wedge s)(H(\boldsymbol{x} \wedge \boldsymbol{y}) - H(\boldsymbol{x})H(\boldsymbol{y})) + \Gamma_U^c(\boldsymbol{x}, \boldsymbol{y}) < \infty,$$
$$\Gamma_U^c(\boldsymbol{x}, \boldsymbol{y}) := \sum_{i=2}^{\infty} \left(E[\tilde{\gamma}_1(\boldsymbol{x})\tilde{\gamma}_i(\boldsymbol{y})] + E[\tilde{\gamma}_1(\boldsymbol{y})\tilde{\gamma}_i(\boldsymbol{x})]\right) < \infty,$$

for each $t, s \geq 0$ *and* $\boldsymbol{x}, \boldsymbol{y} \in [0,1]^K$.

To show the FCLT for the processes $(\boldsymbol{X}^n, \boldsymbol{Y}^n, S^n)$, we define the diffusion-scaled multiparameter sequential empirical processes $\hat{K}^n := \{\hat{K}^n(t, \boldsymbol{x}) : t \geq 0, \boldsymbol{x} \in \mathbb{R}_+^K\}$ by

$$\hat{K}^n(t, \boldsymbol{x}) := \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} \gamma_i(\boldsymbol{x}), \quad t \geq 0, \quad \boldsymbol{x} \in \mathbb{R}_+^K,$$

where for each $i \in \mathbb{N}$, $\gamma_i(\boldsymbol{x}) := \mathbf{1}(\boldsymbol{\eta}^i \leq \boldsymbol{x}) - F(\boldsymbol{x})$. We remark that when the service vectors are i.i.d., an FCLT for $\hat{K}^n$ is used to prove the FCLT for the processes $(\boldsymbol{X}^n, \boldsymbol{Y}^n, S^n)$ in diffusion scale for the infinite-server fork-join queueing system in [24]. Here under the strong mixing (α-mixing) condition, we need the FCLT for $\hat{K}^n$, which can be obtained by applying Theorem 3.1 and Sklar's theorem [38]. Let $\boldsymbol{F} : \mathbb{R}_+^K \to [0,1]^K$ be $\boldsymbol{F}(\boldsymbol{x}) := (F_1(x_1), ..., F_K(x_K))$ for $\boldsymbol{x} \in \mathbb{R}_+^K$. By Sklar's theorem [38], a multidimensional version of probability integral transformation, for any multivariate distribution function G, there exists a multivariate distribution function H_G (called "copula", depending on G)

with uniform marginals on $[0,1]$ such that $G(\boldsymbol{x}) = H_G(G_1(x_1), ..., G_K(x_K))$. For the multivariable distribution function F of service vectors and its copula H, we then can represent $\hat{K}^n(\cdot,\cdot)$ as a composition of $\hat{U}^n(\cdot,\cdot)$ with $\boldsymbol{F}(\cdot)$ in the second component, i.e.,

$$\hat{K}^n(t,\boldsymbol{x}) = \hat{U}^n(t,\boldsymbol{F}(\boldsymbol{x})), \quad t \geq 0, \quad \boldsymbol{x} \in \mathbb{R}_+^K. \tag{3.2}$$

Thus, by Theorem 3.1, we immediately obtain that

$$\hat{K}^n(t,\boldsymbol{x}) = \hat{U}^n(t,\boldsymbol{F}(\boldsymbol{x})) \Rightarrow \hat{K}(t,\boldsymbol{x}) := \hat{U}(t,\boldsymbol{F}(\boldsymbol{x})) \quad \text{in} \quad (\mathbb{D}_{K+1}, J_1) \quad \text{as} \quad n \to \infty, \tag{3.3}$$

where $\hat{K} := \{\hat{K}(t,\boldsymbol{x}) : t \geq 0, \boldsymbol{x}\}$ is a time-changed multiparameter Gaussian process with mean 0 and covariance function

$$Cov(\hat{K}(t,\boldsymbol{x}), \hat{K}(s,\boldsymbol{y})) = (t \wedge s)\Gamma(\boldsymbol{x},\boldsymbol{y}), \tag{3.4}$$

$$\Gamma(\boldsymbol{x},\boldsymbol{y}) := \Gamma_U(\boldsymbol{F}(\boldsymbol{x}),\boldsymbol{F}(\boldsymbol{y})) = (t \wedge s)(F(\boldsymbol{x} \wedge \boldsymbol{y}) - F(\boldsymbol{x})F(\boldsymbol{y})) + \Gamma^c(\boldsymbol{x},\boldsymbol{y}) < \infty,$$

$$\Gamma^c(\boldsymbol{x},\boldsymbol{y}) := \Gamma_U^c(\boldsymbol{F}(\boldsymbol{x}),\boldsymbol{F}(\boldsymbol{y})) = \sum_{i=2}^{\infty} \left(E[\gamma_1(\boldsymbol{x})\gamma_i(\boldsymbol{y})] + E[\gamma_1(\boldsymbol{y})\gamma_i(\boldsymbol{x})]\right) < \infty, \tag{3.5}$$

for each $t, s \geq 0$ and $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}_+^K$. Note that the last term $\Gamma^c(\boldsymbol{x},\boldsymbol{y})$ captures the sequential dependence among the service vectors, and it is evident that $\Gamma^c(\boldsymbol{x},\boldsymbol{y}) \equiv 0$ when the service vectors are i.i.d.

We will represent all the processes $\boldsymbol{X}^n, \boldsymbol{Y}^n, S^n$ as integrals of a multiparameter sequential empirical process $\bar{K}^n := \{\bar{K}^n(t,\boldsymbol{x}) : t \geq 0, \boldsymbol{x} \in \mathbb{R}_+^K\}$ driven by the sequence of service vectors $\{\boldsymbol{\eta}^i : i \geq 1\}$:

$$\bar{K}^n(t,\boldsymbol{x}) := \frac{1}{n}\sum_{i=1}^{\lfloor nt \rfloor} \mathbf{1}(\boldsymbol{\eta}^i \leq \boldsymbol{x}), \quad t \geq 0, \quad \boldsymbol{x} \in \mathbb{R}_+^K. \tag{3.6}$$

For $t \geq 0$ and $k = 1, ..., K$,

$$X_k^n(t) = n\int_0^t \int_{\mathbb{R}_+^K} \mathbf{1}(x_k > \xi^n(t) - \xi^n(s)) d\bar{K}^n\left(\bar{A}^n(s), \boldsymbol{x}\right), \tag{3.7}$$

$$Y_k^n(t) = n\int_0^t \int_{\mathbb{R}_+^K} \left(\mathbf{1}(x_k \leq \xi^n(t) - \xi^n(s)) - \mathbf{1}(x_j \leq \xi^n(t) - \xi^n(s),\ \forall j)\right) d\bar{K}^n\left(\bar{A}^n(s), \boldsymbol{x}\right), \tag{3.8}$$

and

$$S^n(t) = n\int_0^t \int_{\mathbb{R}_+^K} \mathbf{1}(x_j \leq \xi^n(t) - \xi^n(s),\ \forall j) d\bar{K}^n\left(\bar{A}^n(s), \boldsymbol{x}\right). \tag{3.9}$$

The integrals in (3.7), (3.8) and (3.9) are well-defined as Stieltjes integrals for functions of bounded variation as integrators. We remark that the representation like (3.7) were first established by Krichagina and Puhalskii [21] to study $G/GI/\infty$ queues, and Pang and Zhou [36] analyzed the impact of service interruptions on $G/G/\infty$ queues by generalizing the representation in [21] and introducing the cumulative up-time process ξ^n. In the fork-join queueing system, Lu and Pang [24] developed an integral representation for the system dynamics via multiparameter sequential empirical processes driven by the service vectors.

We define fluid-scaled processes $\bar{\boldsymbol{X}}^n$, $\bar{\boldsymbol{Y}}^n$ and $\bar{S}^n$ by

$$\bar{\boldsymbol{X}}^n := \frac{1}{n}\boldsymbol{X}^n, \quad \bar{\boldsymbol{Y}}^n := \frac{1}{n}\boldsymbol{Y}^n, \quad \bar{S}^n := \frac{1}{n}S^n. \tag{3.10}$$

The FWLLN for $\left(\bar{\boldsymbol{X}}^n, \bar{\boldsymbol{Y}}^n, \bar{S}^n\right)$ is stated in the following theorem.

Theorem 3.2 (FWLLN). *Under Assumptions* 1-3*, the fluid-scaled processes converge to deterministic fluid functions,*

$$\left(\bar{A}^n, \bar{\boldsymbol{X}}^n, \bar{\boldsymbol{Y}}^n, \bar{S}^n\right) \Rightarrow \left(\bar{a}, \bar{\boldsymbol{X}}, \bar{\boldsymbol{Y}}, \bar{S}\right) \tag{3.11}$$

in $\mathbb{D}^{2K+2}$ *as* $n \to \infty$, *where the limits are all deterministic continuous functions:* $\bar{a}$ *is the limit in* (2.2), *and for each* $t \geq 0$,

$$\bar{\boldsymbol{X}}(t) := (\bar{X}_1(t), ..., \bar{X}_K(t)), \quad \bar{X}_k(t) := \int_0^t F_k^c(t-s)d\bar{a}(s), \quad \text{for} \quad k = 1, ..., K, \tag{3.12}$$

$$\bar{\boldsymbol{Y}}(t) := (\bar{Y}_1(t), ..., \bar{Y}_K(t)), \quad \bar{Y}_k(t) := \int_0^t (F_{\mathrm{m}}^c(t-s) - F_k^c(t-s))d\bar{a}(s), \quad \text{for} \quad k = 1, ..., K, \tag{3.13}$$

$$\bar{S}(t) := \int_0^t F_{\mathrm{m}}(t-s)d\bar{a}(s). \tag{3.14}$$

We remark that the fluid limits are the same as in Theorem 3.2 of [24], implying that the fluid limits are neither affected by the random environment, nor by the sequential dependence of service vectors. It is worth noting that the fluid limits are affected by the correlation among the service times of the parallel tasks.

We define the diffusion-scaled processes $\boldsymbol{X}^n$, $\boldsymbol{Y}^n$ and S^n by

$$\hat{\boldsymbol{X}}^n := \sqrt{n}(\bar{\boldsymbol{X}}^n - \bar{\boldsymbol{X}}), \quad \hat{\boldsymbol{Y}}^n := \sqrt{n}(\bar{\boldsymbol{Y}}^n - \bar{\boldsymbol{Y}}), \quad \hat{S}^n := \sqrt{n}(\bar{S}^n - \bar{S}). \tag{3.15}$$

We will show the following FCLT. The proof is given in §5. Theorem 3.2 follows directly from this FCLT and thus its proof is omitted.

Theorem 3.3 (FCLT). *Under Assumptions* 1-3, *the diffusion-scaled processes converge in distribution,*

$$\left(\hat{A}^n, \hat{K}^n, \hat{\boldsymbol{X}}^n, \hat{\boldsymbol{Y}}^n, \hat{S}^n\right) \Rightarrow \left(\hat{A}, \hat{K}, \hat{\boldsymbol{X}}, \hat{\boldsymbol{Y}}, \hat{S}\right) \tag{3.16}$$

in $(\mathbb{D}, M_1) \times (\mathbb{D}_{K+1}, J_1) \times (\mathbb{D}^{2K+1}, M_1)$ *as* $n \to \infty$, *where* $\hat{A}$ *is the limit in* (2.1), $\hat{K}$ *is defined in* (3.3), *and for* $t \geq 0$ *and* $k = 1, ..., K$,

$$\hat{\boldsymbol{X}}(t) := \hat{\boldsymbol{M}}_1(t) + \hat{\boldsymbol{M}}_2(t) + \hat{\boldsymbol{M}}_3(t), \quad \hat{\boldsymbol{M}}_i(t) := (\hat{M}_{1,i}(t), ..., \hat{M}_{K,i}(t)), \; i = 1, 2, 3, \tag{3.17}$$

$$\hat{M}_{k,1}(t) := \int_0^t F_k^c(t-s)d\hat{A}(s) = \hat{A}(t) - \int_0^t \hat{A}(s)dF_k^c(t-s), \tag{3.18}$$

$$\hat{M}_{k,2}(t) := \int_0^t \int_{\mathbb{R}_+^K} \mathbf{1}(s + x_k > t)d\hat{K}(\bar{a}(s), \boldsymbol{x}) = -\int_0^t \int_{\mathbb{R}_+^K} \mathbf{1}(s + x_k \leq t)d\hat{K}(\bar{a}(s), \boldsymbol{x}), \tag{3.19}$$

$$\hat{M}_{k,3}(t) := \int_0^t (\hat{J}(t) - \hat{J}(s))\lambda(s)dF_k^c(t-s), \tag{3.20}$$

$$\hat{S}(t) := \hat{V}_1(t) + \hat{V}_2(t) + \hat{V}_3(t), \tag{3.21}$$

$$\hat{V}_1(t) := \int_0^t F_{\mathrm{m}}(t-s)d\hat{A}(s) = -\int_0^t \hat{A}(s)dF_{\mathrm{m}}(t-s), \tag{3.22}$$

$$\hat{V}_2(t) := \int_0^t \int_{\mathbb{R}_+^K} \mathbf{1}(s + x_j \leq t, \; \forall j)d\hat{K}(\bar{a}(s), \boldsymbol{x}), \tag{3.23}$$

$$\hat{V}_3(t) := \int_0^t (\hat{J}(t) - \hat{J}(s))\lambda(s)dF_{\mathrm{m}}(t-s), \tag{3.24}$$

$$\hat{\boldsymbol{Y}}(t) := \hat{\boldsymbol{Z}}_1(t) + \hat{\boldsymbol{Z}}_2(t) + \hat{\boldsymbol{Z}}_3(t), \quad \hat{\boldsymbol{Z}}_i(t) := (\hat{Z}_{1,i}(t), ..., \hat{Z}_{K,i}(t)), \; i = 1, 2, 3, \tag{3.25}$$

$$\hat{Z}_{k,1}(t) := \int_0^t (F_k(t-s) - F_{\mathrm{m}}(t-s))d\hat{A}(s) = \int_0^t \hat{A}(s)d(F_{\mathrm{m}}(t-s) - F_k(t-s)), \tag{3.26}$$

$$\hat{Z}_{k,2}(t) := \int_0^t \int_{\mathbb{R}_+^K} (\mathbf{1}(s + x_k \leq t) - \mathbf{1}(s + x_j \leq t, \; \forall j))d\hat{K}(\bar{a}(s), \boldsymbol{x}) \tag{3.27}$$

$$= -\hat{M}_{k,2}(t) - \hat{V}_2(t). \tag{3.28}$$

$$\hat{Z}_{k,3}(t) \;:=\; \int_0^t (\hat{J}(t)-\hat{J}(s))\lambda(s)d(F_k(t-s)-F_{\rm m}(t-s)) = -\hat{M}_{k,3}(t)-\hat{V}_3(t). \qquad (3.29)$$

Remark 3.1. It is worth noting that the convergences to the processes $\hat{M}_{k,1}$, $\hat{V}_1$ and $\hat{Z}_{k,1}$ in the Skorohod M_1 topology require the continuity of the distribution function $F(\cdot)$ (see Remark 5.1 for a counter example), for $k=1,...,K$. The proofs for the convergences to the processes $\hat{M}_{k,2}$, $\hat{V}_2$ and $\hat{Z}_{k,2}$, and $\hat{M}_{k,3}$, $\hat{V}_3$ and $\hat{Z}_{k,3}$ do not require the continuity of $F(\cdot)$, for $k=1,...,K$.

The limit processes $\hat{M}_{k,1}$, $\hat{Z}_{k,1}$, $k=1,...,K$, and $\hat{V}_1$ are well-defined as stochastic integrals in the sense of integration by parts (see, e.g., page 336 of [6]), that is, for each ω, they can be constructed pathwise via integration by parts. Their existence and continuity are proved by the continuous mapping theorem (Lemma 5.3). They are standard Itô integrals when $\hat{A}$ is a (time-changed) Brownian motion, or more generally, Itô integrals (with respect to semimartingales) if $\hat{A}$ is a semimartingale. The second equalities in (3.18), (3.26) and (3.22) follow from integration by parts. The processes $\hat{\boldsymbol{M}}_2$, $\hat{\boldsymbol{Z}}_2$ and $\hat{V}_2$ are defined in the mean-square sense; see the precise definitions in Definition 4.1. This is in the same way as the limit process with respect to a standard Kiefer process for the $G/GI/\infty$ queue defined in [21, 34]. The second equalities in (3.28) and (3.29) follow from simple algebra. When $\hat{N}$ is Poisson and the arrival rate is constant, we can write $\hat{V}_3$ as

$$\hat{V}_3(t) = -\lambda\int_0^t \hat{J}(s)dF_{\rm m}(s) = -\lambda F_{\rm m}(t)\hat{\mathcal{J}}([0,t]\times\mathbb{R}_+) + \lambda\int_0^t\int_0^\infty xF_{\rm m}(s)\hat{\mathcal{J}}(ds,dx),$$

where $\hat{\mathcal{J}}(s,x)$ is a Poisson random measure defined on $[0,\infty)\times\mathbb{R}_+$, with intensity $\lambda^u ds\times d\tilde{G}(x)$, where $\lambda^u = 1/E[u_i]\in(0,\infty)$ and $\tilde{G}(\cdot)$ is the distribution function of the limiting down times $\{d_k : k\geq 1\}$. Similarly for the processes $\hat{M}_{k,3}$ and $\hat{Z}_{k,3}$, $k=1,...,K$.

We remark that there is a *stochastic decomposition property* for $\hat{X}_k$, $\hat{Y}_k$, $k=1,...,K$, and $\hat{S}$. Each limit process is decomposed into three independent processes capturing variabilities from arrival, service and disruptions respectively. For example, for each $k=1,...,K$, in the representation of $\hat{Y}_k$ in (3.25), $\hat{Z}_{k,1}$, $\hat{Z}_{k,2}$ and $\hat{Z}_{k,3}$ capture the variabilities of arrival, service and disruption processes, respectively, and they are independent of each other. It is also worth noting that the correlation of parallel service times of each job is captured through $F_{\rm m}$ in the limiting processes. In addition, the disruption causes jumps simultaneously for all the processes $\hat{\boldsymbol{X}}$, $\hat{\boldsymbol{Y}}$ and $\hat{S}$, driven by the same process $\hat{J}$, while the jump sizes depend on the distributions of service times at different stations. For example, for each $k=1,...,K$, the jump sizes of $\hat{M}_{k,3}$, $\hat{V}_3$ and $\hat{Z}_{k,3}$ depend on F_k, $F_{\rm m}$ and $F_k - F_{\rm m}$, respectively. The processes $\hat{\boldsymbol{X}}$ and $\hat{\boldsymbol{Y}}$ have upward (positive) jumps and the process $\hat{S}$ has downward (negative) jumps, implying that the disruptions degrade the performance by increasing the congestions and delaying the synchronization processes, and thus, decreasing the throughput. The degradation is precisely characterized by the jump processes in the limit. These observations provide important insights on the impact of the disruptions upon the system performance, as we show next.

3.2. **Characterization of Limit Processes.** In this section, we will characterize the limit processes of $\big(\hat{\boldsymbol{X}}^n, \hat{\boldsymbol{Y}}^n, \hat{S}^n\big)$.

We first characterize the variability induced by the arrival limit process $\hat{A}$. We make a general assumption on the arrival process $\hat{A}$ by assuming it is a Lévy process with a deterministic time change, i.e., $\hat{A}(t) = L(\bar{a}(t))$ for a nondecreasing continuous deterministic function $\bar{a}$ in (2.1) and a Lévy process $L := \{L(t) : t\geq 0\}$ satisfying the square integrable condition $E([L(t))^2] < \infty$ for $t\geq 0$ with a Lévy-Khintchine triplet (b,c,Π) (see, e.g., [2]) and [22]), where $b\in\mathbb{R}$, $c\geq 0$ and Π is a positive measure concentrated on $\mathbb{R}\backslash\{0\}$ satisfying $\int_{\mathbb{R}}[1\wedge x^2]\Pi(dx)<\infty$. A Lévy-Khintchine triplet

(b, c, Π) uniquely determines the Lévy process L, and satisfies the following:

$$E\left[e^{\mathrm{i}\theta L(t)}\right] = \exp\left(\mathrm{i}b\theta t - \frac{1}{2}c^2\theta^2 t + t\int_{\mathbb{R}}[e^{\mathrm{i}\theta x} - 1 - i\theta x\mathbf{1}(|x|<1)]\Pi(dx)\right), \quad \theta\in\mathbb{R}, \quad t\geq 0.$$

Lévy processes with deterministic time change are well defined [22]. Since $\hat{A}(t)$ has mean zero for each $t \geq 0$, we set $b = 0$. In the following lemma, we compute the covariance functions of the limit processes $\big(\hat{\boldsymbol{M}}_1, \hat{\boldsymbol{Z}}_1, \hat{V}_1\big)$ as functionals of the arrival limit process in Theorem 3.3. Its proof is given in §4.

