Stochastic epidemic models with varying infectivity and waning immunity

RAPHAËL FORIEN, GUODONG PANG, ÉTIENNE PARDOUX, AND ARSENE BRICE ZOTSA–NGOUFACK

ABSTRACT. We study an individual-based stochastic epidemic model in which infected individuals become susceptible again following each infection. In contrast to classical compartment models, after each infection, the infectivity is a random function of the time elapsed since one's infection. Similarly, recovered individuals become gradually susceptible after some time according to a random susceptibility function.

We study the large population asymptotic behaviour of the model, by proving a functional law of large numbers (FLLN) and investigating the endemic equilibria properties of the limit. The limit depends on the law of the susceptibility random functions but only on the mean infectivity functions. The FLLN is proved by constructing a sequence of i.i.d. auxiliary processes and adapting the approach from the theory of propagation of chaos. The limit is a generalisation of a PDE model introduced by Kermack and McKendrick, and we show how this PDE model can be obtained as a special case of our FLLN limit.

For the endemic equilibria, if R_0 is lower than (or equal to) some threshold, the epidemic does not last forever and eventually disappears from the population, while if R_0 is larger than this threshold, the epidemic will not disappear and there exists an endemic equilibrium. The value of this threshold turns out to depend on the harmonic mean of the susceptibility a long time after an infection, a fact which was not previously known.

1. INTRODUCTION

Many infectious diseases are such that the immunity acquired after recovery from the illness is eventually lost, after a period whose length varies both with the individual and with the illness. The huge majority of the literature on mathematical epidemiology which considers models with loss of immunity assume that it is lost suddenly (usually at a random time). However, it is rather clear that, in reality, immunity wanes progressively over some period, which can vary from one individual to another. In their pioneering 1927 paper [21], Kermack and McKendrick introduced the first mathematical model of epidemic propagation in which the infectivity of individuals depends on the time elapsed since their infection (this time is often called the age of infection). This model was deterministic and assumed that recovered individuals can no longer be infected. In two subsequent papers, [22, 23], Kermack and McKendrick introduced another deterministic model in which recovered individuals can be infected again, with a probability depending on the time elapsed since their recovery (or recovery age), and in which individuals die and produce new susceptible individuals at some rate. They were interested in studying the conditions under which an infectious disease can become endemic, i.e., when it can establish itself in the population and reach a stable equilibrium for which, at all times, some macroscopic fraction of the population is infected. This kind of equilibrium for the deterministic system is called an endemic equilibrium, as opposed to the disease-free equilibrium, in which the fraction of infected individuals is equal to zero.

Our aim in the present paper is to revisit these questions with the use of a general stochastic model, following the recent developments carried out by the first three authors. In [11], the 1927 model of Kermack and McKendrick was obtained as the infinite population limit of an individual-based
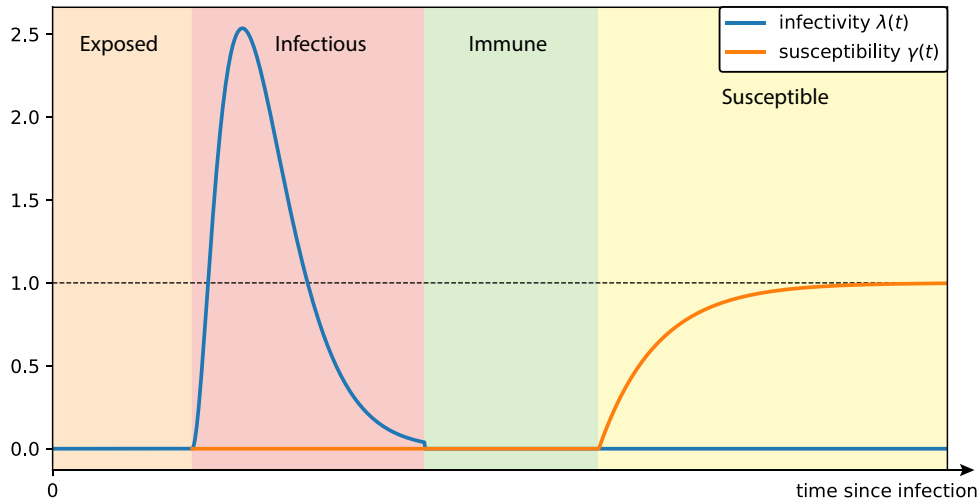
FIGURE 1. Illustration of a typical realization of the random infectivity and susceptibility functions of an individual from the time of infection to the time of recovery, and then to the time of losing immunity and becoming fully susceptible (or in general, partially susceptible).

stochastic model in which the infectivity of individuals is a general random function of their age of infection. More precisely, it was shown that the fraction of susceptible individuals and the average infectivity (also called the force of infection) converge to a deterministic limit which solves the system of non-linear integral equations introduced in [21]. The widespread ordinary differential equations (ODEs) compartmental SIR model is only a very special case of these equations, in which both the infectivity and the recovery rate do not depend on the infection age (resp., on the recovery age). Such models are less realistic as they do not reproduce the dependence of the dynamics of the epidemic on its past [28, 12], and thus neglect the inertia of the evolution of the epidemic. This comes from the fact that ODE models are law of large numbers limit of Markov stochastic models, in which individuals move from one compartment to the next after exponentially distributed times.

In the present paper, we introduce a very general model in which individuals are characterised by their infectivity and their susceptibility, which are assumed to be given as random functions of the age of infection. These random functions are assumed to be i.i.d. among the various individuals in the population, and a new independent pair of random functions is drawn at each new infection. See Figure 1 for a realization of the infectivity and susceptibility of an individual after his/her infection.

Note however that, in contrast with the model in [22, 23], we consider a closed population, in which there are no birth and no death (we do not exclude deaths due to infections, represented by individuals whose susceptibility remains equal to zero after the fatal infection).

We also assume (as in [22, 23]) that all pairs of individuals have contacts at the same rate. In other words, we assume a situation of homogeneous mixing. Of course, this is not quite satisfactory. One may wish to take into account the spatial distribution of the population, as well as the variety of social behaviors of the individuals. However, mathematical models involve necessarily a simplification of the complex reality. We believe that the results presented here constitute a significant progress over the classical models where all rates are constant and immunity is lost instantaneously. In future works, we do intend to combine the complexity of the present work with that of inhomogeneous models, such as spatial models or models on graphons.

Besides the fact that we prefer to present our model in the form of a system of integral equations rather than a system of partial differential equations, our deterministic model is more general

than the model of [22, 23]. The reason is the following. While the law of large numbers limit of the model with infection age dependent infectivity depends only on the mean of the random infectivity function [11], the deterministic limit of the model with random susceptibility depends on the distribution of this random susceptibility function in a much more complicated way. To see this, suppose that we wish to compute the average susceptibility of the individuals at some time t. This average can be obtained by summing the contributions of individuals that have not been infected on the interval $[0, t)$, and of individuals that have been infected at some time $s \in [0, t)$ and have not yet been infected again on the interval $[s, t)$. But the probability that such an individual has not been reinfected by time t depends on the full trajectory of their susceptibility, as well as on the trajectory of the force of infection, on the interval $[s, t)$ (see (3.6)–(3.7) below). This explains why the deterministic limit is more complex than that in [11], and shows that, in their 1932-33 model, Kermack and McKendrick implicitly assumed that the only random component of the susceptibility function is the time of recovery (see the discussion in Section 5.2). As a consequence, we obtain a strict generalization of the model in [22, 23] as the deterministic limit of our stochastic model.

Since in our model there is a flux of new susceptibles, due to waning of immunity, one expects that under certain conditions, there may be a stable endemic equilibrium. We manage to study the existence, uniqueness and some stability properties of an endemic equilibrium (as well as the stability or instability of the disease–free equilibrium) in our deterministic limit. We identify a threshold, which is the harmonic mean of the large time limit of the susceptibility (which is 1 in Figure 1, but may be less than 1 in our general model). If the basic reproduction number R_0 (defined below by (4.1)) is lower than this threshold, then the process converges to the disease-free equilibrium. We also show that under appropriate assumptions, if R_0 is larger than the threshold, and the model converges as $t \to \infty$ to some limit, then this limit is the unique endemic equilibrium, which is fully characterized. We conjecture that under appropriate assumptions, when R_0 is larger than the threshold, then any solution of the deterministic limit starting with a non zero force of infection at time $t = 0$ does converge to the endemic equilibrium.

Let us comment on the method used to prove our law of large number result. One key argument is based upon a coupling of the processes which count the number of infections of each individual up to time t with i.i.d. counting processes, where the renormalized force of infection is replaced by its deterministic limit. This approach, for which the inspiration came from [8], is an application of ideas from the theory of propagation of chaos, see Sznitman [29]. Note both that we were unable to adapt the methodology of [11] to the setting of the present paper, and that the assumptions made here on the random infectivity function are weaker than those made in [11]. Recently, the methodology of the present paper has been applied to the model in [11], thus resulting in the same result under weaker assumptions, see [13]. We also note that our approach differs significantly from the techniques classically used for age-structured population models, as in [25, 20, 31, 14, 15, 9]. In these papers, the authors describe the model as a branching process (sometimes with interaction), Markovian in the age structure, in which the lifespan, birth rate and death rate depend on the age of all individuals in the population. The state of such a model is then described by the empirical measure of the ages of the individuals. These works combine infinite-dimensional stochastic calculus and measure-valued Markov processes analysis in order to prove the convergence of their model.

Using random infectivity and susceptibility functions allows us to build a very general model which is both versatile and tractable. It captures the effect of a progressive loss of immunity when this loss is allowed to be very different from one individual to another. The integral equations that we obtain to describe the large population limit of our model are both compact and extremely general, since most epidemic models with homogeneous mixing and a fixed population size can be written under this form, no matter how many compartments are considered. The effect of the variability of susceptibility on the endemic threshold have received very little attention in the literature, despite some profound implications which we outline in the present work. The fact that the threshold depends on the harmonic mean of the susceptibility reached after an infection

shows that the heterogeneity of immune responses in real populations should not be neglected in public health decisions. Similarly, the variability of the immune response after vaccination (both in time and between individuals) should affect the efficacy of vaccination policies in non-trivial ways, although these questions are outside the scope of the present work.

We want to comment on the terminology which can be used for our model as a compartmental model. Recall that the compartments most classically used in epidemic models include S for Susceptible individuals, E for Exposed (those infected individuals who are not yet infectious), I for infectious and R for Recovered. We claim that our model can be classified as a SEIRS, SIRS, or SIS model. Indeed, referring to Figure 1, we can consider that a given individual passes from the S to the E compartment when he/she becomes infected, then into the I compartment when the attached infectivity first becomes positive, into the R compartment when the infectivity reaches 0 and remains null, and into the S compartment when the attached susceptibility becomes positive. However, without modifying the dynamics of the epidemic, we can merge the E and I compartments into a compartment I (for infected), where the infectivity need not be positive all the time. Similarly, we can merge the R and S compartments into the S compartment (or U, for uninfected, as we suggest below), whose members may have a susceptibility equal to zero.

Models with gradual waning of immunity have been studied since Kermack and McKendrick by only a handful of authors, including Inaba, who in a series of works, see [16, 19, 17, 18], has performed a careful mathematical study of the PDE model from [22, 23], as well as Breda et al. [3] who have considered an integral equation version of the same model. Other authors have pursued the study of the system of ODE/PDEs, see in particular Thieme and Yang [30], Barbarossa and Röst [1] and Carlsson et al. [6]. More recently Khalifi and Britton [24], compare the level of immunity in a model with gradual vs. sudden loss of immunity.

Organization of the paper. The rest of the paper is organized as follows. In Section 2, we define the model in detail. In Section 3, we state the assumptions and the functional law of large numbers (FLLN) and discuss how the results reduce to the known results for the classical SIS and SIRS models. The results on the endemic equilibrium are presented in Section 4. In Section 5, we focus on the generalized SIRS model with a particular set of infectivity and susceptibility random functions and initial conditions, and show how the limit relates to the Kermack and McKendrick PDE model with the corresponding infection-age dependent infectivity and recovery-age dependent susceptibility. The proofs for the FLLN are given in Section 6 and those for the endemic equilibria in Section 7.

Notation. Throughout the paper, all the random variables and processes are defined on a common complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We use $\xrightarrow[N \to +\infty]{\mathbb{P}}$ to denote convergence in probability as the parameter $N \to \infty$. Let $\mathbb{N}$ denote the set of natural numbers and $\mathbb{R}^k(\mathbb{R}^k_+)$ the space of k-dimensional vectors with real (nonnegative) numbers, with $\mathbb{R}(\mathbb{R}_+)$ for $k = 1$. We use $\mathbb{1}_{\{\cdot\}}$ for the indicator function. Let $D = D(\mathbb{R}_+; \mathbb{R})$ be the space of $\mathbb{R}$-valued càdlàg functions defined on $\mathbb{R}_+$, with convergence in D meaning convergence in the Skorohod J_1 topology (see, e.g., [2, Chapter 3]). Also, we use D^k to denote the k-fold product with the product J_1 topology. Let C be the subset of D consisting of continuous functions and D_+ the subset of D of càdlàg functions with values on $\mathbb{R}_+$

2. Model description

We consider a population with fixed size $N \geq 1$. Initially, a random number of individuals within the population are chosen to be infected, while the others are assumed to be susceptible. Each individual is characterized by its current infectivity and susceptibility. The infectivity corresponds to the instantaneous rate at which an individual has a potentially infectious contact with another individual in the population, which is assumed to be chosen uniformly in the population. The latter then becomes infected with a probability equal to its current susceptibility (which will be assumed

to be in $[0, 1]$). At each new successful infection, the newly infected individual draws a random pair of functions (λ, γ) following a given distribution, and, as long as this individual is not infected again, its infectivity (resp. susceptibility) will be given by $\lambda(t - \tau)$ (resp. $\gamma(t - \tau)$), where τ is the time at which the infection has happened. We shall assume that an individual cannot be infectious and susceptible at the same time. More precisely, the susceptibility of an individual must remain equal to zero as long as the infectivity function resulting from this individual's latest infection has not vanished, see Assumption 2.1 below.