Lemma 3.1. *Under the assumptions of Theorem 3.3, if the arrival limit process $\hat{A}(t) = L(\bar{a}(t))$, where L is a square-integrable Lévy process with a Lévy-Khintchine triplet $(0, c, \Pi)$, the multi-dimensional process $\big(\hat{\boldsymbol{M}}_1, \hat{\boldsymbol{Z}}_1, \hat{V}_1\big)$ has mean $\mathbf{0}$, and the covariance functions as follows: for $j, k = 1, ..., K$ and each $t \geq 0$,*

$$Cov\big(\hat{M}_{j,1}(t), \hat{M}_{k,1}(t)\big) = \big(c^2+\nu^2\big)\int_0^t F_j^c(t-s)F_k^c(t-s)d\bar{a}(s), \tag{3.30}$$

$$Cov\big(\hat{M}_{j,1}(t), \hat{Z}_{k,1}(t)\big) = \big(c^2+\nu^2\big)\int_0^t [F_j^c(t-s)F_k(t-s) - F_j^c(t-s)F_{\mathrm{m}}(t-s)]d\bar{a}(s), \tag{3.31}$$

$$Cov\big(\hat{Z}_{j,1}(t), \hat{Z}_{k,1}(t)\big) = \big(c^2+\nu^2\big)\int_0^t \big[F_j(t-s)F_k(t-s) - F_j(t-s)F_{\mathrm{m}}(t-s) \tag{3.32}$$

$$-F_k(t-s)F_{\mathrm{m}}(t-s) + (F_{\mathrm{m}}(t-s))^2\big]\,d\bar{a}(s), \tag{3.33}$$

$$Cov\big(\hat{M}_{j,1}(t), \hat{V}_1(t)\big) = \big(c^2+\nu^2\big)\int_0^t F_j^c(t-s)F_{\mathrm{m}}(t-s)d\bar{a}(s), \tag{3.34}$$

$$Cov\big(\hat{Z}_{j,1}(t), \hat{V}_1(t)\big) = \big(c^2+\nu^2\big)\int_0^t [(F_j(t-s) - F_{\mathrm{m}}(t-s))F_{\mathrm{m}}(t-s)]d\bar{a}(s), \tag{3.35}$$

$$Var\big(\hat{V}_1(t)\big) = \big(c^2+\nu^2\big)\int_0^t (F_{\mathrm{m}}(t-s))^2 d\bar{a}(s), \tag{3.36}$$

where $\nu^2 := \int_{\mathbb{R}} x^2\Pi(dx) < \infty$ and $\nu^2 \geq 0$. In particular, when the arrival limit process $\hat{A}$ is a Brownian motion, i.e., $\hat{A}(t) = c_a B_a(\bar{a}(t))$ for a standard Brownian motion B_a and a positive constant $c_a > 0$, c^2 and ν^2 in (3.30)-(3.36) *are replaced by c_a^2 and 0, respectively.*

We next show the limit processes $\big(\hat{\boldsymbol{M}}_2, \hat{\boldsymbol{Z}}_2, \hat{V}_2\big)$ in (3.19), (3.28) and (3.23), represented as functionals of the multiparameter generalized Kiefer process, are well-defined continuous Gaussian processes. Its proof is given in §4.

Lemma 3.2. *Under the assumptions of Theorem 3.3, the multi-dimensional process $\big(\hat{\boldsymbol{M}}_2, \hat{\boldsymbol{Z}}_2, \hat{V}_2\big)$ in* (3.19), (3.28) *and* (3.23) *is a well-defined Gaussian process with mean $\mathbf{0}$, and covariance functions: for each $t \geq 0$ and $j, k = 1, ..., K$,*

$$Cov\big(\hat{M}_{j,2}(t), \hat{M}_{k,2}(t)\big) = \int_0^t [F_{j,k}(t-s,t-s) - F_j(t-s)F_k(t-s) + \Gamma_{j,k}^c(t-s)]d\bar{a}(s), \tag{3.37}$$

$$\begin{aligned} Cov\big(\hat{Z}_{j,2}(t), \hat{Z}_{k,2}(t)\big) = \int_0^t &[F_{j,k}(t-s,t-s) - F_j(t-s)F_k(t-s) - F_{\mathrm{m}}(t-s) \\ &+ F_k(t-s)F_{\mathrm{m}}(t-s) + F_{\mathrm{m}}(t-s)F_j(t-s) - (F_{\mathrm{m}}(t-s))^2 + \Gamma_{j,k}^c(t-s,t-s) \\ &-\Gamma_{k,\mathrm{m}}^c(t-s,t-s) - \Gamma_{j,\mathrm{m}}^c(t-s,t-s) + \Gamma_{\mathrm{m,m}}^c(t-s,t-s)\big]\,d\bar{a}(s), \end{aligned} \tag{3.38}$$

$$\begin{aligned} Cov\big(\hat{M}_{j,2}(t), \hat{Z}_{k,2}(t)\big) = \int_0^t &[F_{\mathrm{m}}(t-s) - F_{j,k}(t-s,t-s) + F_j(t-s)F_k(t-s) - F_j(t-s)F_{\mathrm{m}}(t-s) \\ &-\Gamma_{j,k}^c(t-s,t-s) + \Gamma_{j,\mathrm{m}}^c(t-s,t-s)\big]\,d\bar{a}(s), \end{aligned} \tag{3.39}$$

$$Cov\big(\hat{M}_{k,2}(t), \hat{V}_2(t)\big) = -\int_0^t \left[F_k^c(t-s)F_{\mathrm{m}}(t-s) + \Gamma^c_{k,\mathrm{m}}(t-s,t-s)\right] d\bar{a}(s), \tag{3.40}$$

$$Cov\big(\hat{Z}_{k,2}(t), \hat{V}_2(t)\big) = \int_0^t [F_{\mathrm{m}}(t-s)(F_{\mathrm{m}}(t-s) - F_k(t-s)) + \Gamma^c_{k,\mathrm{m}}(t-s,t-s) - \Gamma^c_{\mathrm{m},\mathrm{m}}(t-s,t-s)\big] d\bar{a}(s), \tag{3.41}$$

$$Var\big(\hat{V}_2(t)\big) = \int_0^t \left[F_{\mathrm{m}}(t-s)F^c_{\mathrm{m}}(t-s) + \Gamma^c_{\mathrm{m},\mathrm{m}}(t-s,t-s)\right] d\bar{a}(s), \tag{3.42}$$

where $\Gamma^c_{k,\mathrm{m}}(x,y) := \Gamma^c(x\boldsymbol{e}_k, y\boldsymbol{e})$, $\Gamma^c_{j,k}(x,y) := \Gamma^c(x\boldsymbol{e}_j, y\boldsymbol{e}_k)$ *and* $\Gamma^c_{\mathrm{m},\mathrm{m}}(x,y) := \Gamma^c(x\boldsymbol{e}, y\boldsymbol{e})$, *for* $x, y \geq 0$ *and* $j, k = 1, ..., K$.

Note that in the covariance functions (3.37)–(3.42), the sequential dependence is captured in the terms Γ^c, which all become zero when the service vectors are i.i.d.

To understand the impact of the both "componentwise" and "vectorwise" (sequential) dependence for service vectors, we consider the following example of the service vectors with a special structure. From the componentwise perspective, we let $\boldsymbol{\eta}^i$, $i \geq 1$, have the joint continuous distribution function

$$F(\boldsymbol{x}) = (1-\rho)\prod_{k=1}^K G(x_k) + \rho G\left(\min_{k=1,\dots,K}\{x_k\}\right) \tag{3.43}$$

with a marginal continuous distribution function $G(\cdot)$, for $0 \leq \rho < 1$, $x_k \geq 0$ and $k = 1, ..., K$. Namely, the service times at the parallel stations have the same distribution, and are symmetrically correlated with a correlation parameter $\rho \in [0, 1)$, i.e., the correlation of service times at any two parallel stations is ρ. From the vectorwise perspective, we use the first-order discrete vector autoregressive process, "DVAR(1)". Specifically, we let $\boldsymbol{\eta}^1$ be distributed according to F in (3.43), and generate $\{\boldsymbol{\eta}^i : i \geq 2\}$ by

$$\boldsymbol{\eta}^i = \zeta_{i-1}\boldsymbol{\eta}^{i-1} + (1-\zeta_{i-1})\tilde{\boldsymbol{\eta}}^i, \quad i \geq 2, \tag{3.44}$$

where $\{\zeta_i : i \geq 1\}$ is a sequence of i.i.d. Bernoulli random variables with $P(\zeta_1 = 1) = 1 - P(\zeta_1 = 0) = p$, and $\{\tilde{\boldsymbol{\eta}}^i : i \geq 2\}$ is a sequence of i.i.d random vectors, each with joint distribution function F in (3.43). In this special case, by direct calculation, the correlation between η^i_j and η^{i+l}_k is $p^l\rho$, for each $i, l \in \mathbb{N}$ and $j, k = 1, ..., K$ with $j \neq k$, while the correlation between η^i_k and η^{i+l}_k is p^l. The covariance functions of $\big(\hat{\boldsymbol{M}}_2, \hat{\boldsymbol{Z}}_2, \hat{V}_2\big)$ are stated in Corollary 3.1. Its proof is given in §4.

Corollary 3.1. *Under the assumptions of Theorem 3.3, when the sequence of service vectors* $\{\boldsymbol{\eta}^i : i \in \mathbb{N}\}$ *has the structure in* (3.44) *and each service vector has the joint distribution function shown in* (3.43), *we have, for each* $t \geq 0$ *and* $k = 1, ..., K$,

$$Var\big(\hat{M}_{k,2}(t)\big) = (2\Sigma_p + 1)\int_0^t G(t-s)G^c(t-s)d\bar{a}(s),$$

$$Var\big(\hat{Z}_{k,2}(t)\big) = (2\Sigma_p + 1)\int_0^t \Big[(1-\rho)G(t-s)\left(1 - (G(t-s))^{K-1}\right) - (1-\rho)^2(G(t-s))^2\left(1 - (G(t-s))^{K-1}\right)^2\Big] d\bar{a}(s),$$

$$Var\big(\hat{V}_2(t)\big) = (2\Sigma_p + 1)\int_0^t \left[\left(\rho G(t-s) + (1-\rho)(G(t-s))^K\right) \times \left(1 - \rho G(t-s) - (1-\rho)(G(t-s))^K\right)\right] d\bar{a}(s),$$

$$Cov\big(\hat{M}_{k,2}(t), \hat{Z}_{k,2}(t)\big) = -(2\Sigma_p + 1)(1-\rho)\int_0^t \left[G^c(t-s)G(t-s)(1 - (G(t-s))^{K-1})\right] d\bar{a}(s),$$

and for $j,k=1,...,K$ *and* $j\neq k$,

$$Cov\big(\hat{M}_{j,2}(t),\hat{M}_{k,2}(t)\big) = (2\Sigma_p+1)\rho\int_0^t G(t-s)G^c(t-s)d\bar{a}(s), \tag{3.45}$$

$$\begin{aligned}Cov\big(\hat{Z}_{j,2}(t),\hat{Z}_{k,2}(t)\big) = (2\Sigma_p+1)\int_0^t \Big[&(1-\rho)(G(t-s))^2(1-(G(t-s))^{K-2})\\ &-(1-\rho)^2(G(t-s))^2\left(1-(G(t-s))^{K-1}\right)^2\Big]d\bar{a}(s),\end{aligned} \tag{3.46}$$

$$\begin{aligned}Cov\big(\hat{M}_{j,2}(t),\hat{Z}_{k,2}(t)\big) = (2\Sigma_p+1)(1-\rho)\int_0^t \Big[&G(t-s)G^c(t-s)\\ &-G^c(t-s)G(t-s)\left(1-(G(t-s))^{K-1}\right)\Big]d\bar{a}(s),\end{aligned} \tag{3.47}$$

$$Cov\big(\hat{M}_{k,2}(t),\hat{V}_2(t)\big) = -(2\Sigma_p+1)\int_0^t\left[G^c(t-s)\left(\rho G(t-s)+(1-\rho)G^K(t-s)\right)\right]d\bar{a}(s), \tag{3.48}$$

$$\begin{aligned}Cov\big(\hat{Z}_{k,2}(t),\hat{V}_2(t)\big) = (2\Sigma_p+1)(1-\rho)\int_0^t \Big[&G(t-s)\left((G(t-s))^{K-1}-1\right)\\ &\times\left(\rho G(t-s)+(1-\rho)(G(t-s))^K\right)\Big]d\bar{a}(s),\end{aligned} \tag{3.49}$$

where $\Sigma_p := \sum_{i=1}^\infty p^i = p/(1-p)<\infty$.

We remark that there exists a *separation* of the componentwise correlation and vectorwise (sequential) dependence for service vectors in the above special case. Specifically, Σ_p captures all the sequential dependence from the service vectors, while the correlation of the service times of parallel tasks is captured in the correlation coefficient ρ.

Next we characterize the multi-dimensional limit process with jumps $(\hat{\boldsymbol{M}}_3,\hat{\boldsymbol{Z}}_3,\hat{V}_3)$ due to the random environment, when the limit counting process $\hat{N}$ is Poisson. Note that when $\hat{N}$ is Poisson and the arrival rate λ is constant, the processes with jumps $(\hat{\boldsymbol{M}}_3,\hat{\boldsymbol{Z}}_3,\hat{V}_3)$ become, for $t\geq 0$,

$$\hat{M}_{k,3}(t) = -\lambda\int_0^t \hat{J}(s)dF_k^c(s),\ \hat{V}_3(t) = -\lambda\int_0^t\hat{J}(s)dF_{\rm m}(s),\ \hat{Z}_{k,3}(t)=\lambda\int_0^t\hat{J}(s)d(F_{\rm m}(s)-F_k(s)),$$

which are all functionals of the compound Poisson process $\hat{J}(t)$. The proof of the following lemma is provided in §4.

Lemma 3.3. *Under the assumptions of Theorem 3.3, when the limit counting process* $\hat{N}$ *is Poisson with rate* $\lambda^u = 1/E[u_1]<\infty$ *and* $E[d_1^2]<\infty$, *the multi-dimensional process with jumps* $\big(\hat{\boldsymbol{M}}_3,\hat{\boldsymbol{Z}}_3,\hat{V}_3\big)$ *has mean functions: for* $k=1,...,K$ *and* $t\geq0$,

$$\begin{aligned}&E[\hat{M}_{k,3}(t)] = \lambda^u E[d_1]\int_0^t[(t-s)\lambda(s)]dF_k^c(t-s),\ E[\hat{V}_3(t)] = \lambda^uE[d_1]\int_0^t[(t-s)\lambda(s)]dF_{\rm m}(t-s),\\ &E[\hat{Z}_{k,3}(t)] = \lambda^uE[d_1]\int_0^t[(t-s)\lambda(s)]d(F_k(t-s)-F_{\rm m}(t-s)),\end{aligned} \tag{3.50}$$

and covariance functions: for $j,k=1,...,K$ *and each* $t\geq0$,

$$\begin{aligned}Cov\big(\hat{M}_{j,3}(t),\hat{M}_{k,3}(t)\big) &= c_d^2\int_0^t\int_0^t[(t-s_1\vee s_2)\lambda(s_1)\lambda(s_2)]\,dF_j^c(t-s_1)dF_k^c(t-s_2),\\ Cov\big(\hat{M}_{j,3}(t),\hat{Z}_{k,3}(t)\big) &= c_d^2\int_0^t\int_0^t[(t-s_1\vee s_2)\lambda(s_1)\lambda(s_2)]\,dF_j^c(t-s_1)dF_k(t-s_2)\\ &\quad+c_d^2\int_0^t\int_0^t[(t-s_1\vee s_2)\lambda(s_1)\lambda(s_2)]\,dF_j^c(t-s_1)dF_{\rm m}^c(t-s_2),\end{aligned}$$

$$
\begin{aligned}
Cov\big(\hat{Z}_{j,3}(t), \hat{Z}_{k,3}(t)\big) &= c_d^2 \int_0^t \int_0^t [(t - s_1 \vee s_2)\lambda(s_1)\lambda(s_2)]\, dF_j(t-s_1) dF_k(t-s_2) \\
&\quad + c_d^2 \int_0^t \int_0^t [(t - s_1 \vee s_2)\lambda(s_1)\lambda(s_2)]\, dF_j^c(t-s_1) dF_{\mathrm{m}}(t-s_2) \\
&\quad + c_d^2 \int_0^t \int_0^t [(t - s_1 \vee s_2)\lambda(s_1)\lambda(s_2)]\, dF_k^c(t-s_1) dF_{\mathrm{m}}(t-s_2) \\
&\quad + c_d^2 \int_0^t \int_0^t [(t - s_1 \vee s_2)\lambda(s_1)\lambda(s_2)]\, dF_{\mathrm{m}}(t-s_1) dF_{\mathrm{m}}(t-s_2), \\
Cov\big(\hat{M}_{j,3}(t), \hat{V}_3(t)\big) &= c_d^2 \int_0^t \int_0^t [(t - s_1 \vee s_2)\lambda(s_1)\lambda(s_2)]\, dF_j^c(t-s_1) dF_{\mathrm{m}}(t-s_2), \\
Cov\big(\hat{Z}_{j,3}(t), \hat{V}_3(t)\big) &= c_d^2 \int_0^t \int_0^t [(t - s_1 \vee s_2)\lambda(s_1)\lambda(s_2)]\, dF_j(t-s_1) dF_{\mathrm{m}}(t-s_2) \\
&\quad + c_d^2 \int_0^t \int_0^t [(t - s_1 \vee s_2)\lambda(s_1)\lambda(s_2)]\, dF_{\mathrm{m}}(t-s_1) dF_{\mathrm{m}}^c(t-s_2), \\
Var\big(\hat{V}_3(t)\big) &= c_d^2 \int_0^t \int_0^t [(t - s_1 \vee s_2)\lambda(s_1)\lambda(s_2)]\, dF_{\mathrm{m}}(t-s_1) dF_{\mathrm{m}}(t-s_2),
\end{aligned}
$$

where $c_d^2 := \lambda^u E[d_1^2]$.

When $\bar{a}(t) = \lambda t$ for $t \geq 0$, where λ is a positive constant, the covariance functions in Lemma 3.3 can be simplified, for example, for $j, k = 1, ..., K$ and each $t \geq 0$,

$$
\begin{aligned}
Cov\big(\hat{Z}_{j,3}(t), \hat{Z}_{k,3}(t)\big) = \lambda^2 c_d^2 \bigg(&\int_0^t \int_0^t [s_1 \wedge s_2]\, dF_j(s_1) dF_k(s_2) + \int_0^t \int_0^t [s_1 \wedge s_2]\, dF_j^c(s_1) dF_{\mathrm{m}}(s_2) \\
&+ \int_0^t \int_0^t [s_1 \wedge s_2]\, dF_k^c(s_1) dF_{\mathrm{m}}(s_2) + \int_0^t \int_0^t [s_1 \wedge s_2]\, dF_{\mathrm{m}}(s_1) dF_{\mathrm{m}}(s_2) \bigg).
\end{aligned}
$$

When the limiting counting process $\hat{N}$ is Poisson, we can characterize the limiting processes $(\hat{\boldsymbol{X}}, \hat{\boldsymbol{Y}})$ and $\hat{S}$ in Theorem 3.3 as in the following proposition. Its proof directly follows from Lemmas 3.1-3.3 and the stochastic decomposition property.