2.1. **Notations.** The infection process is described by a system of counting processes $(A_k^N(t), t \geq 0, 1 \leq k \leq N)$ where $A_k^N(t)$ counts the number of times that the k-th individual has been infected up to time t (apart from its initial infection if the k-th individual is among the initially infected individuals).

Let $\{(\lambda_{k,i}, \gamma_{k,i}), i \geq 1, 1 \leq k \leq N\}$ be a collection of i.i.d. D^2-valued random variables. $\lambda_{k,i}(t)$ (resp. $\gamma_{k,i}(t)$) is the infectivity (resp. susceptibility) of the k-th individual, t units of time after its i-th infection, provided this individual has not already been infected again at this time.

Also let $\{(\lambda_{k,0}, \gamma_{k,0}), 1 \leq k \leq N\}$ be a collection of i.i.d. D^2-valued random variables independent of the previous one. Similarly, $\lambda_{k,0}(t)$ (resp. $\gamma_{k,0}(t)$) is the infectivity (resp. susceptibility) of the k-th individual at time t if this individual has not already been infected on the interval $(0, t]$. Note that, typically, we shall assume that, at time zero, some fraction of the individuals are "susceptible", hence $\mathbb{P}(\lambda_{k,0} \equiv 0) > 0$. In addition, as in [11], an individual that is infectious at time zero may not have the same remaining infectious period (and infectivity) as an individual who has just been infected (to reflect the fact that it has been infected at some time in the past). As a result the pair $(\lambda_{k,0}, \gamma_{k,0})$ is a priori not distributed as $(\lambda_{k,i}, \gamma_{k,i})$ for $i \geq 1$, but its distribution can in principle remain quite general. We give more concrete constructions of the sequence $\{(\lambda_{k,0}, \gamma_{k,0}), 1 \leq k \leq N\}$ in Remark 2.1 and in Assumption 4.2.

In the following, we use (λ, γ) to denote a generic random variable taking values in D^2 and distributed as $(\lambda_{k,i}, \gamma_{k,i})$ for $i \geq 1$ and similarly we use (λ_0, γ_0) to denote a generic random variable taking values in D^2 and distributed as $(\lambda_{k,0}, \gamma_{k,0})$.

Recall that $A_k^N(t)$ denotes the number of times that the k-th individual has been (re-)infected on the interval $(0, t]$. Hence, the time elapsed since this individual's last infection (or since time 0 if no such infection has occurred), is given by

$$\varsigma_k^N(t) := t - \left(\sup\{s \in (0, t] : A_k^N(s) = A_k^N(s^-) + 1\} \vee 0 \right),$$

where we use the convention $\sup \emptyset = -\infty$. With this notation, the current infectivity and susceptibility of the k-th individual are given by

$$\lambda_{k, A_k^N(t)}(\varsigma_k^N(t)), \qquad \text{and} \qquad \gamma_{k, A_k^N(t)}(\varsigma_k^N(t)).$$

Figure 2 shows a realisation of these processes for two individuals. Let us then define $\overline{\mathfrak{F}}^N(t)$ and $\overline{\mathfrak{S}}^N(t)$ as the average infectivity and susceptibility in the population, i.e.,

$$\overline{\mathfrak{F}}^N(t) := \frac{1}{N} \sum_{k=1}^N \lambda_{k, A_k^N(t)}(\varsigma_k^N(t)), \qquad \overline{\mathfrak{S}}^N(t) := \frac{1}{N} \sum_{k=1}^N \gamma_{k, A_k^N(t)}(\varsigma_k^N(t)).$$

According to our informal description of the model, the instantaneous rate at which the ℓ-th individual infects the k-th individual is

$$\frac{1}{N} \lambda_{\ell, A_\ell^N(t)}(\varsigma_\ell^N(t)) \gamma_{k, A_k^N(t)}(\varsigma_k^N(t))$$

where the $\frac{1}{N}$ factor comes from the probability that the k-th individual is chosen as the target of the infectious contact. Summing over the index ℓ, the instantaneous rate at which the k-th individual is
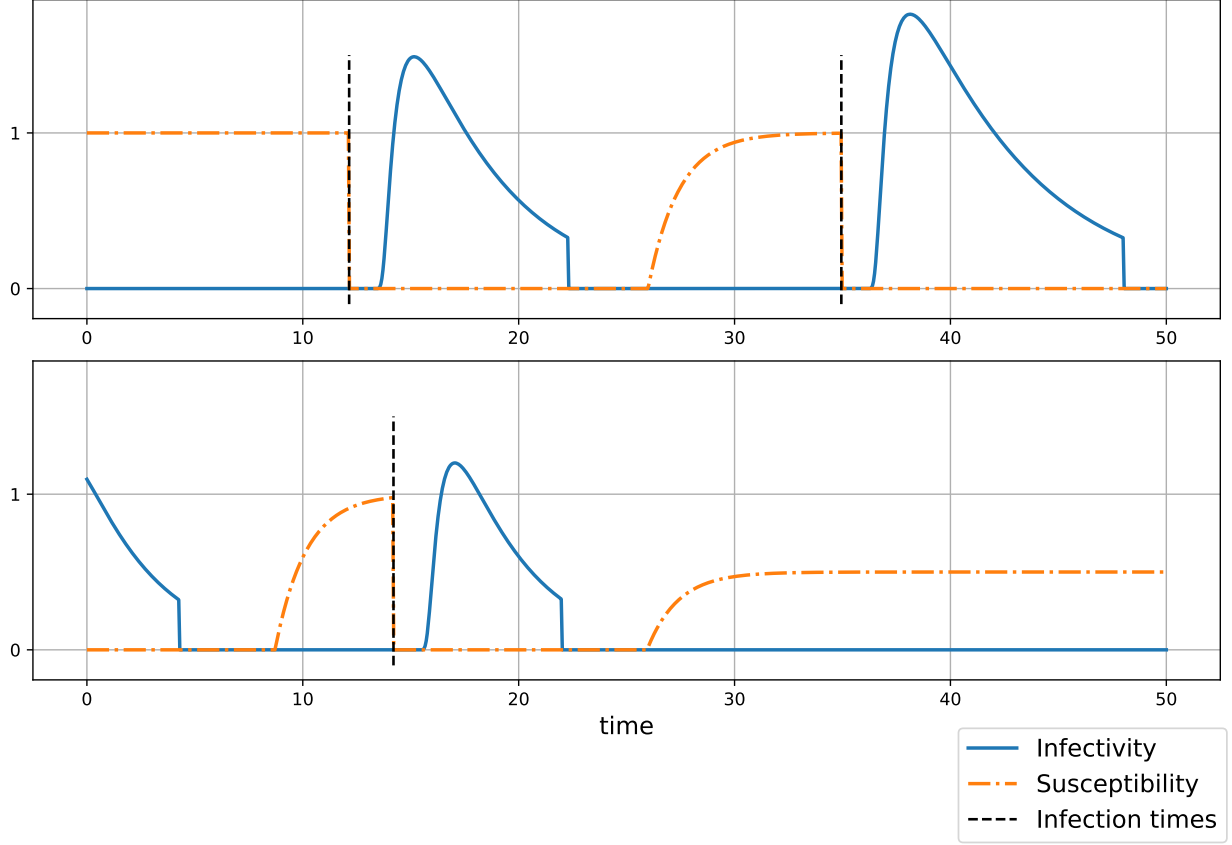
FIGURE 2. Illustration of the evolution of an individual's infectivity and susceptibility through time. Each graphic shows the dynamics of an individual's infectivity (blue) and susceptibility (orange). The top graphic corresponds to an individual which is initially susceptible, and the bottom one to an initially infectious individual. Note that, after being reinfected, the second individual remains partially immune even a long time after infection.

infected or reinfected is given by

$$\gamma_{k,A_k^N(t)}(\varsigma_k^N(t))\,\overline{\mathfrak{F}}^N(t).$$

This leads to the following formal definition of our model.

2.2. **Definition of the model.** Let $\{(\lambda_{k,i}, \gamma_{k,i}), i \geq 1, 1 \leq k \leq N\}$ and $\{(\lambda_{k,0}, \gamma_{k,0}), 1 \leq k \leq N\}$ be two independent families of i.i.d. random variables as above. Also let $(Q_k, 1 \leq k \leq N)$ be an i.i.d. family of standard Poisson random measures on $\mathbb{R}_+^2$, also independent from the two previous families. The family of counting processes $(A_k^N(t), t \geq 0, 1 \leq k \leq N)$ is then defined as the solution of

$$A_k^N(t) = \int_{[0,t]\times\mathbb{R}_+} \mathbb{1}_{u \leq \Upsilon_k^N(r^-)} Q_k(dr, du), \tag{2.1}$$

where

$$\Upsilon_k^N(t) := \gamma_{k,A_k^N(t)}(\varsigma_k^N(t))\,\overline{\mathfrak{F}}^N(t). \tag{2.2}$$

Note that, we construct A_k^N by induction on the jumps times. The next Assumption will be imply that the rate $\Upsilon_k^N(t)$ is bounded almost surely. Consequently the jump times do not accumulate, and the above induction defines $A_k^N(t)$ for all $t \geq 0$.

Assumption 2.1. *We assume that:*

    (i) *There exists a constant $\lambda_* < \infty$ such that for all $t \geq 0$, $0 \leq \lambda(t)$, $\lambda_0(t) \leq \lambda_*$ almost surely, and $0 \leq \gamma(t)$, $\gamma_0(t) \leq 1$ almost surely.*

    (ii) *Almost surely,*

$$\sup\{t \geq 0, \lambda_0(t) > 0\} \leq \inf\{t \geq 0, \gamma_0(t) > 0\} \text{ and } \sup\{t \geq 0, \lambda(t) > 0\} \leq \inf\{t \geq 0, \gamma(t) > 0\}. \tag{2.3}$$

We define,

$$\eta_0 := \sup\{t \geq 0 : \lambda_0(t) > 0\} \text{ and } \eta := \sup\{t \geq 0 : \lambda(t) > 0\}.$$

Similarly for $i \geq 0$ and $1 \leq k \leq N$, we define,

$$\eta_{k,i} := \sup\{t \geq 0 : \lambda_{k,i}(t) > 0\}.$$

Condition (2.3) implies that, as long as an individual remains infectious (i.e. as long as $\varsigma_k^N(t) < \eta_{k,A_k^N(t)}$), he or she cannot be infected. Hence in each infectious-immune-susceptible cycle, the infectious and susceptible periods do not overlap. Note that (2.3) is not necessary for the process to be well defined, and we discuss the consequences of removing this assumption in Remark 3.2.

2.3. **Number of infectious and uninfectious individuals.** For $i \geq 1$, $\eta_{k,i}$ is the duration of the infectious period of the k-th individual following its i-th infection, while $\eta_{k,0} = 0$ if the k-th individual is initially susceptible and $\eta_{k,0} > 0$ is the remaining infectious period of the k-th individual if it is initially infectious. We shall say that the k-th individual is currently infectious (resp. uninfectious) if $\varsigma_k^N(t) < \eta_{k,A_k^N(t)}$ (resp. $\varsigma_k^N(t) \geq \eta_{k,A_k^N(t)}$). Note that, with this definition, an individual may be called infectious even if its current infectivity is equal to zero, for example during an exposed period. In the same way, an individual is called uninfectious if it is *no longer infectious* or has never been infected, hence this group comprises both recovered and susceptible individuals. This choice of two broadly defined compartments allows us to keep the notations tractable, but the equations obtained below can be generalized in a straightforward way to keep track of the number of individuals at different stages of their infection and susceptibility in more detail.

Let $I^N(t)$ (resp. $U^N(t)$) denote the number of infectious (resp. uninfectious) individuals in the population at time $t \geq 0$. Then

$$I^N(t) = \sum_{k=1}^{N} \mathbb{1}_{\varsigma_k^N(t) < \eta_{k,A_k^N(t)}}, \qquad \text{and} \qquad U^N(t) = \sum_{k=1}^{N} \mathbb{1}_{\varsigma_k^N(t) \geq \eta_{k,A_k^N(t)}}. \tag{2.4}$$

Note that, quite obviously, $U^N(t) + I^N(t) = N$ for all $t \geq 0$. We then define

$$\overline{I}(0) := \mathbb{P}(\eta_0 > 0), \qquad\qquad \overline{U}(0) := \mathbb{P}(\eta_0 = 0) = 1 - \overline{I}(0).$$

Recall that $\{(\lambda_{k,0}, \gamma_{k,0}), 1 \leq k \leq N\}$, and hence $(\eta_{k,0}, 1 \leq k \leq N)$, are independent and identically distributed. Thus by the law of large numbers,

$$\left(\frac{1}{N} I^N(0), \frac{1}{N} U^N(0)\right) = \left(\frac{1}{N} \sum_{k=1}^{N} \mathbb{1}_{\eta_{k,0} > 0}, \frac{1}{N} \sum_{k=1}^{N} \mathbb{1}_{\eta_{k,0} = 0}\right) \rightarrow \left(\overline{I}(0), \overline{U}(0)\right)$$

almost surely as $N \rightarrow \infty$.

Remark 2.1. The classical compartmental models

Most compartmental epidemic models can be obtained as special cases of our model, as long as no population structure is assumed. Here are a few examples.

(i) *The SIS model considers that infected individuals instantly become infectious with some deterministic infectivity β, and remain infectious for a random time η, after which they become instantly fully susceptible again. Thus, this model is obtained by assuming that,*

$$\lambda(t) = \beta \mathbb{1}_{0 \leq t < \eta}, \qquad\qquad \gamma(t) = \mathbb{1}_{t \geq \eta}, \qquad\qquad (2.5)$$

and η are non-negative random variables, which follow an exponential distribution in the case of the Markov model. Let us also specify the distribution of (λ_0, γ_0). For this, fix $\overline{I}(0) \in [0, 1]$, set $\overline{S}(0) = 1 - \overline{I}(0)$ and let χ be a random variable taking values in $\{S, I\}$ such that $\mathbb{P}(\chi = S) = \overline{S}(0)$ and $\mathbb{P}(\chi = I) = \overline{I}(0)$. Then set

$$\lambda_0(t) = \begin{cases} 0 & \text{if } \chi = S, \\ \beta \mathbb{1}_{0 \leq t < \eta_0} & \text{if } \chi = I, \end{cases} \qquad \gamma_0(t) = \begin{cases} 1 & \text{if } \chi = S, \\ \mathbb{1}_{t \geq \eta_0} & \text{if } \chi = I, \end{cases} \qquad (2.6)$$

where η_0 is a positive random variable (which follow exponential distributions in the case of the Markov model). (Note that, with this definition, the model starts from a random initial condition, which is not always assumed in the literature, but does not greatly affect its behaviour.)

(ii) *The SIR model instead considers that, at the end of the infectious period, infected individuals recover from the disease, and can no longer be infected again. This model can be obtained from the above by proceeding as for the SIS model, but assuming instead that*

$$\forall t \geq 0, \quad \gamma(t) = 0, \qquad\qquad and \qquad\qquad \gamma_0(t) = \begin{cases} 1 & \text{if } \chi = S, \\ 0 & \text{if } \chi = I. \end{cases} \qquad (2.7)$$

Note that, in this case, if we keep a general distribution for λ and λ_0, this model reduces to the one studied in [11].

(iii) *The SIRS model assumes that, at the end of their infectious period, individuals stay immune to the disease for a random time θ, after which they become fully susceptible again. This model is obtained by assuming that, for $i \geq 1$,*

$$\lambda(t) = \beta \mathbb{1}_{0 \leq t < \eta}, \qquad\qquad \gamma(t) = \mathbb{1}_{t \geq \eta + \theta}, \qquad\qquad (2.8)$$

where (η, θ), is a pair of independent random variables taking values in $\mathbb{R}_+^2$ (which are distributed as pairs of independent exponential variables in the Markov models). The generalization of the definition of the distribution of (λ_0, γ_0) to this case is straightforward.

Note that, in the last two cases, the quantity $U^N(t)$ defined in (2.4) counts both susceptible and removed individuals. The actual number of susceptible and removed individuals in the SIRS model is given by

$$S^N(t) := \sum_{k=1}^{N} \mathbb{1}_{\eta_{k,A_k^N(t)} + \theta_{k,A_k^N(t)} \leq \varsigma_k^N(t)}, \qquad R^N(t) := \sum_{k=1}^{N} \mathbb{1}_{\eta_{k,A_k^N(t)} \leq \varsigma_k^N(t) < \eta_{k,A_k^N(t)} + \theta_{k,A_k^N(t)}}, \qquad (2.9)$$

where

$$\theta_{k,i} = \inf\{\theta > 0, \gamma_{k,i}(\eta_{k,i} + \theta) > 0\},$$

following (2.8).

In the SIR model, the above expression remains exact provided we set $\theta_{k,i} = +\infty$ for $i \geq 1$, $\theta_{k,0} = 0$ if $\chi_k = S$ and $\theta_{k,0} = +\infty$ if $\chi_k = I$.

It is also common to assume that infected individuals do not become infectious right after being infected, but first become exposed (i.e. infected but not yet infectious) before becoming infectious. This results in an additional compartment E, which can also be included in the above examples without difficulty.

Remark 2.2. *In* [8], *Chevallier studied a related model formulated as a system of age-dependent random Hawkes processes. This model considers a system of N neurons which fire at a rate depending both on the times of the previous firings of other neurons and on the time elapsed since their last firing (called the age process). The author proves in* [8] *a propagation of chaos result for the empirical measure of the point processes corresponding to the firing times of the neurons, and for the empirical measure of the age processes of the neurons, as N tends to infinity. Although neither our model or that of Chevallier can be formulated as a special case of the other, the two are closely related (if firing is understood as an analogous of being infected). The main difference between the two frameworks is in the assumptions on the randomness in the interaction between individuals after each firing/infection. For instance, our model would be closer to that of Chevallier if, instead of choosing a different infectivity function after each infection for each individual, we chose a different infectivity function for each directed pair of individuals at the beginning, and kept the same infectivity function for this pair of individual after each infection. Thus, the law of large numbers limit that we prove below can be seen as an extension of Chevallier's result, and indeed some steps of the proof are adapted from* [8]. *See also Theorem 6.1 in Section 6 whose formulation is closer to the propagation of chaos result of* [8]. *In* [7], *the same author also proves a central limit theorem for the empirical measures mentioned above, something which we do not do here, but could be the subject of future work.*

3. Functional law of large numbers

In this section we present the FLLN for the scaled processes $\left(\overline{\mathfrak{S}}^N, \overline{\mathfrak{F}}^N, \overline{U}^N, \overline{I}^N\right)$ where $\overline{U}^N := N^{-1}U^N$ and $\overline{I}^N := N^{-1}I^N$.

Let

$$\overline{\lambda}_0(t) := \mathbb{E}\big[\lambda_{1,0}(t)\big|\eta_{1,0} > 0\big], \quad \overline{\lambda}(t) := \mathbb{E}\left[\lambda_{1,1}(t)\right],$$

$$F_0^c(t) := \mathbb{P}\left(\eta_{1,0} > t\big|\eta_{1,0} > 0\right), \quad F^c(t) := \mathbb{P}\left(\eta > t\right),$$

and recall that $\overline{I}(0) = \mathbb{P}\left(\eta_{1,0} > 0\right)$. Let μ and μ_0 be the laws of $\gamma_{1,1}$ and $\gamma_{1,0}$ in $\mathcal{P}(D)$, respectively. For simplicity, we write γ and γ_0 as processes with the same laws as $\gamma_{1,1}$ and $\gamma_{1,0}$, respectively.

To describe the limits of the FLLN, we introduce the following two-dimensional integral equations. Observe that the solution (x, y) of this system depend on the laws of (λ, γ) and (λ_0, γ_0) only through expectations but for γ and γ_0 it is much more complex.

We consider the following system of integral equations for which we look for a solution $(x, y) \in D_+^2$:

$$\begin{cases} x(t) = \mathbb{E}\left[\gamma_0(t) \exp\left(-\int_0^t \gamma_0(r)y(r)dr\right)\right] \\ \qquad + \int_0^t \mathbb{E}\left[\gamma(t-s)\exp\left(-\int_s^t \gamma(r-s)y(r)dr\right)\right] x(s)y(s)ds, \qquad (3.1) \\ y(t) = \overline{I}(0)\overline{\lambda}_0(t) + \int_0^t \overline{\lambda}(t-s)x(s)y(s)ds. \qquad\qquad\qquad\qquad\qquad (3.2) \end{cases}$$

In (3.1) we take the expectation on the law of γ_0 and γ respectively.

Our first result establishes existence and uniqueness of the solution (x, y) of (3.1)-(3.2). The proof is given in Section 6.

Theorem 3.1. *Under Assumption 2.1, the set of equations* (3.1)–(3.2) *has a unique solution $(\overline{\mathfrak{S}}, \overline{\mathfrak{F}}) \in D^2(\mathbb{R}_+, \mathbb{R}_+)$. The solution belongs to $C^2(\mathbb{R}_+)$ if $t \mapsto (\gamma_0(t), \overline{\lambda}_0(t))$ is continuous.*

We have the following convergence result which we prove in Section 6.

Theorem 3.2. *Under Assumption* 2.1,

$$(\overline{\mathfrak{S}}^N, \overline{\mathfrak{F}}^N) \xrightarrow[N \to +\infty]{\mathbb{P}} (\overline{\mathfrak{S}}, \overline{\mathfrak{F}}) \quad in \quad D^2 \tag{3.3}$$

where $(\overline{\mathfrak{S}}, \overline{\mathfrak{F}})$ *is the unique solution of the system of equations* (3.1)–(3.2).

Given the solution $(\overline{\mathfrak{S}}, \overline{\mathfrak{F}})$,

$$(\overline{U}^N, \overline{I}^N) \xrightarrow[N \to +\infty]{\mathbb{P}} (\overline{U}, \overline{I}) \quad in \quad D^2$$

where $(\overline{U}, \overline{I})$ *is given by*

$$\overline{U}(t) = \mathbb{E}\left[\mathbb{1}_{t \geq \eta_0} \exp\left(-\int_0^t \gamma_0(r)\overline{\mathfrak{F}}(r)dr\right)\right]$$

$$+ \int_0^t \mathbb{E}\left[\mathbb{1}_{t-s \geq \eta} \exp\left(-\int_s^t \gamma(r-s)\overline{\mathfrak{F}}(r)dr\right)\right] \overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)ds, \tag{3.4}$$

$$\overline{I}(t) = \overline{I}(0)F_0^c(t) + \int_0^t F^c(t-s)\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)ds. \tag{3.5}$$

We rewrite (3.1)–(3.2) as

$$\begin{cases} \overline{\mathfrak{S}}(t) = \mathbb{E}\left[\gamma_0(t) \exp\left(-\int_0^t \gamma_0(r)\overline{\mathfrak{F}}(r)dr\right)\right] \\ \qquad + \int_0^t \mathbb{E}\left[\gamma(t-s) \exp\left(-\int_s^t \gamma(r-s)\overline{\mathfrak{F}}(r)dr\right)\right] \overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)ds, \tag{3.6} \\ \overline{\mathfrak{F}}(t) = \overline{I}(0)\overline{\lambda}_0(t) + \int_0^t \overline{\lambda}(t-s)\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)ds. \tag{3.7} \end{cases}$$

We prove the following Lemma in Section 6.

Lemma 3.1. *If the pair* (x, y) *is a solution to the set of equations* (3.1)-(3.2), *then for all* $t \geq 0$,

$$\mathbb{E}\left[\exp\left(-\int_0^t \gamma_0(r)y(r)dr\right)\right] + \int_0^t \mathbb{E}\left[\exp\left(-\int_s^t \gamma(r-s)y(r)dr\right)\right] x(s)y(s)ds = 1. \tag{3.8}$$

Remark 3.1. *Since* $\overline{U}^N(t) + \overline{I}^N(t) = 1$ *for all* $t \geq 0$ *and* $N \geq 1$, *it follows from the above convergence that* $\overline{U}(t) + \overline{I}(t) = 1$ *as well. Let us check that this follows also from the set of equations* (3.6)-(3.7) *satisfied by* $(\overline{\mathfrak{S}}, \overline{\mathfrak{F}})$.

First we note that, by (2.3) *in Assumption* 2.1, $\gamma(t) = 0$ *for all* $t \in [0, \eta)$ *and* $\gamma_0(t) = 0$ *for* $t \in (0, \eta_0)$, *hence*

$$F_0^c(t) = \mathbb{E}\left[\mathbb{1}_{t < \eta_0} \exp\left(-\int_0^t \gamma_0(r)\overline{\mathfrak{F}}(r)dr\right)\right], \quad F^c(t-s) = \mathbb{E}\left[\mathbb{1}_{t-s < \eta} \exp\left(\int_s^t \gamma(r-s)\overline{\mathfrak{F}}(r)dr\right)\right].$$

Hence, summing (3.4) *and* (3.5), *we obtain*

$$\overline{U}(t) + \overline{I}(t) = \mathbb{E}\left[\exp\left(-\int_0^t \gamma_0(r)\overline{\mathfrak{F}}(r)dr\right)\right] + \int_0^t \mathbb{E}\left[\exp\left(-\int_s^t \gamma(r-s)\overline{\mathfrak{F}}(r)dr\right)\right] \overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)ds. \tag{3.9}$$

Therefore, as $(\overline{\mathfrak{S}}, \overline{\mathfrak{F}})$ *is a solution of the set of equations* (3.1)-(3.2), *by Remark* 3.1 *we conclude that* $\overline{U}(t) + \overline{I}(t) = 1$ *for all* $t \geq 0$. *Equation* (3.8) *should thus be seen as stating the conservation of the population size.*

Remark 3.2. *Without condition* (2.3) *of Assumption* 2.1, *the limit obtained in the functional law of large numbers satisfies a different set of equations. More precisely, equation* (3.7) *is replaced by* (3.10) *and* (3.5) *by* (3.11), *where* (3.10) *and* (3.11) *are given below:*

$$
\begin{cases}
\overline{\mathfrak{F}}(t) = \mathbb{E}\left[\lambda_0(t)\exp\left(-\int_0^t \gamma_0(r)\overline{\mathfrak{F}}(r)dr\right)\right] \\
\qquad\qquad + \int_0^t \mathbb{E}\left[\lambda(t-s)\exp\left(-\int_s^t \gamma(r-s)\overline{\mathfrak{F}}(r)dr\right)\right]\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)ds, & (3.10)\\[2mm]
\overline{I}(t) = \mathbb{E}\left[\mathbb{1}_{\eta_0>t}\exp\left(-\int_0^t \gamma_0(r)\overline{\mathfrak{F}}(s)dr\right)\right] \\
\qquad\qquad + \int_0^t \mathbb{E}\left[\mathbb{1}_{\eta>t-s}\exp\left(-\int_s^t \gamma(r-s)\overline{\mathfrak{F}}(r)dr\right)\right]\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)ds. & (3.11)
\end{cases}
$$

Remark 3.3. *We can check that, in the special cases mentioned in Remark* 2.1, *the limiting system of equations obtained in Theorem* 3.2 *coincides with the corresponding models in the literature.*

(i) *In the case of the SIS model, we note that, by* (2.5) *and* (2.6),

$$
\overline{\lambda}(t) = \beta\mathbb{P}(\eta > t), \qquad\qquad \overline{\lambda}_0(t) = \beta\mathbb{P}(\eta_0 > t \mid \eta_0 > 0).
$$

It thus follows from (3.7) *and* (3.5) *that* $\overline{\mathfrak{F}}(t) = \beta\overline{I}(t)$ *for all* $t \geq 0$. *Moreover, comparing* (3.6) *and* (3.4) *and using* (2.5) *and* (2.6), *we see that* $\overline{\mathfrak{S}}(t) = \overline{U}(t)$. *Combining this with the fact that* $\overline{I}(t) + \overline{U}(t) = 1$, *we obtain*

$$
\overline{I}(t) = \overline{I}(0)F_0^c(t) + \beta\int_0^t F^c(t-s)(1-\overline{I}(s))\overline{I}(s)ds, \qquad (3.12)
$$

as stated in Theorem 2.3 of [26].