Proposition 3.1. *Under the assumptions of Theorem 3.3, when the limit counting process* $\hat{N}$ *is Poisson with rate* $\lambda^u = 1/E[u_1] < \infty$ *and* $E[d_1^2] < \infty$*, the limiting processes* $(\hat{\boldsymbol{X}}, \hat{\boldsymbol{Y}})$ *and* $\hat{S}$ *in Theorem* 3.3 *have means* $E[\hat{X}_k(t)] = E[\hat{M}_{k,3}(t)]$, $E[\hat{Y}_k(t)] = E[\hat{Z}_{k,3}(t)]$ *and* $E[\hat{S}(t)] = E[\hat{V}_3(t)]$ *for* $k = 1, ..., K$ *and* $t \geq 0$*, where* $E[\hat{M}_{k,3}(t)]$, $E[\hat{Z}_{k,3}(t)]$ *and* $E[\hat{V}_3(t)]$ *are given in* (3.50)*, and covariance functions are shown as follows: for each* $t \geq 0$ *and* $j, k = 1, ..., K$,

$$
Cov\big(\hat{X}_j(t), \hat{X}_k(t)\big) = \sum_{i=1}^{3} Cov\big(\hat{M}_{j,i}(t), \hat{M}_{k,i}(t)\big), \quad Cov\big(\hat{Y}_j(t), \hat{Y}_k(t)\big) = \sum_{i=1}^{3} Cov\big(\hat{Z}_{j,i}(t), \hat{Z}_{k,i}(t)\big), \tag{3.51}
$$

$$
Cov\big(\hat{X}_j(t), \hat{Y}_k(t)\big) = \sum_{i=1}^{3} Cov\big(\hat{M}_{j,i}(t), \hat{Z}_{k,i}(t)\big), \quad Var\big(\hat{S}(t)\big) = \sum_{i=1}^{3} Var\big(\hat{V}_i(t)\big), \tag{3.52}
$$

where $Cov\big(\hat{M}_{j,i}(t), \hat{M}_{k,i}(t)\big)$, $Cov\big(\hat{Z}_{j,i}(t), \hat{Z}_{k,i}(t)\big)$, $Cov\big(\hat{M}_{j,i}(t), \hat{Z}_{k,i}(t)\big)$ *and* $Var\big(\hat{V}_i(t)\big)$ *are given in Lemmas 3.1-3.3, for each* $t \geq 0$, $i = 1, 2, 3$, *and* $j, k = 1, ..., K$.

4. Proofs for the Characterization of the Limit Processes

In this section, we provide the proofs for Lemmas 3.1-3.3. We first focus on proving Lemma 3.1.

Proof of Lemma 3.1. Since $\hat{A}$ is a time-changed Lévy process with mean 0 satisfying the square-integrable condition, $\hat{A}$ is an L^2-martingale, implying that $\hat{M}_{k,1}$, $\hat{Z}_{k,1}$ and $\hat{V}_1$ have mean zero, for $k = 1, ..., K$. Next, we will only provide the detailed proof for the covariance between $\hat{M}_{j,1}(t)$ and $\hat{Z}_{k,1}(t)$ in (3.31), for each $t \geq 0$ and $j, k = 1, ..., K$, as the other terms can be obtained analogously. By the definition of $\hat{A}$, we have, for $t \geq 0$ and $\theta \in \mathbb{R}$,

$$
\begin{aligned}
E\left[e^{\mathrm{i}\theta \hat{A}(t)}\right] &= E\left[e^{\mathrm{i}\theta L(\bar{a}(t))}\right] \\
&= \exp\left(-\frac{1}{2}c^2\theta^2\bar{a}(t) + \bar{a}(t)\int_{\mathbb{R}}[e^{\mathrm{i}\theta x} - 1 - i\theta x\mathbf{1}(|x|<1)]\Pi(dx)\right). \qquad (4.1)
\end{aligned}
$$

(See, e.g., §1.8 in [1].) By Proposition II.2.17 in [15], we obtain the quadratic variation of $\hat{A}$ as

$$
\langle \hat{A}\rangle(t) = c^2\bar{a}(t) + \bar{a}(t)\int_{\mathbb{R}} x^2\Pi(dx), \quad t \geq 0. \qquad (4.2)
$$

By Theorem I.4.40 in [15], we have

$$
\begin{aligned}
Cov\big(\hat{M}_{j,1}(t), \hat{Z}_{k,1}(t)\big) &= E\left[\hat{M}_{j,1}(t)\hat{Z}_{k,1}(t)\right] \\
&= E\left[\int_0^t [F_j^c(t-s)F_k(t-s) - F_j^c(t-s)F_{\mathrm{m}}(t-s)]d\langle \hat{A}\rangle(s)\right]. \qquad (4.3)
\end{aligned}
$$

Combining (4.2) and (4.3), we can easily see (3.31) holds. This completes the proof. □

We next prove Lemma 3.2. We first provide the definitions of the processes $\hat{\boldsymbol{M}}_2$ in (3.19), $\hat{\boldsymbol{Z}}_2$ in (3.28) and $\hat{V}_2$ in (3.23). Before stating the definitions, we let P_i be the joint distribution of the service vectors $\boldsymbol{\eta}^1$ and $\boldsymbol{\eta}^i$, i.e., $P_i(\boldsymbol{x}, \boldsymbol{y}) := P(\eta^1 \leq \boldsymbol{x}, \eta^i \leq \boldsymbol{y})$, $i \in \mathbb{N}$, and define $P_I(\boldsymbol{x}, \boldsymbol{y}) := F(\boldsymbol{x})F(\boldsymbol{y})$, for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}_+^K$. We next introduce the following notation, which was used in [24]. For a set $\mathcal{J}$, let $|\mathcal{J}|$ be the cardinality of $\mathcal{J}$. Let $\mathcal{J}_k^1$ and $\mathcal{J}_{N-k}^2$ be the partition of $\mathcal{A} := \{1, ..., N\}$, where N is a positive integer, $\mathcal{J}_k^1 \cap \mathcal{J}_{N-k}^2 = \emptyset$, $|\mathcal{J}_k^1| = k$ and $|\mathcal{J}_{N-k}^2| = N - k$. Note that $\mathcal{J}_0^1 = \mathcal{J}_0^2 = \emptyset$. Let $\Phi : \mathbb{R}^N \to \mathbb{R}$. For $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^N$ and $\boldsymbol{x} \leq \boldsymbol{y}$, define $\Phi^{\mathcal{J}_k^1, \mathcal{J}_{N-k}^2}(\boldsymbol{x}; \boldsymbol{y}) := \Phi(\boldsymbol{z})$, where $z_j = x_j$ for $j \in \mathcal{J}_k^1$ and $z_j = y_j$ for $j \in \mathcal{J}_{N-k}^2$. Then, we define

$$
\Delta\Phi(\boldsymbol{x}; \boldsymbol{y}) := \sum_{k=0}^{N}(-1)^k \sum_{\mathcal{J}_k^1, \mathcal{J}_{N-k}^2 \text{ partitions of } \mathcal{A}} \Phi^{\mathcal{J}_k^1, \mathcal{J}_{N-k}^2}(\boldsymbol{x}; \boldsymbol{y}). \qquad (4.4)
$$

This notion "Δ" can be interpreted as the following: for a real-valued function Φ defined on $\mathbb{R}^N$, $\Delta\Phi(\boldsymbol{x}; \boldsymbol{y})$ represents the *increment* of Φ between $\boldsymbol{x}$ and $\boldsymbol{y}$ for each $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^N$ satisfying $\boldsymbol{x} \leq \boldsymbol{y}$. For $N = 1$, $\Delta\Phi(x; y) = \Phi(y) - \Phi(x)$ for $x \leq y$. For $N = 2$, $\Delta\Phi(\boldsymbol{x}; \boldsymbol{y}) = \Phi(y_1, y_2) - \Phi(x_1, y_2) - \Phi(x_2, y_1) + \Phi(x_1, x_2)$ for $\boldsymbol{x} = (x_1, x_2) \leq \boldsymbol{y} = (y_1, y_2)$. In the following proofs, we will use $\Delta\hat{K}(\boldsymbol{x}; \boldsymbol{y})$ and $\Delta F(\boldsymbol{x}; \boldsymbol{y})$ for $N = K + 1$ and $N = K$, respectively, and use $\Delta\Gamma^c(\boldsymbol{x}; \boldsymbol{y})$, $\Delta P_i(\boldsymbol{x}; \boldsymbol{y})$ and $\Delta P_I(\boldsymbol{x}; \boldsymbol{y})$ for $N = 2K$.

Definition 4.1. *For $k = 1, ..., K$, the processes $\hat{M}_{k,2}$ in* (3.19)*, $\hat{V}_2$ in* (3.23) *and $\hat{Z}_{k,2}$ in* (3.28) *are defined as mean-square integrals, i.e., for each $t \geq 0$,*

$$
\begin{aligned}
\lim_{\ell\to\infty} E[(\hat{M}_{k,2}(t) - \hat{M}_{k,2,\ell}(t))^2] &= 0, \qquad (4.5) \\
\lim_{\ell\to\infty} E[(\hat{V}_2(t) - \hat{V}_{2,\ell}(t))^2] &= 0, \qquad (4.6) \\
\lim_{\ell\to\infty} E[(\hat{Z}_{k,2}(t) - \hat{Z}_{k,2,\ell}(t))^2] &= 0, \qquad (4.7)
\end{aligned}
$$

where

$$\begin{aligned}\hat{M}_{k,2,\ell}(t) &:= -\int_0^t \int_{\mathbb{R}_+^K} \mathbf{1}_{k,\ell,t}(s,\boldsymbol{x}) d\hat{K}(\bar{a}(s),\boldsymbol{x}),\\ &= -\sum_{i=1}^{\ell} \left(\hat{K}(\bar{a}(s_i^\ell),(t-s_i^\ell)\boldsymbol{e}_k) - \hat{K}(\bar{a}(s_{i-1}^\ell),(t-s_i^\ell)\boldsymbol{e}_k)\right)\\ &= -\sum_{i=1}^{\ell} \Delta\hat{K}\left((\bar{a}(s_{i-1}^\ell),\mathbf{0});(\bar{a}(s_i^\ell),(t-s_i^\ell)\boldsymbol{e}_k)\right), \qquad (4.8)\\ \hat{V}_{2,\ell}(t) &:= \int_0^t \int_{\mathbb{R}_+^K} \mathbf{1}_{\mathrm{m},\ell,t}(s,\boldsymbol{x}) d\hat{K}(\bar{a}(s),\boldsymbol{x})\\ &= \sum_{i=1}^{\ell} \left(\hat{K}(\bar{a}(s_i^\ell),(t-s_i^\ell)\boldsymbol{e}) - \hat{K}(\bar{a}(s_{i-1}^\ell),(t-s_i^\ell)\boldsymbol{e})\right)\\ &= \sum_{i=1}^{\ell} \Delta\hat{K}\left((\bar{a}(s_{i-1}^\ell),\mathbf{0});(\bar{a}(s_i^\ell),(t-s_i^\ell)\boldsymbol{e})\right), \qquad (4.9)\\ \hat{Z}_{k,2,\ell}(t) &:= -\hat{M}_{k,2,\ell}(t) - \hat{V}_{2,\ell}(t), \qquad (4.10)\end{aligned}$$

and

$$\mathbf{1}_{k,\ell,t}(s,\boldsymbol{x}) := \sum_{i=1}^{\ell} \mathbf{1}(s_{i-1}^\ell < s \le s_i^\ell)\mathbf{1}(x_k \le t - s_i^\ell), \qquad (4.11)$$

$$\mathbf{1}_{\mathrm{m},\ell,t}(s,\boldsymbol{x}) := \sum_{i=1}^{\ell} \mathbf{1}(s_{i-1}^\ell < s \le s_i^\ell)\mathbf{1}(x_j \le t - s_i^\ell,\ \forall j = 1,...,K), \qquad (4.12)$$

with $0 = s_0^\ell < s_1^\ell < ... < s_\ell^\ell = t$ *and* $\max_{1\le i\le \ell} |s_i^\ell - s_{i-1}^\ell| \to 0$ *as* $\ell \to \infty$. *We call* $\{s_i^\ell : 0 \le i \le \ell\}$ *a partition of* $[0,t]$.

Before we show the well-definedness and covariance structure for the multi-dimensional process $(\hat{\boldsymbol{M}}_2, \hat{\boldsymbol{Z}}_2, \hat{V}_2)$ in Lemma 3.2, we also need the following lemmas on the properties of $\Gamma^c(\cdot,\cdot)$ defined in (3.5).

Lemma 4.1. *There exists a finite positive number* C_0 *such that*

$$|\Gamma^c(\boldsymbol{x},\boldsymbol{y})| < C_0,$$

for any $\boldsymbol{x},\boldsymbol{y} \in \mathbb{R}_+^K$, *where* $\Gamma^c(\cdot,\cdot)$ *is defined in* (3.5).

Proof. By the definition of $\Gamma^c(\cdot,\cdot)$ in (3.5), for any $\boldsymbol{x},\boldsymbol{y} \in \mathbb{R}_+^K$,

$$\begin{aligned}|\Gamma^c(\boldsymbol{x},\boldsymbol{y})| &= \left|\sum_{i=2}^{\infty} \left(E[\gamma_1(\boldsymbol{x})\gamma_i(\boldsymbol{y})] + E[\gamma_1(\boldsymbol{y})\gamma_i(\boldsymbol{x})]\right)\right|\\ &\le \sum_{i=2}^{\infty} \left(\left|E[\gamma_1(\boldsymbol{x})\gamma_i(\boldsymbol{y})]\right| + \left|E[\gamma_1(\boldsymbol{y})\gamma_i(\boldsymbol{x})]\right|\right)\\ &= \sum_{i=2}^{\infty} \left(\left|E[\mathbf{1}(\boldsymbol{\eta}^1 \le \boldsymbol{x})\mathbf{1}(\boldsymbol{\eta}^i \le \boldsymbol{y})] - F(\boldsymbol{x})F(\boldsymbol{y})\right| + \left|E[\mathbf{1}(\boldsymbol{\eta}^1 \le \boldsymbol{y})\mathbf{1}(\boldsymbol{\eta}^i \le \boldsymbol{x})] - F(\boldsymbol{x})F(\boldsymbol{y})\right|\right)\\ &\le 2\sum_{i=1}^{\infty} \alpha_i,\end{aligned}$$

where the last inequality follows from the definition of α-mixing coefficients. By Assumption 1, we see that $\sum_{i=1}^{\infty}\alpha_i < \infty$. Thus, Lemma 4.2 holds by taking $C_0 = 2\sum_{i=1}^{\infty}\alpha_i$. $\square$

Lemma 4.2. *For any $\boldsymbol{x},\boldsymbol{y}\in\mathbb{R}_+^K$ with $\boldsymbol{x}\le\boldsymbol{y}$, we have*

$$\Delta\Gamma^c((\boldsymbol{x},\boldsymbol{x});(\boldsymbol{y},\boldsymbol{y})) = \sum_{i=2}^{\infty}\left(2\Delta P_i((\boldsymbol{x},\boldsymbol{x});(\boldsymbol{y},\boldsymbol{y})) - 2\Delta P_I((\boldsymbol{x},\boldsymbol{x});(\boldsymbol{y},\boldsymbol{y}))\right).$$

Proof. Note that, for any $\boldsymbol{x},\boldsymbol{y}\in\mathbb{R}_+^K$,

$$\begin{aligned}\Gamma^c(\boldsymbol{x},\boldsymbol{y}) &= \sum_{i=2}^{\infty}\big(E[\gamma_1(\boldsymbol{x})\gamma_i(\boldsymbol{y})] + E[\gamma_1(\boldsymbol{y})\gamma_i(\boldsymbol{x})]\big)\\ &= \sum_{i=2}^{\infty}\big(E[\mathbf{1}(\boldsymbol{\eta}^1\le\boldsymbol{x})\mathbf{1}(\boldsymbol{\eta}^i\le\boldsymbol{y})] - F(\boldsymbol{x})F(\boldsymbol{y}) + E[\mathbf{1}(\boldsymbol{\eta}^1\le\boldsymbol{y})\mathbf{1}(\boldsymbol{\eta}^i\le\boldsymbol{x})] - F(\boldsymbol{x})F(\boldsymbol{y})\big)\\ &= \sum_{i=2}^{\infty}\big(P_i(\boldsymbol{x},\boldsymbol{y}) + P_i(\boldsymbol{y},\boldsymbol{x}) - 2P_I(\boldsymbol{x},\boldsymbol{y})\big).\end{aligned}$$

By (4.4), Lemma 4.1 and Fubini's Theorem, we see that Lemma 4.2 holds. $\square$

Proof of Lemma 3.2. To show the well-definedness of the multidimensional process $(\hat{\boldsymbol{M}}_2,\hat{\boldsymbol{Z}}_2,\hat{V}_2)$, by (3.19), (3.23) and (3.28), it suffices to prove that $\hat{M}_{k,2}$ and $\hat{V}_2$ are well-defined, $k=1,...,K$. To achieve that, by Definition 4.1, it is sufficient to prove, for each $t\ge 0$,

$$\lim_{l,\ell\to\infty} E\big[(\hat{M}_{k,2,l}(t) - \hat{M}_{k,2,\ell}(t))^2\big] = 0, \tag{4.13}$$

$$\lim_{l,\ell\to\infty} E\big[(\hat{V}_{2,l}(t) - \hat{V}_{2,\ell}(t))^2\big] = 0, \tag{4.14}$$

where we define $\hat{M}_{k,2,l}(t)$, $\hat{V}_{2,l}(t)$ and their associated partition $\{s_i^l : 0\le i\le l\}$ of $[0,t]$ similarly as $\hat{M}_{k,2,\ell}(t)$ in (4.8), $\hat{V}_{2,\ell}(t)$ in (4.9) and the partition $\{s_i^\ell : 0\le i\le \ell\}$ of $[0,t]$ in Definition 4.1 for each $t\ge 0$, respectively.

We provide the detailed proof of (4.14) here, and the proof of (4.13) follows similarly. Without loss of generality, we assume that the partition $\{s_i^\ell : 0\le i\le\ell\}$ of $[0,t]$ is finer than the partition $\{s_i^l : 0\le i\le l\}$ of $[0,t]$. Thus, by definition, we can write

$$\hat{V}_{2,l}(t) - \hat{V}_{2,\ell}(t) = \sum_{i=1}^{l}\sum_{j:s_{i-1}^l < s_j^\ell \le s_i^l}\Delta\hat{K}\left((\bar{a}(s_{j-1}^\ell),(t-s_i^l)\boldsymbol{e});(\bar{a}(s_j^\ell),(t-s_j^\ell)\boldsymbol{e})\right),\quad t\ge 0. \tag{4.15}$$

By direct calculations using (3.4), we can show that for $0\le t_1\le t_2 < t_1'\le t_2'$, $0\le x\le y$ and $0\le x'\le y'$,

$$\begin{aligned}E\left[\Big(\Delta\hat{K}\left((\bar{a}(t_1),x\boldsymbol{e});(\bar{a}(t_2),y\boldsymbol{e})\right)\Big)^2\right] &= (\bar{a}(t_2)-\bar{a}(t_1))\big((\Delta F(x\boldsymbol{e};y\boldsymbol{e}))(1-\Delta F(x\boldsymbol{e};y\boldsymbol{e}))\\ &\quad + \Delta\Gamma^c((x\boldsymbol{e},x\boldsymbol{e});(y\boldsymbol{e},y\boldsymbol{e}))\big),\end{aligned} \tag{4.16}$$

$$E\Big[\Delta\hat{K}((\bar{a}(t_1),x\boldsymbol{e});(\bar{a}(t_2),y\boldsymbol{e}))\Delta\hat{K}((\bar{a}(t_1'),x'\boldsymbol{e});(\bar{a}(t_2'),y'\boldsymbol{e}))\Big] = 0. \tag{4.17}$$

By these two equations and (4.15), we obtain

$$\begin{aligned}&E\left[\big(\hat{V}_{2,l}(t) - \hat{V}_{2,\ell}(t)\big)^2\right]\\ &= \sum_{i=1}^{l}\sum_{j:s_{i-1}^l < s_j^\ell\le s_i^l}(\bar{a}(s_j^\ell) - \bar{a}(s_{j-1}^\ell))\Big((\Delta F((t-s_i^l)\boldsymbol{e};(t-s_j^\ell)\boldsymbol{e}))(1-\Delta F((t-s_i^l)\boldsymbol{e};(t-s_j^\ell)\boldsymbol{e}))\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad + \Delta\Gamma^c(((t-s_i^l)\boldsymbol{e},(t-s_i^l)\boldsymbol{e});((t-s_j^\ell)\boldsymbol{e},(t-s_j^\ell)\boldsymbol{e}))\Big)\\
&\le \sum_{i=1}^{l}\sum_{j:s_{i-1}^l<s_j^\ell\le s_i^l}(\bar a(s_j^\ell)-\bar a(s_{j-1}^\ell))\Big((\Delta F((t-s_i^l)\boldsymbol{e};(t-s_j^\ell)\boldsymbol{e}))\\
&\qquad\qquad + \Delta\Gamma^c(((t-s_i^l)\boldsymbol{e},(t-s_i^l)\boldsymbol{e});((t-s_j^\ell)\boldsymbol{e},(t-s_j^\ell)\boldsymbol{e}))\Big)\\
&\le \sum_{i=1}^{l}(\bar a(s_i^l)-\bar a(s_{i-1}^l))\Big(\Delta F((t-s_i^l)\boldsymbol{e};(t-s_{i-1}^l)\boldsymbol{e})\\
&\qquad\qquad + \Delta\Gamma^c(((t-s_i^l)\boldsymbol{e},(t-s_i^l)\boldsymbol{e});((t-s_{j_i^*}^\ell)\boldsymbol{e},(t-s_{j_i^*}^\ell)\boldsymbol{e}))\Big)\\
&\le \max_{1\le i\le l}\left\{\bar a(s_i^l)-\bar a(s_{i-1}^l)\right\}\left(1+\sum_{i=1}^{l}\Delta\Gamma^c(((t-s_i^l)\boldsymbol{e},(t-s_i^l)\boldsymbol{e});((t-s_{j_i^*}^\ell)\boldsymbol{e},(t-s_{j_i^*}^\ell)\boldsymbol{e}))\right),
\end{aligned}
$$

where $j_i^* := \operatorname*{argmax}_{j:s_{i-1}^l<s_j^\ell\le s_i^l}\{\Delta\Gamma^c(((t-s_i^l)\boldsymbol{e},(t-s_i^l)\boldsymbol{e});((t-s_j^\ell)\boldsymbol{e},(t-s_j^\ell)\boldsymbol{e}))\}$, $i=1,...,l$, and the last inequality follows from the fact that $\sum_{i=1}^l \Delta F((t-s_i^l)\boldsymbol{e};(t-s_{i-1}^l)\boldsymbol{e}) \le F_{\rm m}(t)\le 1$. Note that

$$
\begin{aligned}
&\sum_{i=1}^{l}\Delta\Gamma^c(((t-s_i^l)\boldsymbol{e},(t-s_i^l)\boldsymbol{e});((t-s_{j_i^*}^\ell)\boldsymbol{e},(t-s_{j_i^*}^\ell)\boldsymbol{e}))\\
&\quad = 2\sum_{i=1}^{l}\sum_{q=2}^{\infty}\Big(\Delta P_q(((t-s_i^l)\boldsymbol{e},(t-s_i^l)\boldsymbol{e});((t-s_{j_i^*}^\ell)\boldsymbol{e},(t-s_{j_i^*}^\ell)\boldsymbol{e}))\\
&\qquad\qquad -\Delta P_I(((t-s_i^l)\boldsymbol{e},(t-s_i^l)\boldsymbol{e});((t-s_{j_i^*}^\ell)\boldsymbol{e},(t-s_{j_i^*}^\ell)\boldsymbol{e}))\Big)\\
&\quad = 2\sum_{q=2}^{\infty}\sum_{i=1}^{l}\Big(\Delta P_q(((t-s_i^l)\boldsymbol{e},(t-s_i^l)\boldsymbol{e});((t-s_{j_i^*}^\ell)\boldsymbol{e},(t-s_{j_i^*}^\ell)\boldsymbol{e}))\\
&\qquad\qquad -\Delta P_I(((t-s_i^l)\boldsymbol{e},(t-s_i^l)\boldsymbol{e});((t-s_{j_i^*}^\ell)\boldsymbol{e},(t-s_{j_i^*}^\ell)\boldsymbol{e}))\Big)\\
&\quad = 2\sum_{q=2}^{\infty}\sum_{i=1}^{l}(P(G_{1,i}\cap H_{q,1,i})-P(G_{1,i})P(H_{q,1,i}))\\
&\quad \le 2\sum_{q=2}^{\infty}\sup_{H_{q,1}\in\sigma(\boldsymbol{\eta}^q)}\sum_{i=1}^{l}(P(G_{1,i}\cap H_{q,1})-P(G_{1,i})P(H_{q,1}))\\
&\quad = 2\sum_{q=2}^{\infty}\sup_{H_{q,1}\in\sigma(\boldsymbol{\eta}^q)}(P(G_1\cap H_{q,1})-P(G_1)P(H_{q,1}))\\
&\quad \le 2\sum_{i=1}^{\infty}\alpha_i := C_0,
\end{aligned}
$$

where $G_{1,i} := \{(t-s_i^l)\boldsymbol{e}\le\boldsymbol{\eta}^1\le(t-s_{j_i^*}^\ell)\boldsymbol{e}\}$, $H_{q,1,i} := \{(t-s_i^l)\boldsymbol{e}\le\boldsymbol{\eta}^q\le(t-s_{j_i^*}^\ell)\boldsymbol{e}\}$, $G_1=\cup_{i=1}^l G_{1,i}$, and $\sigma(\boldsymbol{\eta}^q)$ is the σ-algebra generated by $\boldsymbol{\eta}^q$. Here the first equality follows from Lemma 4.2. The second equality holds by Fubini's Theorem and Lemma 4.1, and the third equality follows by the definition of the increment in high dimensions using (4.4). The first inequality holds by the fact that $H_{q,1,i}$, $i=1,...,l$, are in $\sigma(\boldsymbol{\eta}^q)$, and the fourth equality is obtained by noting that $G_{1,i}$, $i=1,...,l$, are mutually exclusive. The last inequality is directly implied by the definition of the strong mixing coefficients. Thus, we have

$$E\left[\left(\hat{V}_{2,l}(t)-\hat{V}_{2,\ell}(t)\right)^2\right] \le \max_{1\le i\le l}\left\{\bar{a}(s_i^l)-\bar{a}(s_{i-1}^l)\right\}(1+C_0).$$

Since $\bar{a}(\cdot)$ is continuous and $\max_{1\le i\le l}(\bar{a}(s_i^l)-\bar{a}(s_{i-1}^l))\to 0$ as $l\to\infty$, we have proved (4.14).

To see the Gaussian property of the multidimensional process $(\hat{\boldsymbol{M}}_2, \hat{\boldsymbol{Z}}_2, \hat{V}_2)$, we first note that, for a fixed $t\ge 0$, $\hat{M}_{k,2,\ell}(t)$, $\hat{Z}_{k,2,\ell}(t)$ and $\hat{V}_{2,\ell}(t)$ have normal distributions with mean zeros, $k=1,...,K$. By the definition of $\hat{M}_{k,2}(t)$, $\hat{Z}_{k,2}(t)$ and $\hat{V}_2(t)$, we see that $\hat{M}_{k,2,\ell}(t)$, $\hat{Z}_{k,2,\ell}(t)$ and $\hat{V}_{2,\ell}(t)$ converge to $\hat{M}_{k,2}(t)$, $\hat{Z}_{k,2}(t)$ and $\hat{V}_2(t)$ in probability as $\ell\to\infty$ for each $t\ge 0$ and $k=1,...,K$. Following from Lemma 4.9.4 of [23], we immediately see that $\hat{M}_{k,2}(t)$, $\hat{Z}_{k,2}(t)$ and $\hat{V}_2(t)$ are normally distributed with mean zeros, for each $t\ge 0$ and $k=1,...,K$.

We now show $\hat{M}_{k,2}(t)$, $\hat{Z}_{k,2}(t)$ and $\hat{V}_2(t)$ are continuous in $t\ge 0$, $k=1,...,K$. Since the proofs for the continuity property of the aforementioned processes are similar, we only consider $\hat{V}_2$. Without loss of generality, we assume $\hat{V}_{2,\ell}(t)$ and $\hat{V}_{2,\ell}(s)$ have the same partition $\{s_i^\ell: 0\le i\le \ell\}$ of $[0,t]$ for $0\le s\le t$. Note from (4.9) that

$$\hat{V}_{2,\ell}(t)-\hat{V}_{2,\ell}(s)=\sum_{i=1}^{\ell}\Delta\hat{K}((\bar{a}(s_{i-1}^\ell),(s-s_i^\ell)\boldsymbol{e});(\bar{a}(s_i^\ell),(t-s_i^\ell)\boldsymbol{e})),$$

where we set $\hat{K}(t,\boldsymbol{x})=0$ for $\boldsymbol{x}<\boldsymbol{0}$. By (4.16) and (4.17), we further obtain

$$\begin{aligned}
&E\left[\left(\hat{V}_{2,\ell}(t)-\hat{V}_{2,\ell}(s)\right)^2\right]\\
&=\sum_{i=1}^{\ell}(\bar{a}(s_i^\ell)-\bar{a}(s_{i-1}^\ell))\Big[(\Delta F((s-s_i^\ell)\boldsymbol{e};(t-s_i^\ell)\boldsymbol{e}))(1-\Delta F((s-s_i^\ell)\boldsymbol{e};(t-s_i^\ell)\boldsymbol{e})\\
&\qquad +\Delta\Gamma^c(((s-s_i^\ell)\boldsymbol{e},(s-s_i^\ell)\boldsymbol{e});((t-s_i^\ell)\boldsymbol{e},(t-s_i^\ell)\boldsymbol{e}))\Big].
\end{aligned}$$

Thus, we have

$$\begin{aligned}
E\left[\left(\hat{V}_2(t)-\hat{V}_2(s)\right)^2\right] &= \lim_{\ell\to\infty}E\left[\left(\hat{V}_{2,\ell}(t)-\hat{V}_{2,\ell}(s)\right)^2\right]\\
&=\int_0^t[\Delta F((s-u)\boldsymbol{e};(t-u)\boldsymbol{e}))(1-\Delta F((s-u)\boldsymbol{e};(t-u)\boldsymbol{e})\\
&\qquad +\Delta\Gamma^c(((s-u)\boldsymbol{e},(s-u)\boldsymbol{e});((t-u)\boldsymbol{e},(t-u)\boldsymbol{e}))]d\bar{a}(u).
\end{aligned}$$

The second equality follows by applying Lebesgue's theorem, while the first equality follows from the definition of $\hat{V}_{2,\ell}(t)$, the fact that $\hat{V}_{2,\ell}(t)$ is normally distributed and applying Lemma 4.9.4 in [23]. Thus, we have shown that $\hat{V}_{2,\ell}(\cdot)$ is continuous in probability. By Lemma 4.9.6 in [23], to show that the sample paths of $\hat{V}_{2,\ell}(\cdot)$ are continuous a.s., it suffices to show that for any partition $\{s_i^\ell: 0\le i\le\ell\}$ of $[0,t]$,

$$\lim_{L\to\infty}\limsup_{\ell\to\infty}P\left(\sum_{i=1}^{\ell}\left(\hat{V}_2(s_i^\ell)-\hat{V}_2(s_{i-1}^\ell)\right)^2\ge L\right)=0. \tag{4.18}$$

By Markov inequality,

$$P\left(\sum_{i=1}^{\ell}\left(\hat{V}_2(s_i^\ell)-\hat{V}_2(s_{i-1}^\ell)\right)^2\ge L\right)$$

$$
\begin{aligned}
&\le \frac{1}{L}\sum_{i=1}^{\ell} E\left[\left(\hat V_2(s_i^\ell)-\hat V_2(s_{i-1}^\ell)\right)^2\right]\\
&= \frac{1}{L}\sum_{i=1}^{\ell}\int_0^t \Big[(\Delta F((s_{i-1}^\ell-u)\boldsymbol{e};(s_i^\ell-u)\boldsymbol{e}))(1-\Delta F((s_{i-1}^\ell-u)\boldsymbol{e};(s_i^\ell-u)\boldsymbol{e}))\\
&\qquad\qquad + \Delta\Gamma^c(((s_{i-1}^\ell-u)\boldsymbol{e},(s_{i-1}^\ell-u)\boldsymbol{e});((s_i^\ell-u)\boldsymbol{e},(s_i^\ell-u)\boldsymbol{e}))\Big]d\bar a(u)\\
&\le \frac{1}{L}\sum_{i=1}^{\ell}\int_0^t \Big[\Delta F((s_{i-1}^\ell-u)\boldsymbol{e};(s_i^\ell-u)\boldsymbol{e})\\
&\qquad\qquad + \Delta\Gamma^c(((s_{i-1}^\ell-u)\boldsymbol{e},(s_{i-1}^\ell-u)\boldsymbol{e});((s_i^\ell-u)\boldsymbol{e},(s_i^\ell-u)\boldsymbol{e}))\Big]d\bar a(u)\\
&\le \frac{1}{L}\bar a(t)(1+C_0),
\end{aligned}
$$

where C_0 is specified in Lemma 4.1. Here the last inequality is implied by $\sum_{i=1}^{\ell}\Delta F((s_{i-1}^\ell-u)\boldsymbol{e};(s_i^\ell-u)\boldsymbol{e})\le F_{\mathrm{m}}(t)\le 1$, together with the fact that

$$
\begin{aligned}
&\sum_{i=1}^{\ell}\Delta\Gamma^c(((s_{i-1}^\ell-u)\boldsymbol{e},(s_{i-1}^\ell-u)\boldsymbol{e});((s_i^\ell-u)\boldsymbol{e},(s_i^\ell-u)\boldsymbol{e}))\\
&\quad= 2\sum_{i=1}^{\ell}\sum_{q=2}^{\infty}\Big(\Delta P_q(((s_{i-1}^\ell-u)\boldsymbol{e},(s_{i-1}^\ell-u)\boldsymbol{e});((s_i^\ell-u)\boldsymbol{e},(s_i^\ell-u)\boldsymbol{e}))\\
&\qquad\qquad -\Delta P_I(((s_{i-1}^\ell-u)\boldsymbol{e},(s_{i-1}^\ell-u)\boldsymbol{e});((s_i^\ell-u)\boldsymbol{e},(s_i^\ell-u)\boldsymbol{e}))\Big)\\
&\quad= 2\sum_{q=2}^{\infty}\sum_{i=1}^{\ell}\Big(\Delta P_q(((s_{i-1}^\ell-u)\boldsymbol{e},(s_{i-1}^\ell-u)\boldsymbol{e});((s_i^\ell-u)\boldsymbol{e},(s_i^\ell-u)\boldsymbol{e}))\\
&\qquad\qquad -\Delta P_I(((s_{i-1}^\ell-u)\boldsymbol{e},(s_{i-1}^\ell-u)\boldsymbol{e});((s_i^\ell-u)\boldsymbol{e},(s_i^\ell-u)\boldsymbol{e}))\Big)\\
&\quad= 2\sum_{q=2}^{\infty}\sum_{i=1}^{\ell}(P(G_{2,i}\cap H_{q,2,i})-P(G_{2,i})P(H_{q,2,i}))\\
&\quad\le 2\sum_{q=2}^{\infty}\sup_{H_{q,2}\in\sigma(\boldsymbol{\eta}^q)}\sum_{i=1}^{\ell}(P(G_{2,i}\cap H_{q,2})-P(G_{2,i})P(H_{q,2}))\\
&\quad= 2\sum_{q=2}^{\infty}\sup_{H_{q,2}\in\sigma(\boldsymbol{\eta}^q)}(P(G_2\cap H_{q,2})-P(G_2)P(H_{q,2}))\\
&\quad\le 2\sum_{i=1}^{\infty}\alpha_i := C_0,
\end{aligned}
$$

where $G_{2,i}:=\{(s_{i-1}^\ell-u)\boldsymbol{e}\le\boldsymbol{\eta}^1\le(s_i^\ell-u)\boldsymbol{e}\}$, $H_{q,2}:=\{(s_{i-1}^\ell-u)\boldsymbol{e}\le\boldsymbol{\eta}^q\le(s_i^\ell-u)\boldsymbol{e}\}$, $G_2=\cup_{i=1}^{\ell}G_{2,i}$, and $\sigma(\boldsymbol{\eta}^q)$ is the σ-algebra generated by $\boldsymbol{\eta}^q$. Here the first equality follows from Lemma 4.2. The second equality holds by Fubini's Theorem and Lemma 4.1, and the third equality follows from (4.4). The first inequality holds by the fact that $H_{q,2,i}$, $i=1,...,\ell$, are in $\sigma(\boldsymbol{\eta}^q)$, and the fourth equality is obtained by noting that $G_{2,i}$, $i=1,...,l$, are mutually exclusive. The last inequality is directly implied by the definition of the strong mixing coefficients. Therefore, we can see that (4.18) holds, which implies that $\hat V_2(\cdot)$ is a continuous process. Therefore, we have proved that the multidimensional process $(\hat{\boldsymbol{M}}_2,\hat{\boldsymbol{Z}}_2,\hat V_2)$ is Gaussian with mean zero and has continuous sample paths.