(ii) *In the case of the SIR model, from the definition of* γ *and* γ_0 *in* (2.7) *and* (3.6), *we see that*

$$
\overline{\mathfrak{S}}(t) = \overline{S}(0)\exp\left(-\int_0^t \overline{\mathfrak{F}}(r)dr\right) \qquad and \qquad \overline{U}(t) = \overline{\mathfrak{S}}(t) + \overline{R}(t).
$$

Combined with the fact that $\overline{\mathfrak{F}}(t) = \beta\overline{I}(t)$, *this yields the statement of Theorem 2.1 in* [26].

(iii) *In the case of the SIRS model, we note that, in view of* (3.4), $\overline{U}(t) = \overline{S}(t) + \overline{R}(t)$, *where*

$$
\overline{S}(t) = \mathbb{E}\left[\mathbb{1}_{\eta_0+\theta_0\leq t}\exp\left(-\int_0^t \gamma_0(r)\overline{\mathfrak{F}}(r)dr\right)\right]
$$
$$
+ \int_0^t \mathbb{E}\left[\mathbb{1}_{\eta+\theta\leq t-s}\exp\left(-\int_s^t \gamma(r-s)\overline{\mathfrak{F}}(r)dr\right)\right]\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)ds,
$$

and

$$
\overline{R}(t) = \mathbb{E}\left[\mathbb{1}_{\eta_0\leq t<\eta_0+\theta_0}\exp\left(-\int_0^t \gamma_0(r)\overline{\mathfrak{F}}(r)dr\right)\right]
$$
$$
+ \int_0^t \mathbb{E}\left[\mathbb{1}_{\eta\leq t-s<\eta+\theta}\exp\left(-\int_s^t \gamma(r-s)\overline{\mathfrak{F}}(r)dr\right)\right]\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)ds.
$$

In fact, $\overline{S}$ *and* $\overline{R}$ *are the limit of* $\frac{1}{N}S^N$ *and* $\frac{1}{N}R^N$, *respectively, where* S^N *and* R^N *are defined in* (2.9) *(recall that, for simplicity, we assumed that no individual is initially in the R compartment). But, from the definition of* γ *in* (2.8), $\gamma(t) = 0$ *for all* $t \in [\eta, \eta+\theta]$ *and* $\gamma_0(t) = 0$ *for all* $t \in [\eta_0, \eta_0+\theta_0)$, *hence*

$$
\overline{S}(t) = \overline{S}(0)\exp\left(-\int_0^t \overline{\mathfrak{F}}(r)dr\right) + \overline{I}(0)\mathbb{E}\left[\mathbb{1}_{\eta_0+\theta_0\leq t}\exp\left(-\int_{\eta_0+\theta_0}^t \overline{\mathfrak{F}}(r)dr\right)\right]
$$

$$+ \int_0^t \mathbb{E}\left[\mathbb{1}_{\eta+\theta \leq t-s} \exp\left(-\int_{s+\eta+\theta}^t \overline{\mathfrak{F}}(r)dr\right)\right] \overline{\mathfrak{F}}(s)\overline{\mathfrak{S}}(s)ds,$$

and

$$\overline{R}(t) = \overline{I}(0)\mathbb{P}(\eta_0 \leq t < \eta_0 + \theta_0 \mid \eta_0 > 0) + \int_0^t \mathbb{P}(\eta \leq t - s < \eta + \theta)\overline{\mathfrak{F}}(s)\overline{\mathfrak{S}}(s)ds.$$

Combined with $\overline{\mathfrak{F}}(t) = \beta\overline{I}(t)$, $\overline{\mathfrak{S}}(t) = \overline{S}(t)$ *and*

$$\overline{S}(t) + \overline{I}(t) + \overline{R}(t) = \overline{U}(t) + \overline{I}(t) = 1,$$

this yields the result stated in Theorem 3.3 in [26].

4. THE ENDEMIC EQUILIBRIUM

When the disease persists in the population, we say that the disease becomes endemic, and if it reaches an equilibrium, it is called the endemic equilibrium. This corresponds to a balance between the number of new infections and new recoveries. In this section, we study the conditions that lead to an endemic equilibrium in the FLLN limit. We first recall the endemic equilibrium behavior of the classical SIS model discussed in Remark 2.1. Given the infectivity rate β, the basic reproduction number is given by $R_0 = \beta/\mathbb{E}[\eta]$. It is well known (see, e.g., [26, Section 4.3], and also [4] for a discussion on the Markovian model) that if $R_0 \leq 1$, $\overline{I}(t) \to 0$ as $t \to \infty$, and if $R_0 > 1$ and $\overline{I}(0) > 0$, $\overline{I}(t) \to 1 - R_0^{-1}$ as $t \to \infty$. This can be easily obtained from the expression of $\overline{I}(t)$ in (3.12). In words, in the case $R_0 \leq 1$, the disease-free steady state is globally asymptotically stable, and in the case $R_0 > 1$, the disease-free steady state is unstable and there is exactly one endemic steady state, which is globally stable. Recall that in this model, $\overline{\lambda}(t) = \beta F^c(t)$. Thus it is equivalent to write

$$R_0 = \int_0^\infty \overline{\lambda}(t)dt, \tag{4.1}$$

since $\int_0^\infty F^c(t)dt = \mathbb{E}[\eta]$. In fact, the expression of R_0 in (4.1) is the definition of the basic reproduction number in the Kermack and McKendrick model with an average infectivity function $\overline{\lambda}(t)$, because it represents the average number of individuals infected by an infectious individual in a fully susceptible population.

We make the following Assumption to ensure that $\overline{\lambda}(t) \to 0$ as $t \to \infty$.

Assumption 4.1. *Almost surely,* $\eta < \infty$.

We make the following assumptions on the random susceptibility and infectivity functions in order to study the equilibria of our model.

Assumption 4.2. *The random functions* $t \mapsto \gamma(t)$ *and* $t \mapsto \gamma_0(t)$ *are non-decreasing a.s. Moreover the pair* (λ_0, γ_0) *is distributed as follows. Let* $\xi \geq 0$ *be a random variable such that* $\xi \leq \eta$ *almost surely. Let* χ *be a Bernoulli random variable with* $\mathbb{P}(\chi = 1) = \overline{I}(0)$, *independent from* $(\lambda, \gamma, \eta, \xi)$. *Then*

$$\lambda_0(t) = \begin{cases} 0 & \text{if } \chi = 0, \\ \lambda(t+\xi) & \text{if } \chi = 1. \end{cases} \quad \text{and} \quad \gamma_0(t) = \begin{cases} 1 & \text{if } \chi = 0, \\ \gamma(t+\xi) & \text{if } \chi = 1. \end{cases}$$

The random variable ξ *represents the age of infection at time zero of the initially infectious individuals.*

We define

$$\gamma_* := \sup_{t \geq 0} \gamma(t) = \lim_{t \to +\infty} \gamma(t), \quad \text{and} \quad \gamma_{0,*} := \lim_{t \to +\infty} \gamma_0(t).$$

In the classical SIS model, the susceptibility functions are given by $\gamma(t) = \mathbb{1}_{t \geq \eta}$ and $\gamma_0(t) = \mathbb{1}_{t \geq \eta_0}$ such that $\gamma_* = 1$ a.s. However, in general, after being infected and recovered, individuals lose

immunity gradually and do not necessarily reach "full" susceptibility (being equal to 1). Thus, γ_* may take any value in $[0,1]$ and is a priori random.

We find that the classification of the endemic equilibria depends on the law of γ_*, more specifically on whether R_0 is smaller than or larger than $\mathbb{E}[1/\gamma_*]$. Note that this expectation $\mathbb{E}[1/\gamma_*]$ may be infinite in general (for example if $\mathbb{P}(\gamma_* = 0) > 0$). We first prove the following result under the condition that $R_0 < \mathbb{E}[1/\gamma_*]$.

Theorem 4.1. *Under Assumption 2.1, 4.2, if $R_0 < \mathbb{E}\left[1/\gamma_*\right]$, there exists $\overline{\mathfrak{S}}_* \in [0,1]$ such that*

$$(\overline{\mathfrak{F}}(t), \overline{\mathfrak{S}}(t)) \to (0, \overline{\mathfrak{S}}_*) \quad as \quad t \to \infty,$$

where

$$\overline{\mathfrak{S}}_* = \lim_{t \to +\infty} \overline{\mathfrak{S}}(t) = \mathbb{E}\left[\gamma_{0,*} \exp\left(-\int_0^{+\infty} \gamma_0(r)\overline{\mathfrak{F}}(r)dr\right)\right]$$
$$+ \int_0^{+\infty} \mathbb{E}\left[\gamma_* \exp\left(-\int_s^{+\infty} \gamma(r-s)\overline{\mathfrak{F}}(r)dr\right)\right]\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)ds. \qquad (4.2)$$

As a consequence, $(\overline{U}(t), \overline{I}(t)) \to (\overline{U}_*, 0)$ as $t \to \infty$, where $\overline{U}_* = 1$. In the special case $\gamma_* = 1$ a.s., we also have $\overline{\mathfrak{S}}_* = 1$.

Remark 4.1. *Without the assumption on the monotonicity of the function γ, when $R_0 < \mathbb{E}\left[(\sup_t \gamma(t))^{-1}\right]$ the same proof as that of Theorem 4.1 shows that as $t \to +\infty$, $\overline{\mathfrak{F}}(t) \to 0$ and $\overline{I}(t) \to 0$.*

Note that in this theorem, we do not assume $\mathbb{E}\left[1/\gamma_*\right] < +\infty$, however, we do assume that $R_0 < \infty$, that is, $\overline{\lambda}(t)$ is integrable.

Remark 4.2. *In [18, Proposition 8.9] one can find a similar result for a model with demography.*

The case $R_0 \geq \mathbb{E}[1/\gamma_*]$ is more complex. We make the following additional assumptions.

Assumption 4.3. *There exists a non-negative random variable t_* such that $\mathbb{E}\left[t_*\right] < +\infty$ and for $t \geq t_*$, $\gamma(t) \geq \frac{\gamma_*}{2}$ a.s.*

Assumption 4.4. *γ_* is deterministic and for any $\delta \in (0,1)$, there exists a deterministic $t_\delta > 0$ such that*

$$\gamma_0(t_\delta) \wedge \gamma(t_\delta) \geq (1-\delta)\gamma_* \text{ almost surely.} \qquad (4.3)$$

Assumption 4.5. *There exists a positive decreasing function h such that $h(0) = 1$ and for all $s, t \in \mathbb{R}_+$, $\overline{\lambda}(s+t) \geq h(s)\overline{\lambda}(t)$. The same holds for $\overline{\lambda}_0$. In addition, $\overline{\lambda}_0$ is continuous and $\overline{\lambda}$ is of bounded total variation.*

Theorem 4.2. (i) *Suppose that Assumptions 2.1, 4.2 and 4.3 hold and $R_0 > \mathbb{E}\left[\frac{1}{\gamma_*}\right]$. If there exists $(\overline{\mathfrak{S}}_*, \overline{\mathfrak{F}}_*)$ such that $(\overline{\mathfrak{S}}(t), \overline{\mathfrak{F}}(t)) \xrightarrow[t \to +\infty]{} (\overline{\mathfrak{S}}_*, \overline{\mathfrak{F}}_*)$, either $\overline{\mathfrak{S}}_* \in [0,1]$ and $\overline{\mathfrak{F}}_* = 0$, or else*

$$\overline{\mathfrak{S}}_* = \frac{1}{R_0}$$

and $\overline{\mathfrak{F}}_*$ is the unique positive solution of the equation

$$\int_0^{+\infty} \mathbb{E}\left[\exp\left(-\int_0^s \gamma\left(\frac{r}{\overline{\mathfrak{F}}_*}\right)dr\right)\right]ds = R_0. \qquad (4.4)$$

In the second case, $(\overline{I}(t), \overline{U}(t)) \to (\overline{I}_*, \overline{U}_*)$ as $t \to \infty$, where $\overline{U}_* = 1 - \overline{I}_*$ and

$$\overline{I}_* = \frac{\mathbb{E}\left[\eta\right]\overline{\mathfrak{F}}_*}{R_0}. \qquad (4.5)$$

If $R_0 = \mathbb{E}\left[\frac{1}{\gamma_*}\right]$, the same statement holds but (4.4) does not admit any positive solution, so, necessarily, $\overline{\mathfrak{F}}_* = 0$ and thus $\overline{I}_* = 0$.