Next we compute the covariance functions of $\hat{M}_{j,2}(t)$, $\hat{Z}_{k,2}(t)$ and $\hat{V}(t)$, for each $t \geq 0$ and $j,k = 1,...,K$. Here we only show (3.47), and the rest of the formulas can be derived similarly. From Definition 4.1, the processes $\hat{M}_{k,2,\ell}(\cdot)$ in (4.8), $\hat{V}_{2,\ell}(\cdot)$ in (4.9) and $\hat{Z}_{k,2,\ell}(\cdot)$ in (4.10) can be written as, for $k = 1,...,K$,

$$\begin{aligned}\hat{M}_{k,2,\ell}(t) &= -\sum_{i=1}^{\ell} \Delta\hat{K}\left((\bar{a}(s_{i-1}^{\ell}),\mathbf{0});(\bar{a}(s_i^{\ell}),(t-s_i^{\ell})\boldsymbol{e}_k)\right)\\ &= -\sum_{i=1}^{\ell}\left[\hat{K}(\bar{a}(s_i^{\ell}),(t-s_i^{\ell})\boldsymbol{e}_k)-\hat{K}(\bar{a}(s_{i-1}^{\ell}),(t-s_i^{\ell})\boldsymbol{e}_k)\right],\quad t\geq 0, \end{aligned}\tag{4.19}$$

$$\begin{aligned}\hat{V}_{2,\ell}(t) &= \sum_{i=1}^{\ell} \Delta\hat{K}\left((\bar{a}(s_{i-1}^{\ell}),\mathbf{0});(\bar{a}(s_i^{\ell}),(t-s_i^{\ell})\boldsymbol{e})\right)\\ &= \sum_{i=1}^{\ell}\left[\hat{K}(\bar{a}(s_i^{\ell}),(t-s_i^{\ell})\boldsymbol{e})-\hat{K}(\bar{a}(s_{i-1}^{\ell}),(t-s_i^{\ell})\boldsymbol{e})\right],\quad t\geq 0, \end{aligned}\tag{4.20}$$

$$\begin{aligned}\hat{Z}_{k,2,\ell}(t) &= -\hat{M}_{k,2,\ell}(t)-\hat{V}_{2,\ell}(t)\\ &= \sum_{i=1}^{\ell}\Big[\hat{K}(\bar{a}(s_i^{\ell}),(t-s_i^{\ell})\boldsymbol{e}_k)-\hat{K}(\bar{a}(s_{i-1}^{\ell}),(t-s_i^{\ell})\boldsymbol{e}_k)\\ &\qquad -\hat{K}(\bar{a}(s_i^{\ell}),(t-s_i^{\ell})\boldsymbol{e})+\hat{K}(\bar{a}(s_{i-1}^{\ell}),(t-s_i^{\ell})\boldsymbol{e})\Big],\quad t\geq 0,\end{aligned}\tag{4.21}$$

where $\{s_i^{\ell} : 0 \leq i \leq \ell\}$ is a partition of $[0,t]$. Then, for $j,k = 1,...,K$,

$$\begin{aligned}&Cov\big(\hat{M}_{j,2,\ell}(t),\hat{Z}_{k,2,\ell}(t)\big)\\ &= Cov\Bigg(-\sum_{i=1}^{\ell}\left[\hat{K}(\bar{a}(s_i^{\ell}),(t-s_i^{\ell})\boldsymbol{e}_j)-\hat{K}(\bar{a}(s_{i-1}^{\ell}),(t-s_i^{\ell})\boldsymbol{e}_j)\right],\\ &\quad\sum_{l=1}^{\ell}\left[\hat{K}(\bar{a}(s_l^{\ell}),(t-s_l^{\ell})\boldsymbol{e}_k)-\hat{K}(\bar{a}(s_{l-1}^{\ell}),(t-s_l^{\ell})\boldsymbol{e}_k)-\hat{K}(\bar{a}(s_l^{\ell}),(t-s_l^{\ell})\boldsymbol{e})+\hat{K}(\bar{a}(s_{l-1}^{\ell}),(t-s_l^{\ell})\boldsymbol{e})\right]\Bigg)\\ &= -\sum_{i=1}^{\ell}\sum_{l=1}^{\ell}Cov\Big(\left[\hat{K}(\bar{a}(s_i^{\ell}),(t-s_i^{\ell})\boldsymbol{e}_j)-\hat{K}(\bar{a}(s_{i-1}^{\ell}),(t-s_i^{\ell})\boldsymbol{e}_j)\right],\\ &\quad\left[\hat{K}(\bar{a}(s_l^{\ell}),(t-s_l^{\ell})\boldsymbol{e}_k)-\hat{K}(\bar{a}(s_{l-1}^{\ell}),(t-s_l^{\ell})\boldsymbol{e}_k)-\hat{K}(\bar{a}(s_l^{\ell}),(t-s_l^{\ell})\boldsymbol{e})+\hat{K}(\bar{a}(s_{l-1}^{\ell}),(t-s_l^{\ell})\boldsymbol{e})\right]\Big).\end{aligned}$$

Using (3.4) with some calculations, if $s_i^{\ell} \leq s_{l-1}^{\ell}$ and $s_l^{\ell} \leq s_{i-1}^{\ell}$,

$$\begin{aligned}&Cov\Big(\left[\hat{K}(\bar{a}(s_i^{\ell}),(t-s_i^{\ell})\boldsymbol{e}_j)-\hat{K}(\bar{a}(s_{i-1}^{\ell}),(t-s_i^{\ell})\boldsymbol{e}_j)\right],\\ &\quad\left[\hat{K}(\bar{a}(s_l^{\ell}),(t-s_l^{\ell})\boldsymbol{e}_k)-\hat{K}(\bar{a}(s_{l-1}^{\ell}),(t-s_l^{\ell})\boldsymbol{e}_k)-\hat{K}(\bar{a}(s_l^{\ell}),(t-s_l^{\ell})\boldsymbol{e})+\hat{K}(\bar{a}(s_{l-1}^{\ell}),(t-s_l^{\ell})\boldsymbol{e})\right]\Big)=0,\end{aligned}$$

and if $s_i^{\ell} = s_l^{\ell}$,

$$\begin{aligned}&Cov\Big(\left[\hat{K}(\bar{a}(s_i^{\ell}),(t-s_i^{\ell})\boldsymbol{e}_k)-\hat{K}(\bar{a}(s_{i-1}^{\ell}),(t-s_i^{\ell})\boldsymbol{e}_k)\right],\\ &\quad\left[\hat{K}(\bar{a}(s_l^{\ell}),(t-s_l^{\ell})\boldsymbol{e}_k)-\hat{K}(\bar{a}(s_{l-1}^{\ell}),(t-s_l^{\ell})\boldsymbol{e}_k)-\hat{K}(\bar{a}(s_l^{\ell}),(t-s_l^{\ell})\boldsymbol{e})+\hat{K}(\bar{a}(s_{l-1}^{\ell}),(t-s_l^{\ell})\boldsymbol{e})\right]\Big)\\ &= (\bar{a}(s_i^{\ell})-\bar{a}(s_{i-1}^{\ell}))\Big[F_{j,k}(t-s_i^{\ell},t-s_i^{\ell})-F_{\mathrm{m}}(t-s_i^{\ell})-F_j(t-s_i^{\ell})F_k(t-s_i^{\ell})\end{aligned}$$

$$+ F_j(t-s_i^\ell)F_{\rm m}(t-s_i^\ell) + \Gamma^c_{j,k}(t-s_i^\ell,t-s_i^\ell) - \Gamma^c_{j,{\rm m}}(t-s_i^\ell,t-s_i^\ell)\Big].$$

Thus, for each $t \ge 0$ and $j,k = 1,...,K$,

$$Cov\big(\hat{M}_{j,2,\ell}(t), \hat{Z}_{k,2,\ell}(t)\big) = \sum_{i=1}^{\ell}(\bar{a}(s_i^\ell) - \bar{a}(s_{i-1}^\ell))\Big[F_{\rm m}(t-s_i^\ell) - F_{j,k}(t-s_i^\ell,t-s_i^\ell) + F_j(t-s_i^\ell)F_k(t-s_i^\ell)$$
$$- F_j(t-s_i^\ell)F_{\rm m}(t-s_i^\ell) - \Gamma^c_{j,k}(t-s_i^\ell,t-s_i^\ell) + \Gamma^c_{j,{\rm m}}(t-s_i^\ell,t-s_i^\ell)\Big].$$

By Lebesgue's theorem, we obtain, for each $t \ge 0$ and $j,k = 1,...,K$,

$$\lim_{\ell\to\infty} Cov\big(\hat{M}_{j,2,\ell}(t), \hat{Z}_{k,2,\ell}(t)\big) = \int_0^t \Big[F_{\rm m}(t-s) - F_{j,k}(t-s,t-s) + F_j(t-s)F_k(t-s)$$
$$- F_j(t-s)F_{\rm m}(t-s) - \Gamma^c_{j,k}(t-s,t-s) + \Gamma^c_{j,{\rm m}}(t-s,t-s)\Big]d\bar{a}(s).$$

Following the analogous argument in Lemma 5.1 of [24], we observe that, for each $t \ge 0$ and $j,k = 1,...,K$,

$$\lim_{\ell\to\infty} Cov\big(\hat{M}_{j,2,\ell}(t), \hat{Z}_{k,2,\ell}(t)\big) = Cov\big(\hat{M}_{j,2}(t), \hat{Z}_{k,2}(t)\big).$$

Therefore, we can see (3.39) holds. The proof for Lemma 3.2 is completed. □

We will next compute the covariance functions of $\hat{M}_{j,2}(t)$, $\hat{Z}_{k,2}(t)$ and $\hat{V}(t)$, for each $t \ge 0$ and $j,k = 1,...,K$, when the sequence of service vectors is generated from the DVAR(1) procedure in (3.44) and each service vector is distributed according to F in (3.43).

Proof of Corollary 3.1. Without loss of generality, we here only show (3.47), as the other covariance formulas can be obtained similarly. Note from (3.5) and (3.44) that, for $t \ge 0$ and $j,k = 1,...,K$,

$$\begin{aligned}\Gamma^c_{j,k}(t,t) &= 2\sum_{i=1}^{\infty}\big(P(\eta_j^1 \le t, \eta_k^{1+i} \le t) - F_j(t)F_k(t)\big)\\ &= 2\sum_{i=1}^{\infty}\big(p^i F_{j,k}(t,t) + (1-p^i)F_j(t)F_k(t) - F_j(t)F_k(t)\big)\\ &= 2\Sigma_p(F_{j,k}(t,t) - F_j(t)F_k(t)),\end{aligned} \tag{4.22}$$

and

$$\begin{aligned}\Gamma^c_{j,{\rm m}}(t,t) &= 2\sum_{i=1}^{\infty}\big(P(\eta_j^1 \le t, \eta_{\rm m}^{1+i} \le t) - F_j(t)F_{\rm m}(t)\big)\\ &= 2\sum_{i=1}^{\infty}\big(p^i F_{\rm m}(t) + (1-p^i)F_j(t)F_{\rm m}(t) - F_j(t)F_{\rm m}(t)\big)\\ &= 2\Sigma_p(F_{\rm m}(t) - F_j(t)F_{\rm m}(t)).\end{aligned} \tag{4.23}$$

Since the joint distribution function F satisfies (3.43), we have, for each $t \ge 0$ and $j,k = 1,...,K$ with $j \ne k$,

$$F_{j,k}(t,t) = \rho G(t) + (1-\rho)(G(t))^2, \quad F_j(t) = F_k(t) = G(t), \tag{4.24}$$

$$F_{\rm m}(t) = \rho G(t) + (1-\rho)(G(t))^K. \tag{4.25}$$

Plugging (4.22)-(4.25) into (3.39) with some simple calculation, we immediately see (3.47) holds. The proof of Lemma 3.2 is completed. □

We next provide the proof for Lemma 3.3.

Proof of Lemma 3.3. Here we only show $Cov(\hat{M}_{j,3}(t), \hat{Z}_{k,3}(t))$, since other mean and covariance functions of $\hat{M}_{j,3}(t)$, $\hat{Z}_{k,3}(t)$ and $\hat{V}_3(t)$, for $t \geq 0$ and $j,k = 1,...,K$, follow from the similar argument. From the assumption that $\hat{N}$ is Poisson, we have, by Fubini's theorem, for $t \geq 0$ and $j,k = 1,...,K$,

$$
\begin{aligned}
Cov\big(\hat{M}_{j,3}(t), \hat{Z}_{k,3}(t)\big) &= E\Big[(\hat{M}_{j,3}(t) - E[\hat{M}_{j,3}(t)])(\hat{Z}_{k,3}(t) - E[\hat{Z}_{k,3}(t)])\Big] \\
&= E\bigg[\int_0^t \int_0^t (\hat{J}(t) - \hat{J}(s_1) - \lambda^u E[d_1](t-s_1)) \\
&\qquad \times (\hat{J}(t) - \hat{J}(s_2) - \lambda^u E[d_1](t-s_2))\lambda(s_1)\lambda(s_2) dF_j^c(t-s_1) dF_k(t-s_2)\bigg] \\
&\quad + E\bigg[\int_0^t \int_0^t (\hat{J}(t) - \hat{J}(s_1) - \lambda^u E[d_1](t-s_1)) \\
&\qquad \times (\hat{J}(t) - \hat{J}(s_2) - \lambda^u E[d_1](t-s_2))\lambda(s_1)\lambda(s_2) dF_j^c(t-s_1) dF_{\mathrm{m}}^c(t-s_2)\bigg] \\
&= \int_0^t \int_0^t E\Big[(\hat{J}(t) - \hat{J}(s_1) - \lambda^u E[d_1](t-s_1))(\hat{J}(t) - \hat{J}(s_2) - \lambda^u E[d_1](t-s_2))\Big] \\
&\qquad \times \lambda(s_1)\lambda(s_2) dF_j^c(t-s_1) dF_k(t-s_2) \\
&\quad + \int_0^t \int_0^t E\Big[(\hat{J}(t) - \hat{J}(s_1) - \lambda^u E[d_1](t-s_1))(\hat{J}(t) - \hat{J}(s_2) - \lambda^u E[d_1](t-s_2))\Big] \\
&\qquad \times \lambda(s_1)\lambda(s_2) dF_j^c(t-s_1) dF_{\mathrm{m}}^c(t-s_2) \\
&= \lambda^u E[d_1^2] \int_0^t \int_0^t \left[((t-s_1) \wedge (t-s_2))\lambda(s_1)\lambda(s_2)\right] dF_j^c(t-s_1) dF_k(t-s_2) \\
&\qquad + \lambda^u E[d_1^2] \int_0^t \int_0^t \left[((t-s_1) \wedge (t-s_2))\lambda(s_1)\lambda(s_2)\right] dF_j^c(t-s_1) dF_{\mathrm{m}}^c(t-s_2).
\end{aligned}
$$

Therefore, the proof of Lemma 3.3 is completed. □

5. Proof of Theorem 3.3

In this section, we prove the FCLT for the processes $\big(\hat{\boldsymbol{X}}^n, \hat{\boldsymbol{Y}}^n, \hat{S}^n\big)$, Theorem 3.3. We first give representations for the processes $\big(\hat{\boldsymbol{X}}^n, \hat{\boldsymbol{Y}}^n, \hat{S}^n\big)$ by the multiparameter sequential empirical processes $\hat{K}^n$.

Lemma 5.1 (Representations of $\hat{\boldsymbol{X}}^n$, $\hat{\boldsymbol{Y}}^n$ and $\hat{S}^n$)**.** *The processes $\hat{\boldsymbol{X}}^n$, $\hat{\boldsymbol{Y}}^n$ and $\hat{S}^n$ in* (3.15) *can be represented as: for each $t \geq 0$ and $k = 1,...,K$,*

$$\hat{\boldsymbol{X}}^n(t) = \hat{\boldsymbol{M}}_1^n(t) + \hat{\boldsymbol{M}}_2^n(t) + \hat{\boldsymbol{M}}_3^n(t), \quad \hat{\boldsymbol{M}}_i^n(t) := \big(\hat{M}_{1,i}^n(t), ..., \hat{M}_{K,i}^n(t)\big), \quad i = 1,2,3, \tag{5.1}$$

$$\hat{\boldsymbol{Y}}^n(t) = \hat{\boldsymbol{Z}}_1^n(t) + \hat{\boldsymbol{Z}}_2^n(t) + \hat{\boldsymbol{Z}}_3^n(t), \quad \hat{\boldsymbol{Z}}_i^n(t) := \big(\hat{Z}_{1,i}^n(t), ..., \hat{Z}_{K,i}^n(t)\big), \quad i = 1,2,3, \tag{5.2}$$

$$\hat{S}^n(t) = \hat{V}_1^n(t) + \hat{V}_2^n(t) + \hat{V}_3^n(t), \tag{5.3}$$

where

$$\hat{M}_{k,1}^n(t) := \int_0^t F_k^c(\xi^n(t) - \xi^n(s)) d\hat{A}^n(s) = \hat{A}^n(t) - \int_0^t \hat{A}^n(s) dF_k^c(\xi^n(t) - \xi^n(s)), \tag{5.4}$$

$$
\begin{aligned}
\hat{M}_{k,2}^n(t) &:= \int_0^t \int_{\mathbb{R}_+^K} \mathbf{1}(x_k > \xi^n(t) - \xi^n(s)) d\hat{K}^n\left(\bar{A}^n(s), \boldsymbol{x}\right) \\
&= -\int_0^t \int_{\mathbb{R}_+^K} \mathbf{1}(x_k \leq \xi^n(t) - \xi^n(s)) d\hat{K}^n\left(\bar{A}^n(s), \boldsymbol{x}\right),
\end{aligned} \tag{5.5}
$$

$$\hat{M}^n_{k,3}(t) := \int_0^t [\sqrt{n}(F^c_k(\xi^n(t) - \xi^n(s)) - F^c_k(t-s))] d\bar{a}(s), \tag{5.6}$$

$$\hat{V}^n_1(t) := \int_0^t F_{\mathrm{m}}(\xi^n(t) - \xi^n(s)) d\hat{A}^n(s) = -\int_0^t \hat{A}^n(s) dF_{\mathrm{m}}(\xi^n(t) - \xi^n(s)), \tag{5.7}$$

$$\hat{V}^n_2(t) := \int_0^t \int_{\mathbb{R}^K_+} \mathbf{1}(x_j \le \xi^n(t) - \xi^n(s),\ \forall\ j) d\hat{K}^n\left(\bar{A}^n(s), \boldsymbol{x}\right), \tag{5.8}$$

$$\hat{V}^n_3(t) := \int_0^t [\sqrt{n}(F_{\mathrm{m}}(\xi^n(t) - \xi^n(s)) - F_{\mathrm{m}}(t-s))] d\bar{a}(s), \tag{5.9}$$

$$\begin{aligned}\hat{Z}^n_{k,1}(t) :=& \int_0^t (F_k(\xi^n(t) - \xi^n(s)) - F_{\mathrm{m}}(\xi^n(t) - \xi^n(s))) d\hat{A}^n(s) \\ =& -\int_0^t \hat{A}^n(s) d(F_k(\xi^n(t) - \xi^n(s)) - F_{\mathrm{m}}(\xi^n(t) - \xi^n(s))),\end{aligned} \tag{5.10}$$

$$\begin{aligned}\hat{Z}^n_{k,2}(t) :=& \int_0^t \int_{\mathbb{R}^K_+} \left(\mathbf{1}(x_k \le \xi^n(t) - \xi^n(s)) - \mathbf{1}(x_j \le \xi^n(t) - \xi^n(s),\ \forall\ j)\right) d\hat{K}^n\left(\bar{A}^n(s), \boldsymbol{x}\right) \\ =& -\hat{M}^n_{k,2}(t) - \hat{V}^n_2(t),\end{aligned} \tag{5.11}$$

$$\begin{aligned}\hat{Z}^n_{k,3}(t) :=& \int_0^t [\sqrt{n}(F_k(\xi^n(t) - \xi^n(s)) - F_k(t-s)) - \sqrt{n}(F_{\mathrm{m}}(\xi^n(t) - \xi^n(s)) - F_{\mathrm{m}}(t-s))] d\bar{a}(s) \\ =& -\hat{M}^n_{k,3}(t) - \hat{V}^n_3(t),\end{aligned} \tag{5.12}$$

and the integrals in (5.4), (5.5), (5.7), (5.8), (5.10) *and* (5.11) *are defined as Stieltjes integrals for functions of bounded variation as integrators.*

Proof. The representations of the processes $\hat{\boldsymbol{X}}^n$, $\hat{\boldsymbol{Y}}^n$ and $\hat{S}^n$ follow from equations (3.7), (3.8), (3.9), (3.15) and direct calculations. The second equalities in (5.4), (5.7) and (5.10) follow from integration by parts, and the second equalities in (5.5), (5.8) and (5.11) are obtained from simple algebra. □

We first prove the convergence of $\left(\hat{\boldsymbol{M}}^n_1, \hat{\boldsymbol{Z}}^n_1, \hat{V}^n_1, \hat{\boldsymbol{M}}^n_3, \hat{\boldsymbol{Z}}^n_3, \hat{V}^n_3\right)$.

Lemma 5.2.

$$\left(\hat{\boldsymbol{M}}^n_1, \hat{\boldsymbol{M}}^n_3, \hat{\boldsymbol{Z}}^n_1, \hat{V}^n_1, \hat{\boldsymbol{Z}}^n_3, \hat{V}^n_3\right) \Rightarrow \left(\hat{\boldsymbol{M}}_1, \hat{\boldsymbol{M}}_3, \hat{\boldsymbol{Z}}_1, \hat{V}_1, \hat{\boldsymbol{Z}}_3, \hat{\boldsymbol{V}}_3\right) \quad in \quad (\mathbb{D}^{4K+2}, M_1) \quad as \quad n \to \infty. \tag{5.13}$$

To show this lemma, we first prove some continuity properties of functional mappings in the M_1 topology. We define the mapping $\Phi : \mathbb{C}_\uparrow \times \mathbb{D}^2 \to \mathbb{D}^{4K+2}$ by

$$\Phi(x, y, z) = (\varpi_k(x, z), \varphi_k(x, z), \psi(x, z), f_k(y), g_k(y), h(y), \quad k = 1, ..., K), \tag{5.14}$$

where the mappings $\varpi_k : \mathbb{C}_\uparrow \times \mathbb{D} \to \mathbb{D}$, $\varphi_k : \mathbb{C}_\uparrow \times \mathbb{D} \to \mathbb{D}$, $\psi : \mathbb{C}_\uparrow \times \mathbb{D} \to \mathbb{D}$, $f_k : \mathbb{D} \to \mathbb{D}$, $g_k : \mathbb{D} \to \mathbb{D}$ and $h : \mathbb{D} \to \mathbb{D}$ are defined by

$$\varpi_k(x, z)(t) := z(t) - \int_0^t z(s) dF^c_k(x(t) - x(s)), \quad t \ge 0, \tag{5.15}$$

$$\varphi_k(x, z)(t) := \int_0^t z(s) d(F_{\mathrm{m}}(x(t) - x(s)) - F_k(x(t) - x(s))), \quad t \ge 0, \tag{5.16}$$

$$\psi(x, z)(t) := -\int_0^t z(s) dF_{\mathrm{m}}(x(t) - x(s)), \quad t \ge 0, \tag{5.17}$$

$$f_k(y)(t) := -\int_0^t [(y(t) - y(s))\lambda(s)] dF^c_k(t-s), \quad t \ge 0, \tag{5.18}$$

$$g_k(y)(t) := -\int_0^t [(y(t) - y(s))\lambda(s)] d(F_k(t-s) - F_{\mathrm{m}}(t-s)), \quad t \ge 0, \tag{5.19}$$

$$h(y)(t) := -\int_0^t [(y(t)-y(s))\lambda(s)]dF_{\rm m}(t-s), \quad t \ge 0, \tag{5.20}$$

for $x \in \mathbb{C}_\uparrow$ and $y, z \in \mathbb{D}$ and $k = 1, ..., K$. We now state the continuity property of the above mappings in the following lemma.

Lemma 5.3. *For $x_n, x \in \mathbb{C}_\uparrow$, $y_n, y \in \mathbb{D}$ and $z_n, z \in \mathbb{D}$, $n \in \mathbb{N}$, if $(x_n, y_n, z_n) \to (x, y, z)$ in $(\mathbb{C}_\uparrow, \|\cdot\|) \times (\mathbb{D}^2, M_1)$ as $n \to \infty$, then $\Phi(x_n, y_n, z_n) \to \Phi(x, y, z)$ in $(\mathbb{D}^{4K+2}, M_1)$ as $n \to \infty$.*

Remark 5.1. We provide a counter example that the continuity of $F(\cdot)$ is necessary in order to prove Lemma 5.3. We focus on the mapping ψ in (5.17). Let $x_n(t) = x(t) = t$ for $t \ge 0$, and

$$F_{\rm m}(t) = \begin{cases} 0 & \text{if} \quad 0 \le t < 1/2, \\ 1/3 & \text{if} \quad 1/2 \le t < 1, \\ 1 & \text{if} \quad t \ge 1. \end{cases}$$

Here $F_{\rm m}$ is the c.d.f of a random variable taking values 1/2 with probability 1/3 and 1 with probability 2/3. We thus have, for $t \ge 0$ and $n \in \mathbb{N}$,

$$\psi(x_n, z_n)(t) = \int_0^t z_n(t-s)dF_{\rm m}(s) = \begin{cases} 0 & \text{if} \quad 0 \le t < 1/2, \\ \frac{1}{3}z_n\left(t-\frac{1}{2}\right) & \text{if} \quad 1/2 \le t < 1, \\ \frac{1}{3}z_n\left(t-\frac{1}{2}\right) + \frac{2}{3}z_n\left(t-1\right) & \text{if} \quad t \ge 1. \end{cases}$$

Similarly, for $t \ge 0$,

$$\psi(x, z)(t) = \int_0^t z(t-s)dF_{\rm m}(s) = \begin{cases} 0 & \text{if} \quad 0 \le t < 1/2, \\ \frac{1}{3}z\left(t-\frac{1}{2}\right) & \text{if} \quad 1/2 \le t < 1, \\ \frac{1}{3}z\left(t-\frac{1}{2}\right) + \frac{2}{3}z\left(t-1\right) & \text{if} \quad t \ge 1. \end{cases}$$

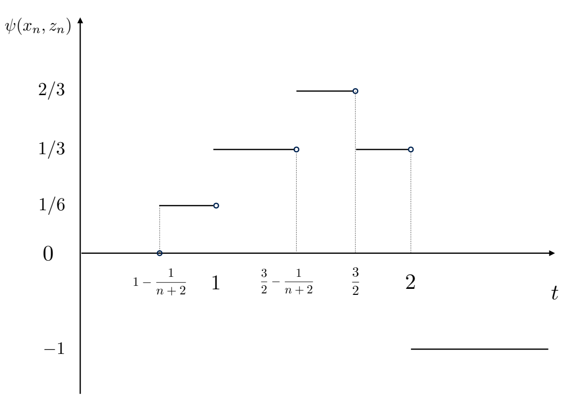
For $n \in \mathbb{N}$, let $z_n(t) = \frac{1}{2}\mathbf{1}\big(t \in \big[\frac{1}{2} - \frac{1}{n+2}, \frac{1}{2}\big)\big) + \mathbf{1}\big(t \in \big[\frac{1}{2}, 1\big)\big) - \mathbf{1}\big(t \in [1, \infty)\big)$, for $t \ge 0$. Let $z(t) = \mathbf{1}\big(t \in \big[\frac{1}{2}, 1\big)\big) - \mathbf{1}\big(t \in [1, \infty)\big)$, for $t \ge 0$. Thus, we obtain that for $t \ge 0$ and $n \in \mathbb{N}$,

$$\psi(x_n, z_n)(t) = \frac{1}{6}\mathbf{1}\Big(t \in \Big[1 - \frac{1}{n+2}, 1\Big)\Big) + \frac{1}{3}\mathbf{1}\Big(t \in \Big[1, \frac{3}{2} - \frac{1}{n+2}\Big)\Big) + \frac{2}{3}\mathbf{1}\Big(t \in \Big[\frac{3}{2} - \frac{1}{n+2}, \frac{3}{2}\Big)\Big)$$
$$+ \frac{1}{3}\mathbf{1}\Big(t \in \Big[\frac{3}{2}, 2\Big)\Big) - \mathbf{1}\big(t \in [2, \infty)\big),$$

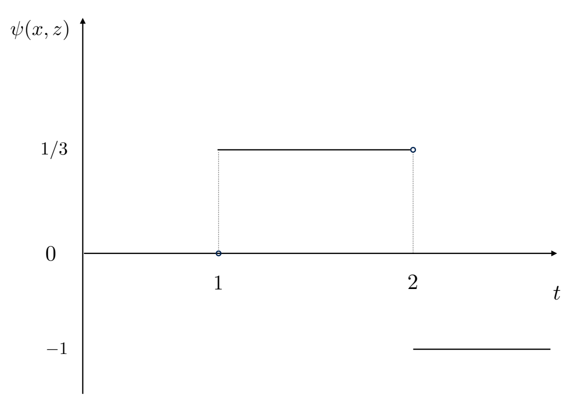
and

$$\psi(x, z)(t) = \frac{1}{3}\mathbf{1}\left(t \in [1, 2)\right) - \mathbf{1}\left(t \in [2, \infty)\right).$$

See the plots of $\psi(x_n, z_n)$ and $\psi(x, z)$ in Figure 1.



(a) The function $\psi(x_n, z_n)$



(b) The function $\psi(x, z)$

FIGURE 1. The functions $\psi(x_n, z_n)$ and $\psi(x, z)$

It is evident that $z_n \to z$ in $(\mathbb{D}, M_1)$ as $n \to \infty$ (refer to the example in page 81 of [41]). Notice that $\psi(x,z)(t) = 1/3$ for $t \in [1, 7/4]$, and thus, $\psi(x,z)(t)$ is a continuous function on $[1, 7/4]$. Therefore, $\psi(x_n, z_n) \to \psi(x,z)$ in $(\mathbb{D}([1,7/4],\mathbb{R}), M_1)$ is equivalent to $\psi(x_n, z_n) \to \psi(x,z)$ in $(\mathbb{C}([1,7/4],\mathbb{R}), \|\cdot\|)$ as $n \to \infty$ (see, e.g., §3.3 of Chapter 3 in [41]). By noting that $\sup_{1\le t\le 7/4} \|\psi(x_n,z_n)(t) - \psi(x,z)(t)\| = 1/3$, we conclude that $\psi(x_n, z_n)$ does not converge to $\psi(x,z)$ in the Skorohod M_1 topology on the domain $[1, 7/4]$ as $n \to \infty$, directly implying that the mapping ψ is *not* continuous in the Skorohod M_1 topology.

Proof of Lemma 5.3. We first observe that as a function in t, $\int_0^t z(s)dF(x(t) - x(s))$ is continuous for any continuous distribution function F, $z \in \mathbb{D}$ and $x \in \mathbb{C}_\uparrow$. (For a proof, we refer to Step 3 in the proof of Lemma 6.3 in [36].) Thus, to prove the continuity of ψ, by Lemma 5.1 in [36] (or Section 3.6 in [12]), it suffices to prove that for $t_n, t \in [0,T]$ such that $t_n \to t$ as $n \to \infty$,

$$\lim_{n\to\infty} \left| \int_0^{t_n} z_n(s) dF_{\mathrm{m}}(x_n(t_n) - x_n(s)) - \int_0^t z(s) dF_{\mathrm{m}}(x(t) - x(s)) \right| = 0. \tag{5.21}$$

Note that

$$\begin{aligned} & \left| \int_0^{t_n} z_n(s) dF_{\mathrm{m}}(x_n(t_n) - x_n(s)) - \int_0^t z(s) dF_{\mathrm{m}}(x(t) - x(s)) \right| \\ \le\ & \left| \int_0^{t_n} z_n(s) d\big(F_{\mathrm{m}}(x_n(t_n) - x_n(s)) - F_{\mathrm{m}}(x(t) - x(s))\big) \right| \\ & + \left| \int_0^{t_n} [z_n(s) - z(s)] dF_{\mathrm{m}}(x(t) - x(s)) \right| + \left| \int_t^{t_n} z(s) dF_{\mathrm{m}}(x(t) - x(s)) \right|. \end{aligned} \tag{5.22}$$

The first integral on the right hand side of (5.22) converges to zero as $n \to \infty$ because of the continuity of F_{m} and $x_n(t_n) \to x(t)$ uniformly as $n \to \infty$ (Recall that x_n and x are continuous). Since $z_n \to z$ in $(\mathbb{D}, M_1)$, $z_n(s) \to z(s)$ as $n \to \infty$ for almost all $s \in [0,T]$, and thus the second integral on the right hand side of (5.22) goes to zero as $n \to \infty$. It is evident that the third integral on the right hand side of (5.22) also goes to zero as $n \to \infty$. Therefore, we have proved (5.21) and the continuity of ψ.

To see the continuities of ϖ_k for $k = 1, ..., K$, we first note that following a similar argument as above (replace F_{m} with F_k), we obtain the continuity of the mapping

$$\varpi_k^{(2)}(x,z)(t) := -\int_0^t z(s) dF_k^c(x(t) - x(s)), \quad t \ge 0.$$

Note that $\varpi_k^{(2)}(x,z)(t)$ is continuous in t so that there is no common jumps of $\varpi_k^{(2)}(x,z)(t)$ and $z(t)$. By noting $\varpi_k(x,z)(t) = z(t) + \varpi_k^{(2)}(x,z)(t)$ and Corollary 12.7.1 in [41] (continuity of addition in $(\mathbb{D}, M_1)$), we obtain the continuities of ϖ_k for $k = 1, ..., K$.

The continuity of φ_k for $k = 1, ..., K$ can be obtained easily from either ψ (replace F_{m} by $F_{\mathrm{m}} - F_k$) or $\varpi_k^{(2)}$. The continuity of the mappings f_k, g_k for $k = 1, ..., K$ and h can be obtained from the continuity of mappings ϖ_k, φ_k and ψ by treating $z(s) = (y(t) - y(s))\lambda(s)$ and $x(s) = s$ for all $s, t \in [0,T]$. Since each of the mappings from (5.15)–(5.20) is proved to be continuous, the lemma is proved. ☐

Proof of Lemma 5.2. Note that, from (5.4), (3.18), (5.10), (3.26), (5.7) and (3.22),

$$\hat{M}_{k,1}^n(t) = \varpi_k(\xi^n, \hat{A}^n)(t), \quad \hat{M}_{k,1}(t) = \varpi_k(e, \hat{A})(t),$$

$$\hat{Z}_{k,1}^n(t) = \varphi_k(\xi^n, \hat{A}^n)(t), \quad \hat{Z}_{k,1}(t) = \varphi_k(e, \hat{A})(t),$$

$$\hat{V}_1^n(t) = \psi(\xi^n, \hat{A}^n)(t), \quad \hat{V}_1(t) = \psi(e, \hat{A})(t),$$

for $t \ge 0$ and $k = 1, ..., K$.

Note that $F_{\rm m}(\cdot)$ is a monotone function, and is thus differentiable *a.e.* Let $f_{\rm m}(\cdot)$ be the *a.e.* derivative of $F_{\rm m}(\cdot)$. Applying the Taylor expansion to the integrands of (5.9), we have

$$\begin{aligned}\hat{V}_3^n(t) &= \int_0^t \sqrt{n}(F_{\rm m}(\xi^n(t)-\xi^n(s)) - F_{\rm m}(t-s))\lambda(s)ds\\ &= \int_0^t \sqrt{n}(\xi^n(t)-\xi^n(s)-t+s)f_{\rm m}(t-s)\lambda(s)ds + o_p(1)\\ &= -\int_0^t \sqrt{n}(\xi^n(t)-\xi^n(s)-t+s)\lambda(s)dF_{\rm m}(t-s) + o_p(1)\\ &= -\int_0^t (\hat{\xi}^n(t)-\hat{\xi}^n(s))\lambda(s)dF_{\rm m}(t-s) + o_p(1).\end{aligned}$$

Thus we see that

$$\hat{V}_3^n(t) = h(\hat{\xi}^n)(t) + o_p(1), \quad \hat{V}_3(t) = h(-\hat{J})(t), \quad t \geq 0.$$

Similarly, we also have that for $t \geq 0$ and $k = 1, ..., K$,

$$\begin{aligned}\hat{M}_{k,3}^n(t) &= f_k(\hat{\xi}^n)(t) + o_p(1), \quad \hat{M}_{k,3}(t) = f_k(-\hat{J})(t),\\ \hat{Z}_{k,3}^n(t) &= g_k(\hat{\xi}^n)(t) + o_p(1), \quad \hat{Z}_{k,3}(t) = g_k(-\hat{J})(t).\end{aligned}$$

By Assumption 1, Lemma 2.1 and the continuity of the mapping Φ (Lemma 5.3) as well as the continuous mapping theorem, we immediately obtain the convergence in (5.13). □

In the next two lemmas, we prove the convergence of $\big(\hat{\boldsymbol{M}}_2^n, \hat{\boldsymbol{Z}}_2^n, \hat{V}_2^n\big)$ by Theorem 13.5 in [5]. Since the limits of $\hat{\boldsymbol{M}}_2^n$, $\hat{\boldsymbol{Z}}_2^n$ and $\hat{V}_2^n$ are continuous, we will apply the criterion in Theorem 13.5 in [5] to prove the convergence of $\big(\hat{\boldsymbol{M}}_2^n, \hat{\boldsymbol{Z}}_2^n, \hat{V}_2^n\big)$ in the Skorohod J_1 topology.

Remark 5.2. We emphasize that in the proof for the convergence of $\big(\hat{\boldsymbol{M}}_2^n, \hat{\boldsymbol{Z}}_2^n, \hat{V}_2^n\big)$, the continuity condition on the distribution function $F(\cdot)$ is not required, as can be seen below. The results in Lemmas 5.5–5.6 hold for any general distribution function $F(\cdot)$.

In Lemma 5.5 we prove the convergence of finite dimensional distributions which is the first condition in Theorem 13.5 in [5]. We first present the following lemma on random sequences satisfying the strong mixing (α-mixing) condition, which is a special case of Lemma 2.1 in [10] (choosing $p = 2$ and $C = 1$ and the random variables bounded by 1 *a.s.*).

Lemma 5.4. *Suppose that $\{\zeta_n : n \geq 1\}$ is a sequence of random variables satisfying the α-mixing condition with the mixing coefficients $\{\alpha_n^{\zeta}\colon n \geq 1\}$. Let $\Psi_1 \in \mathcal{F}_k := \sigma\{\zeta_j : 1 \leq j \leq k\}$ and $\Psi_2 \in \mathcal{G}_{k+n} := \sigma\{\zeta_j : j \geq k+n\}$. Moreover, assume that $|\Psi_1| \leq 1$ a.s. and $|\Psi_2| \leq 1$ a.s. Then*

$$\big|E[\Psi_1\Psi_2] - E[\Psi_1]E[\Psi_2]\big| \leq 6(\alpha_n^{\zeta})^{1/2}. \tag{5.23}$$

Lemma 5.5. *The finite-dimensional distributions of $\big(\hat{\boldsymbol{M}}_2^n, \hat{\boldsymbol{Z}}_2^n, \hat{V}_2^n\big)$ converge to those of $\big(\hat{\boldsymbol{M}}_2, \hat{\boldsymbol{Z}}_2, \hat{V}_2\big)$ as $n \to \infty$.*

Proof. We first introduce some additional processes. For $t \geq 0$, we divide the interval $[0, t]$ by the sequence $\{s_i^l : 0 \leq i \leq l\}$: $0 = s_0^l < s_1^l < ... < s_l^l = t$ satisfying $\max_{1\leq i\leq l} |s_i^l - s_{i-1}^l| \to 0$, as $l \to \infty$. We define, for $t \geq 0$ and $k = 1, ..., K$,

$$\tilde{M}_{k,2,l}^n(t) := -\int_0^t \int_{\mathbb{R}_+^K} \mathbf{1}_{k,l,t}(s, \boldsymbol{x}) d\hat{K}^n(\bar{a}(s), \boldsymbol{x}), \tag{5.24}$$

$$\hat{M}_{k,2,l}^n(t) := -\int_0^t \int_{\mathbb{R}_+^K} \mathbf{1}_{k,l,t}(s, \boldsymbol{x}, \xi^n) d\hat{K}^n(\bar{A}^n(s), \boldsymbol{x}), \tag{5.25}$$

$$\tilde{Z}^n_{k,2,l}(t) := \int_0^t \int_{\mathbb{R}^K_+} [\mathbf{1}_{k,l,t}(s,\boldsymbol{x}) - \mathbf{1}_{\mathrm{m},l,t}(s,\boldsymbol{x})] d\hat{K}^n(\bar{a}(s),\boldsymbol{x}), \tag{5.26}$$

$$\hat{Z}^n_{k,2,l}(t) := \int_0^t \int_{\mathbb{R}^K_+} [\mathbf{1}_{k,l,t}(s,\boldsymbol{x},\xi^n) - \mathbf{1}_{\mathrm{m},l,t}(s,\boldsymbol{x},\xi^n)] d\hat{K}^n(\bar{A}^n(s),\boldsymbol{x}), \tag{5.27}$$

$$\tilde{V}^n_{2,l}(t) := \int_0^t \int_{\mathbb{R}^K_+} \mathbf{1}_{\mathrm{m},l,t}(s,\boldsymbol{x}) d\hat{K}^n(\bar{a}(s),\boldsymbol{x}), \tag{5.28}$$

$$\hat{V}^n_{2,l}(t) := \int_0^t \int_{\mathbb{R}^K_+} \mathbf{1}_{\mathrm{m},l,t}(s,\boldsymbol{x},\xi^n) d\hat{K}^n(\bar{A}^n(s),\boldsymbol{x}), \tag{5.29}$$

where $\mathbf{1}_{k,l,t}(\cdot,\cdot)$ and $\mathbf{1}_{\mathrm{m},l,t}(\cdot,\cdot)$ are defined in (4.11) and (4.12), respectively, and

$$\mathbf{1}_{k,l,t}(s,\boldsymbol{x},\xi^n) := \sum_{i=1}^{l} \mathbf{1}(s^l_{i-1} < s \le s^l_i) \mathbf{1}(x_k \le \xi^n(t) - \xi^n(s^l_i)),$$
$$\mathbf{1}_{\mathrm{m},l,t}(s,\boldsymbol{x},\xi^n) := \sum_{i=1}^{l} \mathbf{1}(s^l_{i-1} < s \le s^l_i) \mathbf{1}(x_j \le \xi^n(t) - \xi^n(s^l_i),\ \forall j = 1,...,K).$$