(ii) Assume in addition that Assumption 4.4 and Assumption 4.5 hold and that $\overline{\mathfrak{F}}(0) > 0$. Then there exists $c > 0$ such that for all $t > 0$, $\overline{\mathfrak{F}}(t) \geq c$. In particular $\overline{\mathfrak{F}}(t)$ cannot tend to zero as $t \to \infty$.

Remark 4.3. *Assumption 4.4 ensures that after some time the average susceptibility returns above $\frac{1}{R_0}$ if there are not too many re-infections and Assumption 4.5 ensures that the force of infection does not decrease too rapidly.*

Corollary 4.1. *Suppose that Assumptions 2.1, 4.2 and 4.3 hold and $R_0 > \mathbb{E}\left[\frac{1}{\gamma_*}\right]$. Assume that $\gamma(t) = \gamma_* \mathbb{1}_{t \geq \zeta}$, where ζ is a random variable satisfying $\zeta \geq \eta$ almost surely and $\mathbb{E}[\zeta] < +\infty$ and γ_* is a random variable taking values in $(0, 1]$. Then,*

$$\overline{\mathfrak{F}}_* = \frac{R_0 - \mathbb{E}\left[\frac{1}{\gamma_*}\right]}{\mathbb{E}[\zeta]}, \tag{4.6}$$

and

$$\overline{I}_* = \frac{\mathbb{E}[\eta]\overline{\mathfrak{F}}_*}{R_0} = \frac{\mathbb{E}[\eta]}{\mathbb{E}[\zeta]}\left(1 - R_0^{-1}\mathbb{E}\left[\frac{1}{\gamma_*}\right]\right) \tag{4.7}$$

where R_0 is given in (4.1).

Proof. In this case, equation (4.4) becomes

$$\int_0^{+\infty} \mathbb{E}\left[\exp\left(-\gamma_* \int_0^s \mathbb{1}_{\frac{r}{\overline{\mathfrak{F}}_*} \geq \zeta} dr\right)\right] ds = R_0.$$

By a change of variables and Fubini's theorem, the left hand side is equal to

$$\overline{\mathfrak{F}}_* \int_0^{+\infty} \mathbb{E}\left[\exp\left(-\overline{\mathfrak{F}}_* \gamma_* \int_0^s \mathbb{1}_{r \geq \zeta} dr\right)\right] ds = \overline{\mathfrak{F}}_* \int_0^{+\infty} \mathbb{E}\left[\mathbb{1}_{\zeta > s} + \mathbb{1}_{\zeta \leq s} \exp\left(-\overline{\mathfrak{F}}_* \gamma_*(s - \zeta)\right)\right] ds$$

$$= \overline{\mathfrak{F}}_* \mathbb{E}[\zeta] + \overline{\mathfrak{F}}_* \mathbb{E}\left[\int_\zeta^{+\infty} \exp\left(-\overline{\mathfrak{F}}_* \gamma_*(s - \zeta)\right) ds\right]$$

$$= \overline{\mathfrak{F}}_* \mathbb{E}[\zeta] + \mathbb{E}\left[\frac{1}{\gamma_*}\right].$$

This gives the expression in (4.6). $\qquad\square$

Remark 4.4. *For the classical models discussed in Remark 2.1, the above allows us to recover previously known results.*

(i) *In the SIS model, we have $\gamma_* = 1$ and $\eta = \zeta$ almost surely, so we obtain that $\overline{I}(t) \to 0$ as $t \to \infty$ if $R_0 \leq 1$, and that, if $R_0 > 1$, the only other possible limit for $\overline{I}(t)$ is given by*

$$\overline{I}_* = \frac{\mathbb{E}[\eta]}{\mathbb{E}[\zeta]}\left(1 - R_0^{-1}\mathbb{E}\left[\frac{1}{\gamma_*}\right]\right) = 1 - \frac{1}{R_0}.$$

(ii) *For the SIRS model, we have $\gamma_* = 1$ and $\zeta = \eta + \theta$. As a result, applying Corollary 4.1, we see that, when $R_0 > 1$, the only possible positive limit for $\overline{I}(t)$ is*

$$\overline{I}_* = \frac{\mathbb{E}[\eta]}{\mathbb{E}[\eta] + \mathbb{E}[\theta]}\left(1 - \frac{1}{R_0}\right).$$

In the same way, we can also deduce that $\overline{R}_* := \lim_{t \to +\infty} \overline{R}(t)$ is given by

$$\overline{R}_* = 1 - \overline{\mathfrak{S}}_* - \overline{I}_* = \frac{\mathbb{E}[\theta]}{\mathbb{E}[\eta] + \mathbb{E}[\theta]}\left(1 - \frac{1}{R_0}\right).$$

We refer to Proposition 4.2 of [26] for a previous derivation of this equilibrium in the case of a deterministic ODE model.

Our results in (4.6)–(4.7) *thus extend those for classical compartmental models. Note that γ_* is random and takes values in* $[0,1]$, *indicating potentially partial susceptibility after recovery.*

For the SIS and SIRS models discussed in Remark 4.4, it is known that, if $R_0 > 1$ and $\overline{I}(0) > 0$, then $\overline{I}(t)$ does indeed converge to the endemic equilibrium $\overline{I}_*$ as $t \to \infty$. We are not yet able to prove such a result for our general model.

We make the following conjecture on the convergence to the equilibrium in the case $R_0 > \mathbb{E}\left[1/\gamma_*\right]$.

Conjecture 4.1. *Under Assumptions 2.1, 4.2–4.5, if $R_0 > \mathbb{E}\left[\frac{1}{\gamma_*}\right]$ and $\overline{\mathfrak{F}}(0) > 0$, then*

$$(\overline{\mathfrak{F}}(t), \overline{\mathfrak{S}}(t)) \to (\overline{\mathfrak{F}}_*, \overline{\mathfrak{S}}_*) \quad as \quad t \to \infty,$$

where $\overline{\mathfrak{S}}_* = 1/R_0$ and $\overline{\mathfrak{F}}_*$ is the unique positive solution of (4.4).

This conjecture is equivalent to another. Indeed, by applying the Arzelà-Ascoli Theorem to the set of functions $\{(\tau_j \overline{\mathfrak{S}}, \tau_j \overline{\mathfrak{F}}), j \in \mathbb{N}\}$ where $\tau_j x(t) := x(t + t_j)$ and $(t_j)_j \subset \mathbb{R}_+, t_j \to +\infty$ as $j \to +\infty$, there exists a subsequence of pairs $(\tau_j \overline{\mathfrak{S}}, \tau_j \overline{\mathfrak{F}})$ denoted again $(\tau_j \overline{\mathfrak{S}}, \tau_j \overline{\mathfrak{F}})$ such that $(x_j, y_j) := (\tau_j \overline{\mathfrak{S}}, \tau_j \overline{\mathfrak{F}}) \to (x, y)$ uniformly on compact sets as $j \to +\infty$. Note that the pair $(x_j(t), y_j(t))$ satisfies the following system of equations: for $t \geq -t_j$,

$$\begin{cases} x_j(t) = \mathbb{E}\left[\gamma_0(t + t_j)\exp\left(-\int_0^{t+t_j} \gamma_0(r)\overline{\mathfrak{F}}(r)dr\right)\right] \\ \qquad\qquad + \int_{-t_j}^t \mathbb{E}\left[\gamma(t-s)\exp\left(-\int_s^t \gamma(r-s)y_j(r)dr\right)\right] x_j(s)y_j(s)ds, \qquad (4.8) \\ y_j(t) = \overline{I}(0)\overline{\lambda}_0(t + t_j) + \int_{-t_j}^t \overline{\lambda}(t-s)x_j(s)y_j(s)ds. \qquad\qquad\qquad\qquad (4.9) \end{cases}$$

As a result, as the first terms of the right hand side of (4.8) and (4.9) tend to zero when $j \to +\infty$, and $(x_j(t), y_j(t)) \to (x(t), y(t))$ for all $t \in \mathbb{R}$, we deduce by the dominated convergence theorem that the pair (x, y) satisfies the following set of equations,

$$\begin{cases} y(t) = \int_{-\infty}^t \overline{\lambda}(t-s)x(s)y(s)ds, \qquad\qquad\qquad\qquad\qquad\qquad (4.10) \\ \int_{-\infty}^t \mathbb{E}\left[\exp\left(-\int_s^t \gamma(r-s)y(r)dr\right)\right] x(s)y(s)ds = 1. \qquad\qquad (4.11) \end{cases}$$

We can remark that the constant pair $(\frac{1}{R_0}, \overline{\mathfrak{F}}_*)$ where $\overline{\mathfrak{F}}_*$ is the unique solution of (4.4) is a solution of (4.10)-(4.11). Hence if this solution is unique, all converging subsequences of $(\tau_j \overline{\mathfrak{S}}, \tau_j \overline{\mathfrak{F}})$ have the same limit, from which we can easily conclude the convergence of $(\overline{\mathfrak{F}}(t), \overline{\mathfrak{S}}(t))$ as $t \to \infty$.

Thus, Conjecture 4.1 is equivalent to the following.

Conjecture 4.2. *Under Assumptions 4.2–4.5, if $R_0 > \mathbb{E}\left[\frac{1}{\gamma_*}\right]$ and $\overline{\mathfrak{F}}(0) > 0$, the set of equations* (4.10)-(4.11) *has a unique positive and bounded solution on $\mathbb{R}$.*

5. Relating to the Kermack and McKendrick PDEs for the SIRS model

5.1. **The FLLN limits with a special set of susceptibility/infectivity functions and initial conditions.** We consider the special family of susceptibility and infectivity functions:

$$\lambda(t) = \widetilde{\lambda}(t)\mathbb{1}_{t<\eta} \quad \text{and} \quad \gamma(t) = \widetilde{\gamma}(t-\eta)\mathbb{1}_{t>\eta}$$

where $\widetilde{\lambda}(t)$ and $\widetilde{\gamma}(t)$ are deterministic functions, representing the infectivity and susceptibility functions, and the infectious period η is a random variable with cumulative distribution function F. Then we have $\overline{\lambda}(t) = \widetilde{\lambda}(t)F^c(t)$. In addition, let f be the density function of F and $\mu_F(t) := \frac{f(t)}{F^c(t)}$ be the hazard rate function of F. Then one can also write

$$F^c(t) = \exp\left(-\int_0^t \mu_F(s)ds\right), \tag{5.1}$$

For an initially infected individual, we recall that ξ is the elapsed times since the individual was last infected before time zero. We also assume that the infectivity function of an initially infected individual is the same as that of a newly infected individual but shifted by the time elapsed since infection before time zero, that is,

$$\lambda_0(t) = \widetilde{\lambda}(t + \xi)\mathbb{1}_{t<\eta_0}$$

where η_0 is the duration of the remaining infectious period after time zero, whose distribution depends on the elapsed infection time ξ. To specify the distribution of ξ, we assume that there exists a function $\overline{I}(0, \tau)$ such that $\overline{I}(0) = \int_0^\infty \overline{I}(0, \tau)d\tau$, that is, $\overline{I}(0, \tau)$ is the density of the distribution of $\overline{I}(0)$ over the ages of infection. Then we can specify the distribution of ξ as

$$\mathbb{P}(\xi > x) = \frac{1}{\overline{I}(0)} \int_x^{+\infty} \overline{I}(0, r)dr. \tag{5.2}$$

And the conditional distribution of η_0 given ξ is given by

$$\mathbb{P}(\eta_0 > t|\xi) = \frac{F^c(\xi + t)}{F^c(\xi)} = \exp\left(-\int_\xi^{\xi+t} \mu_F(r)dr\right). \tag{5.3}$$

It is then clear that

$$\overline{\lambda}_0(t) = \mathbb{E}\left[\widetilde{\lambda}(t + \xi)\mathbb{1}_{t<\eta_0}\right] = \frac{1}{\overline{I}(0)} \int_0^{+\infty} \widetilde{\lambda}(t + \tau)\overline{I}(0, \tau) \exp\left(-\int_\tau^{t+\tau} \mu_F(s)ds\right) d\tau. \tag{5.4}$$

Next, to specify the susceptibility $\gamma_0(t)$ associated with an individual, we consider the three groups of individuals at time zero, fully susceptible $\overline{S}(0)$, initially infected $\overline{I}(0)$, and initially recovered $\overline{R}(0)$ such that $\overline{S}(0) + \overline{I}(0) + \overline{R}(0) = 1$. Note that we have combined the fully susceptible and initially recovered individuals as one group in the model description in Section 2. Thus, the model discussed in this section is in fact a generalized SIRS model. Moreover, we shall in this section consider that individuals in the R compartment have recovered from the disease but are not necessarily immune. In accordance with [16] (see Section 5.2 below), once an individual recovers from the infection, after some potential immune period, the immunity is gradually lost, and the individual may become infected again, and the process is repeated at each new infection with a same realization of the random infectivity and susceptibility functions. This means that, the susceptible individuals have never been infected. For the initially fully susceptible individuals, their susceptibility γ_0 equals 1. For the initially infected individuals, their susceptibility starts to take effect after the remaining infection period η_0, i.e. $\gamma_0(t) = \mathbb{1}_{t \geq \eta_0}\widetilde{\gamma}(t - \eta_0)$. For the initially recovered individuals, their susceptibility depends on the elapsed time since the last recovery, which we denote by ϑ, so that, for these individuals, $\gamma_0(t) = \widetilde{\gamma}(t + \vartheta)$. Let χ be a random variable indicating the group of an individual at time $t = 0$, with the following law

$$\mathcal{L}(\chi) = \overline{S}(0)\delta_S + \overline{I}(0)\delta_I + \overline{R}(0)\delta_R.$$

Then the initial susceptibility $\gamma_0(t)$ of an individual can be written as

$$\gamma_0(t) = \mathbb{1}_{\chi=S} + \widetilde{\gamma}(t - \eta_0)\mathbb{1}_{t \geq \eta_0, \chi=I} + \widetilde{\gamma}(t + \vartheta)\mathbb{1}_{\chi=R}.$$

To specify the distribution of ϑ, we assume that there exists a function $\overline{R}(0,\theta)$ such that $\overline{R}(0) = \int_0^\infty \overline{R}(0,\theta)d\theta$, that is, $\overline{R}(0,\theta)$ is the density of the distribution of $\overline{R}(0)$ over the age of recovery. Then we can specify the distribution of ϑ by

$$\mathbb{P}(\vartheta > x) = \frac{1}{\overline{R}(0)} \int_x^{+\infty} \overline{R}(0,r)dr. \tag{5.5}$$

Thus, we have

$$\mathbb{E}\left[\gamma_0(t) \exp\left(-\int_0^t \gamma_0(r)\overline{\mathfrak{F}}(r)dr\right)\right] = \overline{S}(0) \exp\left(-\int_0^t \overline{\mathfrak{F}}(r)dr\right)$$

$$+ \int_0^t \int_0^\infty \widetilde{\gamma}(t-\tau)\mu_F(\tau+r)\overline{I}(0,r) \exp\left(-\int_r^{\tau+r} \mu_F(s)ds\right) \exp\left(-\int_\tau^t \widetilde{\gamma}(s-\tau)\overline{\mathfrak{F}}(s)ds\right) drd\tau$$

$$+ \int_0^\infty \widetilde{\gamma}(t+\tau)\overline{R}(0,\tau) \exp\left(-\int_0^t \widetilde{\gamma}(s+\tau)\overline{\mathfrak{F}}(s)ds\right) d\tau.$$

Note that, concerning the second term, we have to condition on ξ in order to compute its expression.

We denote, $\overline{S}(t)$ the proportion of susceptible individuals who have never been infected, $\overline{I}(t)$ the proportion of infected individuals and $\overline{R}(t)$ the proportion of recovered individuals at time t.

Therefore we obtain the following expression of the limit $(\overline{\mathfrak{S}}, \overline{\mathfrak{F}})$.

Proposition 5.1. *For the generalized SIRS model described above, the limit $(\overline{\mathfrak{S}}, \overline{\mathfrak{F}})$ is given by*

$$\overline{\mathfrak{S}}(t) = \overline{S}(0) \exp\left(-\int_0^t \overline{\mathfrak{F}}(r)dr\right)$$

$$+ \int_0^t \int_0^\infty \widetilde{\gamma}(t-\tau)\mu_F(\tau+r)\overline{I}(0,r) \exp\left(-\int_r^{\tau+r} \mu_F(s)ds\right) \exp\left(-\int_\tau^t \widetilde{\gamma}(s-\tau)\overline{\mathfrak{F}}(s)ds\right) drd\tau$$

$$+ \int_0^\infty \widetilde{\gamma}(t+\tau)\overline{R}(0,\tau) \exp\left(-\int_0^t \widetilde{\gamma}(s+\tau)\overline{\mathfrak{F}}(s)ds\right) d\tau$$

$$+ \int_0^t \left(\int_0^{t-r} \widetilde{\gamma}(t-r-\tau)\mu_F(\tau)F^c(\tau) \exp\left(-\int_{\tau+r}^t \widetilde{\gamma}(s-r-\tau)\overline{\mathfrak{F}}(s)ds\right) d\tau\right) \overline{\mathfrak{S}}(r)\overline{\mathfrak{F}}(r)dr, \tag{5.6}$$

and

$$\overline{\mathfrak{F}}(t) = \int_0^{+\infty} \widetilde{\lambda}(t+\tau)\overline{I}(0,\tau) \exp\left(-\int_\tau^{t+\tau} \mu_F(s)ds\right) d\tau$$

$$+ \int_0^t \widetilde{\lambda}(t-\tau)\overline{\mathfrak{S}}(\tau)\overline{\mathfrak{F}}(\tau) \exp\left(-\int_0^{t-\tau} \mu_F(s)ds\right) d\tau. \tag{5.7}$$

In addition,

$$\overline{S}(t) = \overline{S}(0) \exp\left(-\int_0^t \overline{\mathfrak{F}}(r)dr\right),$$

$$\overline{I}(t) = \int_0^{+\infty} \exp\left(-\int_\tau^{t+\tau} \mu_F(s)ds\right) \overline{I}(0,\tau)d\tau + \int_0^t F^c(t-s)\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)ds$$

$$\overline{R}(t) = \int_0^{+\infty} \exp\left(-\int_0^t \widetilde{\gamma}(\tau+r)\overline{\mathfrak{F}}(r)dr\right) \overline{R}(0,\tau)d\tau$$

$$+ \int_0^{+\infty} \int_0^t \exp\left(-\int_s^t \widetilde{\gamma}(r-s)\overline{\mathfrak{F}}(r)dr\right) \mu_F(\tau+s) \exp\left(-\int_\tau^{\tau+s} \mu_F(r)dr\right) ds\overline{I}(0,\tau)d\tau$$

$$+ \int_0^t \int_s^t \exp\left(-\int_u^t \widetilde{\gamma}(r-u)\overline{\mathfrak{F}}(r)dr\right)\mu_F(u-s)\exp\left(-\int_0^{u-s}\mu_F(r)dr\right)du\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)ds$$

5.2. **The associated Kermack and McKendrick PDE model.** In [16], the model introduced by Kermack and McKendrick in [22, 23] was reformulated as follows. Let $\overline{S}(t)$ denote the proportion of susceptible individuals who have never been infected, let $\overline{I}(t,\tau)$ denote the density of infectious individuals with infection-age $\tau \geq 0$ and $\overline{R}(t,\theta)$ the density of recovered individuals with recovery-age $\theta \geq 0$. Then, given an initial condition $(\overline{S}(0), \overline{I}(0,\cdot), \overline{R}(0,\cdot))$ such that

$$\overline{S}(0) + \int_0^\infty \overline{I}(0,\tau)d\tau + \int_0^\infty \overline{R}(0,\theta)d\theta = 1,$$

the system evolves according to the following set of partial differential equations:

$$\begin{cases} \dfrac{d\overline{S}}{dt}(t) = -\overline{S}(t)\displaystyle\int_0^{+\infty}\widetilde{\lambda}(\tau)\overline{I}(t,\tau)d\tau \\[2ex] \dfrac{\partial\overline{I}}{\partial t}(t,\tau) + \dfrac{\partial\overline{I}}{\partial\tau}(t,\tau) = -\mu_F(\tau)\overline{I}(t,\tau) \\[2ex] \dfrac{\partial\overline{R}}{\partial t}(t,\tau) + \dfrac{\partial\overline{R}}{\partial\tau}(t,\tau) = -\overline{R}(t,\tau)\widetilde{\gamma}(\tau)\displaystyle\int_0^{+\infty}\widetilde{\lambda}(r)\overline{I}(t,r)dr \\[2ex] \overline{I}(t,0) = \left(\overline{S}(t) + \displaystyle\int_0^{+\infty}\widetilde{\gamma}(\theta)\overline{R}(t,\theta)d\theta\right)\displaystyle\int_0^{+\infty}\widetilde{\lambda}(\tau)\overline{I}(t,\tau)d\tau \\[2ex] \overline{R}(t,0) = \displaystyle\int_0^{+\infty}\mu_F(\tau)\overline{I}(t,\tau)d\tau. \end{cases} \quad (5.8)$$

In this case we set

$$\overline{I}(t) = \int_0^\infty \overline{I}(t,\tau)d\tau \text{ and } \overline{R}(t) = \int_0^\infty \overline{R}(t,\tau)d\tau.$$

Integrating (5.8) along the characteristics, we obtain:

$$\begin{cases} \overline{S}(t) = \overline{S}(0)\exp\left(-\displaystyle\int_0^t\int_0^{+\infty}\widetilde{\lambda}(\tau)\overline{I}(s,\tau)d\tau ds\right), \\[3ex] \overline{I}(t,\tau) = \overline{I}(0,\tau-t)\dfrac{F^c(\tau)}{F^c(\tau-t)}\mathbb{1}_{\tau>t} + \overline{I}(t-\tau,0)F^c(\tau)\mathbb{1}_{t\geq\tau}, \\[3ex] \overline{R}(t,\tau) = \overline{R}(t-\tau,0)\exp\left(-\displaystyle\int_0^\tau\widetilde{\gamma}(s)\int_0^{+\infty}\widetilde{\lambda}(r)\overline{I}(t+s-\tau,r)drds\right)\mathbb{1}_{\tau<t} \\[2ex] \qquad\qquad + \overline{R}(0,\tau-t)\exp\left(-\displaystyle\int_{\tau-t}^\tau\widetilde{\gamma}(s)\int_0^{+\infty}\widetilde{\lambda}(r)\overline{I}(t-\tau+s,r)drds\right)\mathbb{1}_{\tau\geq t}, \\[3ex] \overline{I}(t,0) = \left(\overline{S}(t) + \displaystyle\int_0^{+\infty}\widetilde{\gamma}(\theta)\overline{R}(t,\theta)d\theta\right)\displaystyle\int_0^{+\infty}\widetilde{\lambda}(\tau)\overline{I}(t,\tau)d\tau \\[3ex] \overline{R}(t,0) = \displaystyle\int_0^{+\infty}\overline{I}(0,\tau)\dfrac{f(t+\tau)}{F^c(\tau)}d\tau + \displaystyle\int_0^t f(t-\tau)\overline{I}(\tau,0)d\tau, \end{cases} \quad (5.9)$$

We show in the following theorem how the PDE solution $(\overline{S}(t), \overline{I}(t,\tau), \overline{R}(t,\tau))$ and $(\overline{\mathfrak{S}}, \overline{\mathfrak{F}})$ are related.

Theorem 5.1. *If $(\overline{S}(t), \overline{I}(t, \cdot), \overline{R}(t, \cdot), t \geq 0)$ is a solution of the system (5.8), and $(\overline{\mathfrak{S}}(\cdot), \overline{\mathfrak{F}}(\cdot))$ is given by*

$$
\begin{cases}
\overline{\mathfrak{S}}(t) = \overline{S}(t) + \int_0^{+\infty} \widetilde{\gamma}(\tau) \overline{R}(t, \tau) d\tau, \\[2mm]
\overline{\mathfrak{F}}(t) = \int_0^{+\infty} \widetilde{\lambda}(\tau) \overline{I}(t, \tau) d\tau,
\end{cases}
\tag{5.10}
$$

then $(\overline{\mathfrak{S}}(\cdot), \overline{\mathfrak{F}}(\cdot))$ solves (5.6)-(5.7). Conversely, given $(\overline{\mathfrak{S}}(\cdot), \overline{\mathfrak{F}}(\cdot))$, the solution to (5.6)-(5.7), the following is a solution to (5.8).

$$
\begin{cases}
\overline{S}(t) = \overline{S}(0) \exp\left(- \int_0^t \overline{\mathfrak{F}}(s) ds \right), & (5.11) \\[3mm]
\overline{I}(t, \tau) = \overline{I}(0, \tau - t) \dfrac{F^c(\tau)}{F^c(\tau - t)} \mathbb{1}_{\tau > t} + \overline{\mathfrak{F}}(t - \tau) \overline{\mathfrak{S}}(t - \tau) F^c(\tau) \mathbb{1}_{t \geq \tau}, & (5.12) \\[3mm]
\overline{R}(t, \tau) = \overline{R}(t - \tau, 0) \exp\left(- \int_0^\tau \widetilde{\gamma}(s) \overline{\mathfrak{F}}(t + s - \tau) ds \right) \mathbb{1}_{\tau < t} & \\[3mm]
\qquad\qquad + \overline{R}(0, \tau - t) \exp\left(- \int_{\tau - t}^\tau \widetilde{\gamma}(s) \overline{\mathfrak{F}}(t - \tau + s) ds \right) \mathbb{1}_{\tau \geq t}, & (5.13) \\[3mm]
\overline{R}(t, 0) = \int_0^{+\infty} \overline{I}(0, \tau) \dfrac{f(t + \tau)}{F^c(\tau)} d\tau + \int_0^t f(t - \tau) \overline{\mathfrak{S}}(\tau) \overline{\mathfrak{F}}(\tau) d\tau, & (5.14)
\end{cases}
$$

with F given by (5.1) and f is the density of F. If moreover $\overline{R}(0, \cdot)$ is bounded and f is locally bounded, then there exists a unique non-negative solution $(\overline{S}, \overline{I}(\cdot, \cdot), \overline{R}(\cdot, \cdot))$ of (5.8) such that $\overline{R}(\cdot, 0)$ is locally bounded.