We set $\tilde{M}^n_{k,2,l} := \{\tilde{M}^n_{k,2,l}(t) : t \ge 0\}$, $\hat{M}^n_{k,2,l} := \{\hat{M}^n_{k,2,l}(t) : t \ge 0\}$, $\tilde{Z}^n_{k,2,l} := \{\tilde{Z}^n_{k,2,l}(t) : t \ge 0\}$, $\hat{Z}^n_{k,2,l} := \{\hat{Z}^n_{k,2,l}(t) : t \ge 0\}$, $k = 1,...,K$, $\tilde{V}^n_{2,l} := \{\tilde{V}^n_{2,l}(t) : t \ge 0\}$ and $\hat{V}^n_{2,l} := \{\hat{V}^n_{2,l}(t) : t \ge 0\}$. Note that, for $k = 1,...,K$ and $t \ge 0$, $\tilde{M}^n_{k,2,l}(t)$, $\hat{M}^n_{k,2,l}(t)$, $\tilde{Z}^n_{k,2,l}(t)$, $\hat{Z}^n_{k,2,l}(t)$, $\tilde{V}^n_{2,l}(t)$ and $\hat{V}^n_{2,l}(t)$ can be rewritten as

$$\tilde{M}^n_{k,2,l}(t) = -\sum_{i=1}^{l} \Delta\hat{K}^n\left((\bar{a}(s^l_{i-1}),\mathbf{0}); (\bar{a}(s^l_i), (t - s^l_i)\boldsymbol{e}_k)\right),$$
$$\hat{M}^n_{k,2,l}(t) = -\sum_{i=1}^{l} \Delta\hat{K}^n\left((\bar{A}^n(s^l_{i-1}),\mathbf{0}); (\bar{A}^n(s^l_i), (\xi^n(t) - \xi^n(s^l_i))\boldsymbol{e}_k)\right),$$
$$\tilde{V}^n_{2,l}(t) = \sum_{i=1}^{l} \Delta\hat{K}^n\left((\bar{a}(s^l_{i-1}),\mathbf{0}); (\bar{a}(s^l_i), (t - s^l_i)\boldsymbol{e})\right),$$
$$\hat{V}^n_{2,l}(t) = \sum_{i=1}^{l} \Delta\hat{K}^n\left((\bar{A}^n(s^l_{i-1}),\mathbf{0}); (\bar{A}^n(s^l_i), (\xi^n(t) - \xi^n(s^l_i))\boldsymbol{e})\right),$$
$$\tilde{Z}^n_{k,2,l}(t) = \sum_{i=1}^{l} \left\{\Delta\hat{K}^n\left((\bar{a}(s^l_{i-1}),\mathbf{0}); (\bar{a}(s^l_i), (t - s^l_i)\boldsymbol{e}_k)\right) - \Delta\hat{K}^n\left((\bar{a}(s^l_{i-1}),\mathbf{0}); (\bar{a}(s^l_i), (t - s^l_i)\boldsymbol{e})\right)\right\},$$
$$\hat{Z}^n_{k,2,l}(t) = \sum_{i=1}^{l} \left\{\Delta\hat{K}^n\left((\bar{A}^n(s^l_{i-1}),\mathbf{0}); (\bar{A}^n(s^l_i), (\xi^n(t) - \xi^n(s^l_i))\boldsymbol{e}_k)\right)\right.$$
$$\left. -\Delta\hat{K}^n\left((\bar{A}^n(s^l_{i-1}),\mathbf{0}); (\bar{A}^n(s^l_i), (\xi^n(t) - \xi^n(s^l_i))\boldsymbol{e})\right)\right\}.$$

Set $\tilde{\boldsymbol{M}}^n_{2,l} := (\tilde{M}^n_{1,2,l}, ..., \tilde{M}^n_{K,2,l})$, $\hat{\boldsymbol{M}}^n_{2,l} := (\hat{M}^n_{1,2,l}, ..., \hat{M}^n_{K,2,l})$, $\tilde{\boldsymbol{Z}}^n_{2,l} := (\tilde{Z}^n_{1,2,l}, ..., \tilde{Z}^n_{K,2,l})$ and $\hat{\boldsymbol{Z}}^n_{2,l} := (\hat{Z}^n_{1,2,l}, ..., \hat{Z}^n_{K,2,l})$. For $t^k_{i,1}, t^k_{i',2}, t_j \ge 0$, $c^k_{i,1}, c^k_{i',2}, c_j \in \mathbb{R}$, and positive integers $I_{k,1}, I_{k,2}$ and I_3, where $i = 1,...,I_{k,1}$, $i' = 1,...,I_{k,2}$, $j = 1,...,I_3$ and $k = 1,...,K$, with the weak convergence of $\hat{K}^n$ in (3.3), we see that, as $n \to \infty$,

$$\sum_{k=1}^{K}\left[\sum_{i=1}^{I_{k,1}} c_{i,1}^k \tilde{M}_{k,2,l}^n(t_{i,1}^k) + \sum_{i'=1}^{I_{k,2}} c_{i',2}^k \tilde{Z}_{k,2,l}^n(t_{i',2}^k)\right] + \sum_{j=1}^{I_3} c_j \tilde{V}_{2,l}^n(t_j)$$
$$\Rightarrow \sum_{k=1}^{K}\left[\sum_{i=1}^{I_{k,1}} c_{i,1}^k \hat{M}_{k,2,l}(t_{i,1}^k) + \sum_{i'=1}^{I_{k,2}} c_{i',2}^k \hat{Z}_{k,2,l}(t_{i',2}^k)\right] + \sum_{j=1}^{I_3} c_j \hat{V}_{2,l}(t_j),$$

where we recall $\hat{M}_{k,2,l}$, $\hat{Z}_{k,2,l}$ and $\hat{V}_{2,l}$ are defined in (4.8), (4.10) and (4.9), respectively, for $k = 1, ..., K$. By the Cramer-Wold theorem (see Theorem 3.9.5 in [11]), we have

$$\big(\tilde{\boldsymbol{M}}_{2,l}^n, \tilde{\boldsymbol{Z}}_{2,l}^n, \tilde{V}_{2,l}^n\big) \xrightarrow{d_f} \big(\hat{\boldsymbol{M}}_{2,l}, \hat{\boldsymbol{Z}}_{2,l}, \hat{V}_{2,l}\big) \quad \text{as} \quad n \to \infty,$$

where $\hat{\boldsymbol{M}}_{2,l} := \big(\hat{M}_{1,2,l}, ..., \hat{M}_{K,2,l}\big)$ and $\hat{\boldsymbol{Z}}_{2,l} := \big(\hat{Z}_{1,2,l}, ..., \hat{Z}_{K,2,l}\big)$. Then, we have

$$\big(\hat{\boldsymbol{M}}_1^n, \hat{\boldsymbol{Z}}_1^n, \hat{V}_1^n, \hat{\boldsymbol{M}}_3^n, \hat{\boldsymbol{Z}}_3^n, \hat{V}_3^n, \tilde{\boldsymbol{M}}_{2,l}^n, \tilde{\boldsymbol{Z}}_{2,l}^n, \tilde{V}_{2,l}^n\big) \xrightarrow{d_f} \big(\hat{\boldsymbol{M}}_1, \hat{\boldsymbol{Z}}_1, \hat{V}_1, \hat{\boldsymbol{M}}_3, \hat{\boldsymbol{Z}}_3, \hat{V}_3, \hat{\boldsymbol{M}}_{2,l}, \hat{\boldsymbol{Z}}_{2,l}, \hat{V}_{2,l}\big) \quad \text{as} \quad n \to \infty,$$

since $\big(\hat{\boldsymbol{M}}_1^n, \hat{\boldsymbol{Z}}_1^n, \hat{V}_1^n, \hat{\boldsymbol{M}}_3^n, \hat{\boldsymbol{Z}}_3^n, \hat{V}_3^n\big)$ and $\big(\tilde{\boldsymbol{M}}_{2,l}^n, \tilde{\boldsymbol{Z}}_{2,l}^n, \tilde{V}_{2,l}^n\big)$ are independent.

Now it suffices to show the difference between $\big(\hat{\boldsymbol{M}}_{2,l}^n, \hat{\boldsymbol{Z}}_{2,l}^n, \hat{V}_{2,l}^n\big)$ and $\big(\tilde{\boldsymbol{M}}_{2,l}^n, \tilde{\boldsymbol{Z}}_{2,l}^n, \tilde{V}_{2,l}^n\big)$ is asymptotically negligible in probability as $l \to \infty$, and the difference between $\big(\hat{\boldsymbol{M}}_{2,l}^n, \hat{\boldsymbol{Z}}_{2,l}^n, \hat{V}_{2,l}^n\big)$ and $\big(\hat{\boldsymbol{M}}_2^n, \hat{\boldsymbol{Z}}_2^n, \hat{V}_2^n\big)$ is asymptotically negligible in probability as $n \to \infty$ and $l \to \infty$, i.e., for any $\epsilon > 0$, $T > 0$ and each $k = 1, ..., K$,

$$\lim_{n\to\infty} P\left(\sup_{0\le t\le T} |\hat{M}_{k,2,l}^n(t) - \tilde{M}_{k,2,l}^n(t)| > \epsilon\right) = 0, \tag{5.30}$$

$$\lim_{n\to\infty} P\left(\sup_{0\le t\le T} |\hat{Z}_{k,2,l}^n(t) - \tilde{Z}_{k,2,l}^n(t))| > \epsilon\right) = 0, \tag{5.31}$$

$$\lim_{n\to\infty} P\left(\sup_{0\le t\le T} |\hat{V}_{2,l}^n(t) - \tilde{V}_{2,l}^n(t)| > \epsilon\right) = 0, \tag{5.32}$$

and,

$$\lim_{l\to\infty} \limsup_{n\to\infty} P\left(|\hat{M}_{k,2}^n(t) - \hat{M}_{k,2,l}^n(t)| > \epsilon\right) = 0, \tag{5.33}$$

$$\lim_{l\to\infty} \limsup_{n\to\infty} P\left(|\hat{Z}_{k,2}^n(t) - \hat{Z}_{k,2,l}^n(t)| > \epsilon\right) = 0, \tag{5.34}$$

$$\lim_{l\to\infty} \limsup_{n\to\infty} P\left(|\hat{V}_2^n(t) - \hat{V}_{2,l}^n(t)| > \epsilon\right) = 0. \tag{5.35}$$

The claims in (5.30)-(5.32) follow directly by noting that $\hat{U}$ in (3.1) and $\bar{a}$ in (2.2) are continuous, together with (3.2), Assumption 2, Theorem 3.1 and Lemma 2.1.

Next we show (5.33)–(5.35) hold. We will focus on (5.33), and (5.34) and (5.35) follow from a similar argument. We first observe that

$$\hat{M}_{k,2}^n(t) - \hat{M}_{k,2,l}^n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{A^n(t)} \chi_{k,i,l}^n(t),$$

where

$$\chi_{k,i,l}^n(t) = \sum_{p=1}^{l} \big(\mathbf{1}(s_{p-1}^l < \tau_i^n \le s_p^l)\big[\mathbf{1}(\xi^n(t) - \xi^n(s_p^l) < \eta_k^i \le \xi^n(t) - \xi^n(\tau_i^n))$$

$$- F_k(\xi^n(t) - \xi^n(\tau_i^n)) + F_k(\xi^n(t) - \xi^n(s_p^l))]\Big),$$

and $\{s_i^l : 0 \le i \le l\}$ is a partition of $[0,t]$. For fixed $t > 0$, let $\varsigma = \varsigma(t) > 0$ such that $\varsigma > \bar{a}(t)$. Thus, we obtain, for any $\epsilon > 0$,

$$\begin{aligned} &P\left(|\hat{M}_{k,2}^n(t) - \hat{M}_{k,2,l}^n(t)| > \epsilon\right) \\ &\le P(A^n(t) > n\varsigma) + P\Big(A^n(t) \le n\varsigma,\ |\hat{M}_{k,2}^n(t) - \hat{M}_{k,2,l}^n(t)| > \epsilon\Big) \\ &\le P(A^n(t) > n\varsigma) + \frac{1}{\epsilon^4} E\big[\mathbf{1}(A^n(t) \le n\varsigma)|\hat{M}_{k,2}^n(t) - \hat{M}_{k,2,l}^n(t)|^4\big]. \end{aligned}$$

By Assumption 2, we have $\limsup_{n\to\infty} P(A^n(t) > n\varsigma) = 0$. For the second term, we have

$$\begin{aligned} &E\big[\mathbf{1}(A^n(t) \le n\varsigma)|\hat{M}_{k,2}^n(t) - \hat{M}_{k,2,l}^n(t)|^4\big] \\ &= \frac{1}{n^2} E\left[\sum_{i=1}^{A^n(t)\wedge\lfloor n\varsigma\rfloor} \chi_{k,i,l}^n(t)^4\right] + \frac{4}{n^2} E\left[\sum_{i,j=1,\ i\neq j}^{A^n(t)\wedge\lfloor n\varsigma\rfloor} \chi_{k,i,l}^n(t)\chi_{k,j,l}^n(t)^3\right] \\ &\quad + \frac{6}{n^2} E\left[\sum_{i,j=1,\ i\neq j}^{A^n(t)\wedge\lfloor n\varsigma\rfloor} \chi_{k,i,l}^n(t)^2\chi_{k,j,l}^n(t)^2\right]. \end{aligned} \tag{5.36}$$

By conditioning, we have that for the first term on the right hand side of (5.36),

$$\begin{aligned} \frac{1}{n^2} E\left[\sum_{i=1}^{A^n(t)\wedge\lfloor n\varsigma\rfloor} \chi_{k,i,l}^n(t)^4\right] &= \frac{1}{n^2} E\left[\sum_{i=1}^{A^n(t)\wedge\lfloor n\varsigma\rfloor} E\big[\chi_{k,i,l}^n(t)^4 | A^n(s), \xi^n(s) : 0 \le s \le t\big]\right] \\ &\le \frac{16\varsigma}{n} \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

By Lemma 5.4 and conditioning, together with $E[\chi_{k,i,l}^n(t)] = 0$, we obtain

$$\begin{aligned} \frac{4}{n^2} E\left[\sum_{i,j=1,\ i\neq j}^{A^n(t)\wedge\lfloor n\varsigma\rfloor} \chi_{k,i,l}^n(t)\chi_{k,j,l}^n(t)^3\right] &\le \frac{8}{n^2} E\left[\sum_{i,j=1,\ i<j}^{A^n(t)\wedge\lfloor n\varsigma\rfloor} 6\alpha_{j-i}^{1/2}\right] \\ &\le \frac{48}{n^2} \sum_{i,j=1,\ i<j}^{\lfloor n\varsigma\rfloor} \alpha_{j-i}^{1/2} \to 0 \quad \text{as} \quad n \to \infty, \end{aligned}$$

where the convergence follows from the Assumption 1. Now for the third term, by Lemma 5.4 and conditioning, we obtain

$$\frac{6}{n^2} E\left[\sum_{i,j=1,\ i\neq j}^{A^n(t)\wedge\lfloor n\varsigma\rfloor} \chi_{k,i,l}^n(t)^2\chi_{k,j,l}^n(t)^2\right] \le \frac{12}{n^2} E\left[\sum_{i,j=1,\ i<j}^{A^n(t)\wedge\lfloor n\varsigma\rfloor} \Big(E[\chi_{k,i,l}^n(t)^2]E[\chi_{k,j,l}^n(t)^2] + 6\alpha_{j-i}^{1/2}\Big)\right].$$

It is clear that under Assumption 1,

$$\frac{1}{n^2} \sum_{i,j=1,\ i<j}^{\lfloor n\varsigma\rfloor} \alpha_{j-i}^{1/2} \to 0 \quad \text{as} \quad n \to \infty.$$

It suffices to show that

$$\lim_{l\to\infty} \limsup_{n\to\infty} \frac{1}{n^2} E\left[\sum_{i,j=1,\ i<j}^{A^n(t)\wedge\lfloor n\varsigma\rfloor} E[\chi_{k,i,l}^n(t)^2]E[\chi_{k,j,l}^n(t)^2]\right] = 0.$$

We observe that by conditioning,

$$E[\chi^n_{k,i,l}(t)^2] = E\Bigg[\sum_{p=1}^{l}\big(\mathbf{1}(s^l_{p-1} < \tau^n_i \le s^l_p)\big[F_k(\xi^n(t)-\xi^n(\tau^n_i)) - F_k(\xi^n(t)-\xi^n(s^l_p))\big] \times \big[1 - F_k(\xi^n(t)-\xi^n(\tau^n_i)) + F_k(\xi^n(t)-\xi^n(s^l_p))\big]\big)\Bigg]$$

$$\le E\Bigg[\sum_{p=1}^{l}\big(\mathbf{1}(s^l_{p-1} < \tau^n_i \le s^l_p)\big[F_k(\xi^n(t)-\xi^n(\tau^n_i)) - F_k(\xi^n(t)-\xi^n(s^l_p))\big]\Bigg],$$

and similarly,

$$E[\chi^n_{k,j,l}(t)^2] \le E\Bigg[\sum_{p=1}^{l}\big(\mathbf{1}(s^l_{p-1} < \tau^n_j \le s^l_p)\big[F_k(\xi^n(t)-\xi^n(\tau^n_j)) - F_k(\xi^n(t)-\xi^n(s^l_p))\big]\Bigg].$$

Thus, we have

$$\frac{1}{n^2}E\Bigg[\sum_{i,j=1,\ i<j}^{A^n(t)\wedge\lfloor n\varsigma\rfloor} E[\chi^n_{k,i,l}(t)^2]E[\chi^n_{k,j,l}(t)^2]\Bigg]$$

$$\le \frac{1}{n^2}E\Bigg[\sum_{i,j=1,\ i<j}^{A^n(t)\wedge\lfloor n\varsigma\rfloor}\Big[\sum_{p=1}^{l}\Big(\mathbf{1}(s^l_{p-1} < \tau^n_i \le s^l_p)\big[F_k(\xi^n(t)-\xi^n(\tau^n_i)) - F_k(\xi^n(t)-\xi^n(s^l_p))\big]\Big)\Big] \times \Big[\sum_{p=1}^{l}\big(\mathbf{1}(s^l_{p-1} < \tau^n_j \le s^l_p)\big[F_k(\xi^n(t)-\xi^n(\tau^n_i)) - F_k(\xi^n(t)-\xi^n(s^l_p))\big]\big)\Big]\Bigg]$$

$$\le \frac{1}{n^2}E\Bigg[\Bigg(\sum_{i=1}^{A^n(t)\wedge\lfloor n\varsigma\rfloor}\Big[\sum_{p=1}^{l}\Big(\mathbf{1}(s^l_{p-1} < \tau^n_i \le s^l_p)\big[F_k(\xi^n(t)-\xi^n(\tau^n_i)) - F_k(\xi^n(t)-\xi^n(s^l_p))\big]\Big)\Big]\Bigg)^2\Bigg]$$

$$= E\Bigg[\mathbf{1}(\bar{A}^n(t)\le\varsigma)\Bigg(\sum_{p=1}^{l}\int_{s^l_{p-1}}^{s^l_p}\big[F_k(\xi^n(t)-\xi^n(u)) - F_k(\xi^n(t)-\xi^n(s^l_p))\big]d\bar{A}^n(u)\Bigg)^2\Bigg].$$

Now it suffices to show that

$$\lim_{l\to\infty}\limsup_{n\to\infty} E\Bigg[\mathbf{1}(\bar{A}^n(t)\le\varsigma)\Bigg(\sum_{p=1}^{l}\int_{s^l_{p-1}}^{s^l_p}\big[F_k(\xi^n(t)-\xi^n(u)) - F_k(\xi^n(t)-\xi^n(s^l_p))\big]d\bar{A}^n(u)\Bigg)^2\Bigg] = 0. \tag{5.37}$$

By the convergence of $\bar{A}^n \Rightarrow \bar{a}$ under Assumption 2 and $\xi^n \Rightarrow e$ in Lemma 2.1 under Assumption 3, the continuous mapping theorem and the uniform integrability of

$$\Bigg\{\mathbf{1}(\bar{A}^n(t)\le\varsigma)\Bigg(\sum_{p=1}^{l}\int_{s^l_{p-1}}^{s^l_p}\big[F_k(\xi^n(t)-\xi^n(u)) - F_k(\xi^n(t)-\xi^n(s^l_p))\big]d\bar{A}^n(u)\Bigg)^2 :\ n\ge 1\Bigg\},$$

we immediately have

$$\limsup_{n\to\infty} E\Bigg[\mathbf{1}(\bar{A}^n(t)\le\varsigma)\Bigg(\sum_{p=1}^{l}\int_{s^l_{p-1}}^{s^l_p}\big[F_k(t-u) - F_k(t-s^l_p)\big]d\bar{A}^n(u)\Bigg)^2\Bigg] = \Bigg(\sum_{p=1}^{l}\int_{s^l_{p-1}}^{s^l_p}\big[F_k(t-u) - F_k(t-s^l_p)\big]d\bar{a}(u)\Bigg)^2.$$

Then, by Lebesgue's theorem, the right hand side of the above equation converges to zero as $l \to \infty$. Therefore, we have proved (5.37), which completes the proof of (5.33)-(5.35). □

We next complete the proof of the convergence of the processes $\big(\hat{\boldsymbol{M}}^n_2, \hat{\boldsymbol{Z}}^n_2, \hat{V}^n_2\big)$.