Proof. We first prove the equivalence.

Plugging (5.10) in (5.9) we obtain the expressions in (5.11)-(5.14). Therefore plugging the expressions in (5.11)-(5.14) in (5.10), by the expression of F in (5.1), change of variables, and Fubini's theorem, we obtain (5.6) and (5.7).

Conversely from (5.6) and (5.7), by change of variables and Fubini's theorem, we obtain

$$
\overline{\mathfrak{S}}(t) = \overline{S}(0) \exp\left(- \int_0^t \overline{\mathfrak{F}}(s) ds \right) + \int_0^\infty \widetilde{\gamma}(\tau) \overline{R}(0, \tau - t) \exp\left(- \int_{\tau - t}^\tau \widetilde{\gamma}(s) \overline{\mathfrak{F}}(t - \tau + s) ds \right) \mathbb{1}_{\tau > t} d\tau
$$

$$
+ \int_0^\infty \widetilde{\gamma}(t - \tau) \Bigg[\int_0^{+\infty} \mu_F(\tau + r) \overline{I}(0, r) \exp\left(- \int_r^{\tau + r} \mu_F(s) ds \right) dr
$$

$$
+ \int_0^\tau \mu_F(\tau - r) \overline{\mathfrak{S}}(r) \overline{\mathfrak{F}}(r) F^c(\tau - r) dr \Bigg] \exp\left(- \int_0^{t - \tau} \widetilde{\gamma}(s) \overline{\mathfrak{F}}(s + \tau) ds \right) \mathbb{1}_{t \geq \tau} d\tau,
$$

and

$$
\overline{\mathfrak{F}}(t) = \int_0^\infty \widetilde{\lambda}(\tau) \Bigg[\overline{I}(0, \tau - t) \exp\left(- \int_{\tau - t}^\tau \mu_F(s) ds \right) \mathbb{1}_{\tau > t}
$$

$$
+ \overline{\mathfrak{S}}(t - \tau) \overline{\mathfrak{F}}(t - \tau) \exp\left(- \int_0^\tau \mu_F(s) ds \right) \mathbb{1}_{t \geq \tau} \Bigg] d\tau.
$$

Then by the expression of F in (5.1), and using the expressions in (5.11)-(5.14), we obtain the representations of $(\overline{\mathfrak{S}}, \overline{\mathfrak{F}})$ in (5.10). Combining (5.10) with (5.11)-(5.14) we obtain (5.9). So $(\overline{S}(\cdot), \overline{I}(\cdot, \cdot), \overline{R}(\cdot, \cdot))$ solves (5.8). This completes the proof of the equivalence.

We will now finally prove uniqueness of the solution of equation (5.8) on the interval $[0, T]$, with T arbitrary.

We recall that
$$\overline{I}(t) = \int_0^\infty \overline{I}(t, \tau)d\tau \text{ and } \overline{R}(t) = \int_0^\infty \overline{R}(t, \tau)d\tau.$$

From (5.9) plugging $\overline{R}(t, \tau)$ and $\overline{I}(t, \tau)$ in $\overline{R}(t, 0)$ and $\overline{I}(t, 0)$ we obtain

$$\overline{R}(t, 0) = \int_0^{+\infty} \overline{I}(0, \tau)\frac{f(t + \tau)}{F^c(\tau)}d\tau + \int_0^t f(t - \tau)\overline{I}(\tau, 0)d\tau \tag{5.15}$$

and

$$
\begin{aligned}
\overline{I}(t, 0) = &\left\{ \int_0^t \widetilde{\lambda}(\tau)\overline{I}(t - \tau, 0)F^c(\tau)d\tau + \int_t^\infty \widetilde{\lambda}(\tau)\overline{I}(0, \tau - t)\frac{F^c(\tau)}{F^c(\tau - t)}d\tau \right\} \\
&\times \left\{ \int_0^t \widetilde{\gamma}(\tau)\overline{R}(t - \tau, 0) \exp\left(-\int_0^\tau \widetilde{\gamma}(s) \int_0^{t+s-\tau} \widetilde{\lambda}(r)\overline{I}(t + s - \tau - r, 0)drds \right) \right. \\
&\qquad\qquad \times \exp\left(-\int_0^\tau \widetilde{\gamma}(s) \int_{t+s-\tau}^\infty \widetilde{\lambda}(r)\overline{I}(0, r - t - s + \tau)drds \right) d\tau \\
&+ \int_t^\infty \widetilde{\gamma}(\tau)\overline{R}(0, \tau - t) \exp\left(-\int_{\tau-t}^\tau \widetilde{\gamma}(s) \int_0^{t+s-\tau} \widetilde{\lambda}(r)\overline{I}(t + s - \tau - r, 0)drds \right) \\
&\qquad\qquad \times \exp\left(-\int_{\tau-t}^\tau \widetilde{\gamma}(s) \int_{t+s-\tau}^\infty \widetilde{\lambda}(r)\overline{I}(0, r - t - s + \tau)drds \right) d\tau \\
&+ \overline{S}(0) \exp\left(-\int_0^t \int_0^s \widetilde{\lambda}(\tau)\overline{I}(s - \tau, 0)F^c(\tau)d\tau ds \right) \exp\left(-\int_0^t \int_s^{+\infty} \widetilde{\lambda}(\tau)\overline{I}(0, \tau - s)\frac{F^c(\tau)}{F^c(\tau - s)}d\tau ds \right) \right\}.
\end{aligned}
\tag{5.16}
$$

Let $(\overline{S}_1, \overline{I}_1(\cdot, \cdot), \overline{R}_1(\cdot, \cdot))$ and $(\overline{S}_2, \overline{I}_2(\cdot, \cdot), \overline{R}_2(\cdot, \cdot))$ be two solutions of (5.8) with the same initial conditions. We set
$$\Delta\overline{R} = \overline{R}_1 - \overline{R}_2 \text{ and } \Delta\overline{I} = \overline{I}_1 - \overline{I}_2.$$

Consequently from (5.15)
$$\Delta\overline{R}(t, 0) = \int_0^t f(t - \tau)\Delta\overline{I}(\tau, 0)d\tau, \tag{5.17}$$

On the other hand, recalling that,
$$\overline{I}(t) = \int_0^\infty \overline{I}(t, \tau)d\tau \text{ and } \overline{R}(t) = \int_0^\infty \overline{R}(t, \tau)d\tau,$$

as $\widetilde{\lambda}(r) \leq \lambda_*$,
$$\int_0^{+\infty} \widetilde{\lambda}(\tau)\overline{I}(t, \tau)d\tau \leq \lambda_*\overline{I}(t) \leq \lambda_*$$

and $\widetilde{\gamma} \leq 1$,
$$\overline{S}(t) + \int_0^{+\infty} \widetilde{\gamma}(\theta)\overline{R}(t, \theta)d\theta \leq \overline{S}(t) + \overline{R}(t) \leq 1.$$

Therefore, using the fact that there exists $C > 0$, $\overline{R}(t, 0) \leq C$, and $F^c(\tau) \leq 1$, it follows that

$$
\begin{aligned}
|\Delta\overline{I}(t, 0)| \leq &\lambda_* \int_0^t |\Delta\overline{I}(t - \tau, 0)|d\tau + \lambda_* \int_0^t |\Delta\overline{R}(t - \tau, 0)|d\tau + C\lambda_*^2 \int_0^t \int_0^\tau \int_0^{t+s-\tau} |\Delta\overline{I}(t + s - \tau - r, 0)|drdsd\tau \\
&+ \lambda_*^2 \int_t^\infty \overline{R}(0, \tau - t) \int_{\tau-t}^\tau \int_0^{t+s-\tau} |\Delta\overline{I}(t + s - \tau - r, 0)|drdsd\tau + \lambda_*^2 \int_0^t \int_0^s |\Delta\overline{I}(s - \tau, 0)|d\tau ds \\
&\leq \left(\lambda_* + C\lambda_*^2 t^2 + 2\lambda_*^2 t \right) \int_0^t |\Delta\overline{I}(\tau, 0)|d\tau + \lambda_* \int_0^t |\Delta\overline{R}(\tau, 0)|d\tau. \tag{5.18}
\end{aligned}
$$

Adding the inequalities (5.17) and (5.18), since f is bounded on $[0, T]$, we deduce uniqueness from the Gronwall Lemma. $\qquad\square$

6. Proofs for the FLLN

In this section, we prove the FLLN. We start with the proof of Lemma 3.1.

Proof of Lemma 3.1. We can note that, if we multiply the equation for $x(t)$ by $y(t)$, we obtain

$$y(t)x(t) = \mathbb{E}\left[y(t)\gamma_0(t)\exp\left(-\int_0^t \gamma_0(r)y(r)dr\right)\right]$$
$$+ \int_0^t \mathbb{E}\left[y(t)\gamma(t-s)\exp\left(-\int_s^t \gamma(r-s)y(r)dr\right)\right]x(s)y(s)ds.$$

As a result,

$$\frac{d}{dt}\left(\mathbb{E}\left[\exp\left(-\int_0^t \gamma_0(r)y(r)dr\right)\right] + \int_0^t \mathbb{E}\left[\exp\left(-\int_s^t \gamma(r-s)y(r)dr\right)\right]x(s)y(s)ds\right) = 0.$$

Note that this function may not be differentiable everywhere since there may be an at most countable set of points where its right and left derivatives do not coincide. Nevertheless this function is almost everywhere differentiable and since its derivative is equal to zero almost everywhere, it is constant.

Hence, integrating between 0 and t, we obtain the result. $\qquad\square$

We next prove Theorem 3.1.

Proof of Theorem 3.1. Let $(x, y) \in D_+^2$ be a solution to the set of equation (3.1)–(3.2). From Assumption 2.1, $\gamma_0(t) \le 1$ and $\gamma(t-s) \le 1$. Hence the combination of (3.1) and (3.8) implies that $x(t) \le 1$. Moreover, from Assumption 2.1 if $\lambda(t) > 0\,(\lambda_0(t) > 0)$ then $\gamma(s) = 0\,(\gamma_0(s) = 0)$ for $0 \le s \le t$, we have

$$y(t) = \overline{I}(0)\overline{\lambda}_0(t) + \int_0^t \overline{\lambda}(t-s)x(s)y(s)ds$$
$$= \mathbb{E}\left[\lambda_0(t)\exp\left(-\int_0^t \gamma_0(r)y(r)dr\right)\right] + \int_0^t \mathbb{E}\left[\lambda(t-s)\exp\left(-\int_s^t \gamma(r-s)y(r)dr\right)\right]x(s)y(s)ds$$
$$\le \lambda^*,$$

where the last inequality follows from (3.8) and the fact that $\lambda_0(t), \lambda(t-s) \le \lambda^*$. Consequently, if (x, y) solves (3.1)–(3.2), then for $t \ge 0$, $x(t) \le 1$ and $y(t) \le \lambda^*$.

Suppose now that we have two solutions (x^1, y^1) and (x^2, y^2) of equations (3.1)–(3.2). Then we have

$$x^1(t) - x^2(t) = \mathbb{E}\left[\gamma_0(t)\left(\exp\left(-\int_0^t \gamma_0(r)y^1(r)dr\right) - \exp\left(-\int_0^t \gamma_0(r)y^2(r)dr\right)\right)\right]$$
$$+ \int_0^t \mathbb{E}\left[\gamma(t-s)\left(\exp\left(-\int_s^t \gamma(r-s)y^1(r)dr\right)x^1(s)y^1(s)\right.\right.$$
$$\left.\left. - \exp\left(-\int_s^t \gamma(r-s)y^2(r)dr\right)x^2(s)y^2(s)\right)\right]ds,$$
$$y^1(t) - y^2(t) = \int_0^t \overline{\lambda}(t-s)\left[x^1(s)y^1(s) - x^2(s)y^2(s)\right]ds.$$

We first deduce from the second relation, taking into account that for $i = 1, 2$, $x^i(s) \leq 1$ and $y^i(s) \leq \lambda^*$, that for any $t > 0$,

$$\left|y^1(t) - y^2(t)\right| \leq \lambda^* \int_0^t \left[\left|x^1(s) - x^2(s)\right|\left|y^1(s)\right| + \left|y^1(s) - y^2(s)\right|\left|x^2(s)\right|\right] ds$$

$$\leq \lambda^* \max(\lambda^*, 1) \int_0^t \left[\left|x^1(s) - x^2(s)\right| + \left|y^1(s) - y^2(s)\right|\right] ds. \qquad (6.1)$$

We now exploit the first relation. Since $|\exp(-a) - \exp(-b)| \leq |a - b|$, $\forall a, b \in \mathbb{R}_+$ and $\gamma \leq 1$, we have for $0 \leq t \leq T$,

$$\left|x^1(t) - x^2(t)\right| \leq \int_0^t \left|y^1(s) - y^2(s)\right| ds$$

$$+ \int_0^t \mathbb{E}\left[\left|\exp\left(-\int_s^t \gamma(r - s)y^1(r)dr\right) x^1(s)y^1(s)\right.\right.$$

$$\left.\left. - \exp\left(-\int_s^t \gamma(r - s)y^2(r)dr\right) x^2(s)y^2(s)\right|\right] ds$$

$$\leq \int_0^t \left|y^1(s) - y^2(s)\right| ds$$

$$+ \int_0^t \mathbb{E}\left[\left|\exp\left(-\int_s^t \gamma(r - s)y^1(r)dr\right)\right.\right.$$

$$\left.\left. - \exp\left(-\int_s^t \gamma(r - s)y^2(r)dr\right)\right|\right] x^1(s)y^1(s)ds$$

$$+ \int_0^t \mathbb{E}\left[\exp\left(-\int_s^t \gamma(r - s)y^2(r)dr\right)\right] \left|x^1(s)y^1(s) - x^2(s)y^2(s)\right| ds$$

$$\leq \int_0^t \left|y^1(s) - y^2(s)\right| ds + T \int_0^t \left|y^1(s) - y^2(s)\right| ds$$

$$+ \int_0^t \left|x^1(s)y^1(s) - x^2(s)y^2(s)\right| ds.$$

Hence, again, given $T > 0$, there exists a constant C such that for any $0 \leq t \leq T$,

$$\left|x^1(t) - x^2(t)\right| \leq C \int_0^t \left[\left|x^1(s) - x^2(s)\right| + \left|y^1(s) - y^2(s)\right|\right] ds. \qquad (6.2)$$

Uniqueness follows from (6.1), (6.2) and the Gronwall Lemma.