Lemma 5.6.

$$\big(\hat{\boldsymbol{M}}^n_2, \hat{\boldsymbol{Z}}^n_2, \hat{V}^n_2\big) \Rightarrow \big(\hat{\boldsymbol{M}}_2, \hat{\boldsymbol{Z}}_2, \hat{V}_2\big) \quad in \quad (\mathbb{D}^{2K+1}, J_1) \quad as \quad n \to \infty. \tag{5.38}$$

Proof. We apply Theorem 13.5 in [5]. By Lemma 5.5, since the limits are continuous, it suffices to show the third condition is satisfied, that is, for $0 \le r \le s \le t \le T$ and $T > 0$, and for $\epsilon > 0$ and $k = 1, \dots, K$,

$$P\left(\left|\hat{M}^n_{k,2}(t) - \hat{M}^n_{k,2}(s)\right| \wedge \left|\hat{M}^n_{k,2}(s) - \hat{M}^n_{k,2}(r)\right| \ge \epsilon\right) \le \frac{1}{\epsilon^4}(H_{M,k}(t) - H_{M,k}(r))^2, \tag{5.39}$$

$$P\left(\left|\hat{Z}^n_{k,2}(t) - \hat{Z}^n_{k,2}(s)\right| \wedge \left|\hat{Z}^n_{k,2}(s) - \hat{Z}^n_{k,2}(r)\right| \ge \epsilon\right) \le \frac{1}{\epsilon^4}(H_{Z,k}(t) - H_{Z,k}(r))^2,$$

$$P\left(\left|\hat{V}^n_2(t) - \hat{V}^n_2(s)\right| \wedge \left|\hat{V}^n_2(s) - \hat{V}^n_2(r)\right| \ge \epsilon\right) \le \frac{1}{\epsilon^4}(H_V(t) - H_V(r))^2,$$

where $H_{M,k}(\cdot)$, $H_{Z,k}(\cdot)$ and $H_V(\cdot)$ are nondecreasing and continuous functions on $[0,T]$ (to be determined below). We focus on the proof for $\hat{M}^n_{k,2}$ since the proof for $\hat{V}^n_2$ follows from a similar argument, and the property for $\hat{Z}^n_{k,2}$ follows from results for $\hat{M}^n_{k,2}$ and $\hat{V}^n_2$.

We first observe that it is convenient to represent $\hat{M}^n_{k,2}(t)$ for $k = 1, ..., K$ in (5.5) as

$$\hat{M}^n_{k,2}(t) := -\frac{1}{\sqrt{n}} \sum_{i=1}^{A^n(t)} \big(\mathbf{1}(\eta^i_k \le \xi^n(t) - \xi^n(\tau^n_i)) - F_k(\xi^n(t) - \xi^n(\tau^n_i))\big), \quad t \ge 0. \tag{5.40}$$

Fix $\kappa > 0$ such that $\kappa > \bar{a}(T)$. We write

$$\begin{aligned}
&P\left(\left|\hat{M}^n_{k,2}(t) - \hat{M}^n_{k,2}(s)\right| \wedge \left|\hat{M}^n_{k,2}(s) - \hat{M}^n_{k,2}(r)\right| \ge \epsilon\right) \\
&\le P\left(A^n(T) > n\kappa\right) + P\left(A^n(T) \le n\kappa,\ \left|\hat{M}^n_{k,2}(t) - \hat{M}^n_{k,2}(s)\right| \wedge \left|\hat{M}^n_{k,2}(s) - \hat{M}^n_{k,2}(r)\right| \ge \epsilon\right) \\
&\le P\left(A^n(T) > n\kappa\right) + \frac{1}{\epsilon^4} E\left[\mathbf{1}(A^n(T) \le n\kappa)\left|\hat{M}^n_{k,2}(t) - \hat{M}^n_{k,2}(s)\right|^2 \left|\hat{M}^n_{k,2}(s) - \hat{M}^n_{k,2}(r)\right|^2\right] \\
&\le P\left(A^n(T) > n\kappa\right) + \frac{1}{\epsilon^4} \left(E\left[\mathbf{1}(A^n(T) \le n\kappa)\left|\hat{M}^n_{k,2}(t) - \hat{M}^n_{k,2}(s)\right|^4\right]\right)^{1/2} \\
&\qquad \times \left(E\left[\mathbf{1}(A^n(T) \le n\kappa)\left|\hat{M}^n_{k,2}(s) - \hat{M}^n_{k,2}(r)\right|^2\right]\right)^{1/2},
\end{aligned}$$

where the last inequality follows from Cauchy-Schwartz inequality. Since $P(\bar{A}^n(T) \ge \kappa) \to 0$ as $n \to \infty$, it suffices to show that for all $0 \le s \le t \le T$ and $T > 0$,

$$E\left[\mathbf{1}(\bar{A}^n(T) \le \kappa)\left|\hat{M}^n_{k,2}(t) - \hat{M}^n_{k,2}(s)\right|^4\right] \le (H_{M,k}(t) - H_{M,k}(s))^2.$$

We have

$$\begin{aligned}
&E\left[\mathbf{1}(\bar{A}^n(T) \le \kappa)\left|\hat{M}^n_{k,2}(t) - \hat{M}^n_{k,2}(s)\right|^4\right] \\
&= E\Bigg[\mathbf{1}(\bar{A}^n(T) \le \kappa)\Bigg|\frac{1}{\sqrt{n}} \sum_{i=1}^{A^n(t)} \left[\mathbf{1}(\eta^i_k + \xi^n(\tau^n_i) \le \xi^n(t)) - F_k(\xi^n(t) - \xi^n(\tau^n_i))\right] \\
&\qquad - \frac{1}{\sqrt{n}} \sum_{i=1}^{A^n(s)} \left[\mathbf{1}(\eta^i_k + \xi^n(\tau^n_i) \le \xi^n(s)) - F_k(\xi^n(s) - \xi^n(\tau^n_i))\right]\Bigg|^4\ \Bigg]
\end{aligned}$$

$$
\begin{aligned}
&= E\Bigg[\mathbf{1}(\bar{A}^n(T) \le \kappa)\Bigg|\frac{1}{\sqrt{n}} \sum_{i=1}^{A^n(s)} \zeta^n_{i,k}(s,t) \\
&\qquad\qquad + \frac{1}{\sqrt{n}} \sum_{i=A^n(s)+1}^{A^n(t)} \big[\mathbf{1}(\eta^i_k + \xi^n(\tau^n_i) \le \xi^n(t)) - F_k(\xi^n(t) - \xi^n(\tau^n_i))\big]\Bigg|^4\Bigg] \\
&\le 8E\Bigg[\mathbf{1}(\bar{A}^n(T) \le \kappa)\Bigg|\frac{1}{\sqrt{n}} \sum_{i=1}^{A^n(s)} \zeta^n_{i,k}(s,t)\Bigg|^4\Bigg] \\
&\quad + 8E\Bigg[\mathbf{1}(\bar{A}^n(T) \le \kappa)\Bigg|\frac{1}{\sqrt{n}} \sum_{i=A^n(s)+1}^{A^n(t)} \big[\mathbf{1}(\eta^i_k + \xi^n(\tau^n_i) \le \xi^n(t)) - F_k(\xi^n(t) - \xi^n(\tau^n_i))\big]\Bigg|^4\Bigg], \quad (5.41)
\end{aligned}
$$

where

$$
\zeta^n_{i,k}(s,t) := \mathbf{1}(\eta^i_k + \xi^n(\tau^n_i) \in (\xi^n(s), \xi^n(t)]) - F_k(\xi^n(t) - \xi^n(\tau^n_i)) + F_k(\xi^n(s) - \xi^n(\tau^n_i)).
$$

We next provide an upper bound for the two terms on the right hand of (5.41). For the first term, we have

$$
\begin{aligned}
&E\Bigg[\mathbf{1}(\bar{A}^n(T) \le \kappa)\Bigg|\frac{1}{\sqrt{n}} \sum_{i=1}^{A^n(s)} \zeta^n_{i,k}(s,t)\Bigg|^4\Bigg] \\
&= \frac{1}{n^2}E\Bigg[\mathbf{1}(\bar{A}^n(T) \le \kappa) \sum_{i=1}^{A^n(s)} (\zeta^n_{i,k}(s,t))^4\Bigg] + \frac{4}{n^2}E\Bigg[\mathbf{1}(\bar{A}^n(T) \le \kappa) \sum_{i,j=1,i\neq j}^{A^n(s)} \zeta^n_{i,k}(s,t)(\zeta^n_{j,k}(s,t))^3\Bigg] \\
&\qquad + \frac{6}{n^2}E\Bigg[\mathbf{1}(\bar{A}^n(T) \le \kappa) \sum_{i,j=1,i\neq j}^{A^n(s)} (\zeta^n_{i,k}(s,t))^2(\zeta^n_{j,k}(s,t))^2\Bigg]. \quad (5.42)
\end{aligned}
$$

It is easy to see that the first term on the right hand side of (5.42) is in the order of $O(1/n)$ and converges to 0 as $n \to \infty$. For the second term, by Lemma 5.4 and conditioning, since $E[\zeta^n_{i,k}(s,t)] = 0$, we have that for $i, j \ge 1$ and $j > i$,

$$
E\big[\zeta^n_{i,k}(s,t)(\zeta^n_{j,k}(s,t))^3\big] \le 6\alpha^{1/2}_{j-i}.
$$

Thus,

$$
\begin{aligned}
&\frac{4}{n^2}E\Bigg[\mathbf{1}(\bar{A}^n(T) \le \kappa) \sum_{i,j=1,i\neq j}^{A^n(s)} \zeta^n_{i,k}(s,t)(\zeta^n_{j,k}(s,t))^3\Bigg] \\
&\le \frac{8}{n^2}E\Bigg[\mathbf{1}(\bar{A}^n(T) \le \kappa) \sum_{i,j=1,i<j}^{A^n(s)} 6\alpha^{1/2}_{j-i}\Bigg] \\
&\le \frac{48}{n^2} \sum_{i,j=1,i<j}^{\lfloor n\kappa \rfloor} \alpha^{1/2}_{j-i} \to 0 \quad \text{as} \quad n \to \infty, \quad (5.43)
\end{aligned}
$$

where the convergence holds under Assumption 1 with $\alpha_n \to 0$ as $n \to \infty$.

Similarly, by Lemma 5.4 and conditioning, we have that for $i, j \ge 1$ and $j > i$,

$$
E\big[(\zeta^n_{i,k}(s,t))^2(\zeta^n_{j,k}(s,t))^2\big] \le E\big[(\zeta^n_{i,k}(s,t))^2\big]E\big[(\zeta^n_{j,k}(s,t))^2\big] + 6\alpha^{1/2}_{j-i}.
$$

Thus,

$$
\begin{aligned}
&\frac{6}{n^2}E\bigg[\mathbf{1}(\bar{A}^n(T)\le\kappa)\sum_{i,j=1,i\neq j}^{A^n(s)}(\zeta_{i,k}^n(s,t))^2(\zeta_{j,k}^n(s,t))^2\bigg]\\
&\le\frac{12}{n^2}E\bigg[\mathbf{1}(\bar{A}^n(T)\le\kappa)\sum_{i,j=1,i<j}^{A^n(s)}E\big[(\zeta_{i,k}^n(s,t))^2\big]E\big[(\zeta_{j,k}^n(s,t))^2\big]\bigg]\\
&\qquad\qquad+\frac{72}{n^2}E\bigg[\mathbf{1}(\bar{A}^n(T)\le\kappa)\sum_{i,j=1,i<j}^{A^n(t)}\alpha_{j-i}^{1/2}\bigg].
\end{aligned}
\tag{5.44}
$$

The second term on the right hand side of (5.44) converges to 0 as in (5.43). Notice that conditional on the ξ^n process,

$$
\begin{aligned}
E\big[(\zeta_{i,k}^n(s,t))^2|\xi^n(u):0\le u\le t\big]&=(F_k(\xi^n(t)-\xi^n(\tau_i^n))-F_k(\xi^n(s)-\xi^n(\tau_i^n))\\
&\qquad\times(1-(F_k(\xi^n(t)-\xi^n(\tau_i^n))-F_k(\xi^n(s)-\xi^n(\tau_i^n)))\\
&\le F_k(\xi^n(t)-\xi^n(\tau_i^n))-F_k(\xi^n(s)-\xi^n(\tau_i^n)).
\end{aligned}
$$

Thus, the first term on the right hand side of (5.44) is bounded above as follows:

$$
\begin{aligned}
&\frac{12}{n^2}E\bigg[\mathbf{1}(\bar{A}^n(T)\le\kappa)\sum_{i,j=1,i<j}^{A^n(s)}E\big[(\zeta_{i,k}^n(s,t))^2\big]E\big[(\zeta_{j,k}^n(s,t))^2\big]\bigg]\\
&\le\frac{12}{n^2}E\bigg[\mathbf{1}(\bar{A}^n(T)\le\kappa)\sum_{i,j=1,i<j}^{A^n(s)}(F_k(\xi^n(t)-\xi^n(\tau_i^n))-F_k(\xi^n(s)-\xi^n(\tau_i^n)))\\
&\qquad\qquad\qquad\times\Big(F_k(\xi^n(t)-\xi^n(\tau_j^n))-F_k(\xi^n(s)-\xi^n(\tau_j^n))\Big)\bigg]\\
&\le\frac{6}{n^2}E\bigg[\mathbf{1}(\bar{A}^n(T)\le\kappa)\Big(\sum_{i=1}^{A^n(s)}(F_k(\xi^n(t)-\xi^n(\tau_i^n))-F_k(\xi^n(s)-\xi^n(\tau_i^n)))\Big)^2\bigg]\\
&=6E\bigg[\mathbf{1}(\bar{A}^n(T)\le\kappa)\Big(\int_0^s(F_k(\xi^n(t)-\xi^n(u)-F_k(\xi^n(s)-\xi^n(u))\,d\bar{A}^n(u)\Big)^2\bigg]\\
&\xrightarrow{n\to\infty}6\Big(\int_0^s[F_k(t-u)-F_k(s-u)]d\bar{a}(u)\Big)^2\\
&\le6\Big(\int_0^T[F_k(t-u)-F_k(s-u)]d\bar{a}(u)\Big)^2,
\end{aligned}
\tag{5.45}
$$

where the convergence follows from Assumption 2 and $\xi^n\Rightarrow e$ in Lemma 2.1 under Assumption 3.

Therefore we have shown that the first term on the right hand of (5.41) is bounded above by

$$
C_1\Big(\int_0^T[F_k(t-u)-F_k(s-u)]d\bar{a}(u)\Big)^2
$$

for all n with some sufficiently large constant $C_1>0$. Similar calculations and arguments show that the second term on the right hand of (5.41) is bounded above by $C_2\big(\bar{a}(t)-\bar{a}(s)\big)^2$ for all n with some sufficiently large constant $C_2>0$. Thus, to prove the claim in (5.39) for $\hat{M}_{k,2}^n$, we can choose the function $H_{M,k}(\cdot)$ as

$$
H_{M,k}(t)=C\Big(\int_0^T F_k(t-u)d\bar{a}(u)+\bar{a}(t)\Big)
$$

for $t \in [0,T]$ and some sufficiently large constant $C > 0$. It is evident that $H_{M,k}(\cdot)$ is nondecreasing and continuous on $[0,T]$. This completes the proof of the lemma. □

Completing the proof of Theorem 3.3. By Lemmas 5.2 and 5.5, we have shown that

$$\big(\hat{\boldsymbol{M}}_1^n, \hat{\boldsymbol{Z}}_1^n, \hat{V}_1^n, \hat{\boldsymbol{M}}_3^n, \hat{\boldsymbol{Z}}_3^n, \hat{V}_3^n, \hat{\boldsymbol{M}}_2^n, \hat{\boldsymbol{Z}}_2^n, \hat{V}_2^n\big) \xrightarrow{d_f} \big(\hat{\boldsymbol{M}}_1, \hat{\boldsymbol{Z}}_1, \hat{V}_1, \hat{\boldsymbol{M}}_3, \hat{\boldsymbol{Z}}_3, \hat{V}_3, \hat{\boldsymbol{M}}_2, \hat{\boldsymbol{Z}}_2, \hat{V}_2\big) \quad \text{as} \quad n \to \infty.$$

By the continuous mapping theorem, we have

$$\big(\hat{\boldsymbol{M}}_1^n + \hat{\boldsymbol{M}}_2^n + \hat{\boldsymbol{M}}_3^n, \hat{\boldsymbol{Z}}_1^n + \hat{\boldsymbol{Z}}_2^n + \hat{\boldsymbol{Z}}_3^n, \hat{V}_1^n + \hat{V}_2^n + \hat{V}_3^n\big) \xrightarrow{d_f} \big(\hat{\boldsymbol{M}}_1 + \hat{\boldsymbol{M}}_2 + \hat{\boldsymbol{M}}_3, \hat{\boldsymbol{Z}}_1 + \hat{\boldsymbol{Z}}_2 + \hat{\boldsymbol{Z}}_3, \hat{V}_1 + \hat{V}_2 + \hat{V}_3\big)$$

as $n \to \infty$. Thus,

$$\big(\hat{\boldsymbol{X}}^n, \hat{\boldsymbol{Y}}^n, \hat{S}^n\big) \xrightarrow{d_f} \big(\hat{\boldsymbol{X}}, \hat{\boldsymbol{Y}}, \hat{S}\big) \quad \text{as} \quad n \to \infty.$$

Lemmas 5.2 and 5.6 also imply that the processes $\big(\hat{\boldsymbol{X}}^n, \hat{\boldsymbol{Y}}^n, \hat{S}^n\big)$ are tight (applying Theorem 11.6.7 in [41]). This completes the proof of Theorem 3.3.

□

6. Concluding Remarks

In this paper, we have studied an infinite-server fork-join queueing system with NES and sequentially correlated (strong mixing) service vectors which is subject to renewal alternating service disruptions. By proving the FCLT, we have obtained the mean and covariance approximations in transient and steady states for the service and synchronization processes. They provide important insights on the impact of the interruptions, and the "component-wise" and "vector-wise" correlations among service times of the parallel tasks, upon the service and synchronization processes.

Many interesting problems remain to be studied. First, we have assumed that the down times in the random environment are asymptotically negligible comparing with the service times in this paper. If the down times are of the same order as the service times, we can easily show an FWLLN for the service and synchronization processes, where the fluid limit will be stochastic, and in fact, a deterministic system in the random environment. It will be interesting to study the steady state of the stochastic fluid limit and the convergence rate to the steady state. Second, we have assumed that all service stations are simultaneously disrupted. It may be interesting to investigate the situation where only some of the service stations are affected during the down times. In that regard, our analysis is the worst-case scenario, and can be used to bound the associated performance measures. For example, the results on the synchronized process will become a lower bound. Third, it will be interesting to study such fork-join queueing systems in the Halfin-Whitt regime, and fork-join models with multiple classes of jobs. In addition, we have assumed that the sequence of the service times of all parallel tasks is weakly dependent, satisfying the strong mixing (α-mixing) condition. It will be interesting to consider cases with strong dependence among service times. This is a fundamental problem, even for infinite-server queues. Pang and Zhou [37] has recently studied infinite-server queues with arrival dependent services (which result in strong dependence among service times). These problems will be interesting future work.

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