Now local existence follows by an approximation procedure, which exploits the estimates (6.1) and (6.2). Indeed, Picard iteration procedure works for an existence of integral equation as for the more technical ODEs. For detail, see Brunner [5]. Global existence then follows from the above a priori estimates, which forbid explosion. Theorem 3.1 is established. □

We next prove Theorem 3.2. We first construct a system of stochastic equations driven by Poisson random measures (PRMs), and then use a approach of the type of propagation of chaos as in [29].

Let $m \in D_+$, $(\lambda_i, \gamma_i)_{i \geq 0}$ a collection of i.i.d. random elements of D_+, Q a standard PRM on $\mathbb{R}_+^2$ independent of the previous collection. We define for $t \geq 0$,

$$\begin{cases} A^{(m)}(t) := \int_0^t \int_0^{+\infty} \mathbb{1}_{u \leq \Upsilon^{(m)}(r-)} Q(du, dr) \\ \Upsilon^{(m)}(t) := \gamma_{A^{(m)}(t)}(\varsigma^{(m)}(t))m(t) \end{cases} \qquad (6.3)$$

where $\varsigma^{(m)}$ is defined in the same manner as ς_1^N with $A^{(m)}$ instead of A_1^N.

Let

$$\overline{\mathfrak{F}}^{(m)}(t) = \mathbb{E}\left[\lambda_{A^{(m)}(t)}(\varsigma^{(m)}(t))\right], \quad \text{and} \quad \overline{\mathfrak{S}}^{(m)}(t) = \mathbb{E}\left[\gamma_{A^{(m)}(t)}(\varsigma^{(m)}(t))\right].$$

Lemma 6.1. *There exists a unique function $m^* \in D_+$ such that*

$$\overline{\mathfrak{F}}^{(m^*)} = m^*.$$

Moreover, $(\overline{\mathfrak{S}}^{(m^*)}, \overline{\mathfrak{F}}^{(m^*)})$ solves the set of equations (3.1)–(3.2).

Proof. Let us denote the jump times of the process $A^{(m)}$ by $(\tau_i^{(m)})_{i \geq 0}$ with $\tau_0^{(m)} = 0$. From Assumption 2.1, for $t \geq \tau_i^{(m)}$, if $\lambda_i(t - \tau_i^{(m)}) \neq 0$, we have

$$\int_{\tau_i^{(m)}}^t \gamma_i(r - \tau_i^{(m)})\overline{\mathfrak{F}}^{(m)}(r)dr = 0,$$

hence $A^{(m)}(t) = A^{(m)}(\tau_i^{(m)}) = i$. Therefore for $0 \leq i \leq A^{(m)}(t)$, $\lambda_i(t - \tau_i^{(m)}) = \lambda_i(t - \tau_i^{(m)})\mathbb{1}_{A^{(m)}(t)=i}$ a.s. We also recall that $\overline{\lambda}_0(t) = \mathbb{E}[\lambda_0(t)|\eta_0 > 0]$ and $\overline{I}(0) = \mathbb{P}(\eta_0 > 0)$ and we deduce that

$$\overline{\mathfrak{F}}^{(m)}(t) = \mathbb{E}\left[\lambda_{A^{(m)}(t)}(\varsigma^{(m)}(t))\right]$$

$$= \mathbb{E}\left[\sum_{i \geq 0}\lambda_i(t - \tau_i^{(m)})\mathbb{1}_{A^{(m)}(t)=i}\right]$$

$$= \mathbb{E}\left[\lambda_0(t) + \sum_{i=1}^{A^{(m)}(t)}\lambda_i(t - \tau_i^{(m)})\right]$$

$$= \mathbb{E}[\lambda_0(t)|\eta_0 > 0]\,\mathbb{P}(\eta_0 > 0) + \sum_{i \geq 1}\mathbb{E}\left[\lambda_i(t - \tau_i^{(m)})\mathbb{1}_{\tau_i^{(m)} \leq t}\right]$$

$$= \overline{I}(0)\overline{\lambda}_0(t) + \sum_{i \geq 1}\mathbb{E}\left[\lambda_i(t - \tau_i^{(m)})\mathbb{1}_{\tau_i^{(m)} \leq t}\right],$$

and as λ_i and $\tau_i^{(m)}$ are independent, we further obtain that

$$\overline{\mathfrak{F}}^{(m)}(t) = \overline{I}(0)\overline{\lambda}_0(t) + \sum_{i \geq 1}\mathbb{E}\left[\overline{\lambda}(t - \tau_i^{(m)})\mathbb{1}_{\tau_i^{(m)} \leq t}\right]$$

$$= \overline{I}(0)\overline{\lambda}_0(t) + \mathbb{E}\left[\int_0^t \overline{\lambda}(t - s)A^{(m)}(ds)\right].$$

Note that the process $t \mapsto A^{(m)}(t) - \int_0^t \Upsilon^{(m)}(s)ds$ is a martingale. So is also

$$\int_0^t \overline{\lambda}(t' - s)\left[A^{(m)}(ds) - \Upsilon^{(m)}(s)ds\right], \quad t \geq 0.$$

Hence its expectation is zero, in particular in the case $t' = t$. As a result,

$$\overline{\mathfrak{F}}^{(m)}(t) = \overline{I}(0)\overline{\lambda}_0(t) + \int_0^t \overline{\lambda}(t - s)m(s)\overline{\mathfrak{S}}^{(m)}(s)ds. \tag{6.4}$$

In addition,

$$\gamma_{A^{(m)}(t)}(\varsigma^{(m)}(t)) = \gamma_0(t)\mathbb{1}_{A^{(m)}(t)=0} + \sum_{i=1}^{+\infty}\gamma_i(t - \tau_i^{(m)})\mathbb{1}_{\tau_i^{(m)} \leq t}\mathbb{1}_{A^{(m)}(t)=i}.$$

Let
$$\mathcal{F}_t = \sigma\left(\left\{(\lambda_i)_{0 \leq i \leq A^{(m)}(t)}, (\gamma_i)_{0 \leq i \leq A^{(m)}(t)}, A^{(m)}(t'), 0 \leq t' \leq t\right\}\right).$$

Since $Q\big|_{]\tau_i^{(m)},t]}$ is independent of $\mathcal{F}_{\tau_i^{(m)}}$, we have

$$\mathbb{P}\left(A^{(m)}(t) = i \Big| \mathcal{F}_{\tau_i^{(m)}}\right) \mathbb{1}_{\tau_i^{(m)} \leq t}$$

$$= \mathbb{P}\left(Q\left(\{(s,u) \in \mathbb{R}_+^2, \tau_i^{(m)} < s \leq t, \gamma_i((s - \tau_i^{(m)})^-)m(s^-) \geq u\}\right) = 0 \Big| \mathcal{F}_{\tau_i^{(m)}}\right) \mathbb{1}_{\tau_i^{(m)} \leq t}$$

$$= \exp\left(-\int_{\tau_i^{(m)}}^t \gamma_i(r - \tau_i^{(m)})m(r)dr\right) \mathbb{1}_{\tau_i^{(m)} \leq t}.$$

Thus, since γ_0 and Q are also independent, we obtain

$$\overline{\mathfrak{S}}^{(m)}(t) = \mathbb{E}\left[\gamma_0(t)\mathbb{1}_{A^{(m)}(t)=0}\right] + \sum_{i \geq 1} \mathbb{E}\left[\gamma_i(t - \tau_i^{(m)})\mathbb{1}_{\tau_i^{(m)} \leq t}\mathbb{P}\left(A^{(m)}(t) = i \Big| \mathcal{F}_{\tau_i^{(m)}}\right)\right]$$

$$= \mathbb{E}\left[\gamma_0(t)\exp\left(-\int_0^t \gamma_0(r)m(r)dr\right)\right]$$

$$+ \sum_{i \geq 1} \mathbb{E}\left[\gamma_i(t - \tau_i^{(m)})\mathbb{1}_{\tau_i^{(m)} \leq t}\exp\left(-\int_{\tau_i^{(m)}}^t \gamma_i(r - \tau_i^{(m)})m(r)dr\right)\right].$$

Moreover, since γ_i and $\tau_i^{(m)}$ are independent, recalling that the law of γ is denoted by μ, we further obtain

$$\overline{\mathfrak{S}}^{(m)}(t) = \mathbb{E}\left[\gamma_0(t)\exp\left(-\int_0^t \gamma_0(r)m(r)dr\right)\right]$$

$$+ \sum_{i \geq 1} \mathbb{E}\left[\int_D \gamma(t - \tau_i^{(m)})\exp\left(-\int_{\tau_i^{(m)}}^t \gamma(r - \tau_i^{(m)})m(r)dr\right)\mu(d\gamma)\mathbb{1}_{\tau_i^{(m)} \leq t}\right].$$

In addition, as $(\tau_i^{(m)})_i$ are the jump time of $A^{(m)}$, by Fubini's theorem, we have

$$\overline{\mathfrak{S}}^{(m)}(t) = \mathbb{E}\left[\gamma_0(t)\exp\left(-\int_0^t \gamma_0(r)m(r)dr\right)\right]$$

$$+ \int_D \mathbb{E}\left[\sum_{i=1}^{A^{(m)}(t)} \gamma(t - \tau_i^{(m)})\exp\left(-\int_{\tau_i^{(m)}}^t \gamma(r - \tau_i^{(m)})m(r)dr\right)\right]\mu(d\gamma)$$

$$= \mathbb{E}\left[\gamma_0(t)\exp\left(-\int_0^t \gamma_0(r)m(r)dr\right)\right]$$

$$+ \int_D \mathbb{E}\left[\int_0^t \gamma(t - s)\exp\left(-\int_s^t \gamma(r - s)m(r)dr\right)A^{(m)}(ds)\right]\mu(d\gamma). \quad (6.5)$$

For any given $\gamma \in D$, we define

$$g_\gamma(s,t) := \gamma(t - s)\exp\left(-\int_s^t \gamma(r - s)m(r)dr\right).$$

Since

$$A^{(m)}(t) = \int_0^t \int_0^{+\infty} \mathbb{1}_{\Upsilon^{(m)}(s^-) \geq u}Q(du, ds),$$

by the same argument as in (6.4) we obtain

$$\mathbb{E}\left[\int_0^t g_\gamma(s,t)A^{(m)}(ds)\right] = \mathbb{E}\left[\int_0^t g_\gamma(s,t)\Upsilon^{(m)}(s)ds\right]$$
$$= \int_0^t g_\gamma(s,t)\mathbb{E}\left[\Upsilon^{(m)}(s)\right]ds$$
$$= \int_0^t g_\gamma(s,t)\overline{\mathfrak{S}}^{(m)}(s)m(s)ds. \tag{6.6}$$

Then, from (6.5) and (6.6), we deduce that

$$\overline{\mathfrak{S}}^{(m)}(t) = \mathbb{E}\left[\gamma_0(t)\exp\left(-\int_0^t \gamma_0(r)m(r)dr\right)\right]$$
$$+ \int_0^t \int_D \gamma(t-s)\exp\left(-\int_s^t \gamma(r-s)m(r)dr\right)\mu(d\gamma)m(s)\overline{\mathfrak{S}}^{(m)}(s)ds$$
$$= \mathbb{E}\left[\gamma_0(t)\exp\left(-\int_0^t \gamma_0(r)m(r)dr\right)\right]$$
$$+ \int_0^t \mathbb{E}\left[\gamma(t-s)\exp\left(-\int_s^t \gamma(r-s)m(r)dr\right)\right]m(s)\overline{\mathfrak{S}}^{(m)}(s)ds. \tag{6.7}$$

Hence from (6.4) and (6.7), $\overline{\mathfrak{F}}^{(m)} = m$ if and only if $(\overline{\mathfrak{S}}^{(m)}, m)$ solves (3.1)–(3.2). Consequently, by Theorem 3.1, there exists a unique element $m^* \in D_+$ such that $\overline{\mathfrak{F}}^{(m^*)} = m^*$ and that $(\overline{\mathfrak{S}}^{(m^*)}, m^*)$ solves (3.1)–(3.2). This concludes the proof of the lemma. $\qquad\square$

We next consider the sequence $(Q_k)_{k\geq 1}$ of Poisson random measures introduced in section 2 and for each $k \geq 1$, we define the process $\{A_k(t), t \geq 0\}$:

$$A_k(t) = \int_0^t \int_0^{+\infty} \mathbb{1}_{u\leq\Upsilon_k(r^-)}Q_k(du,dr),$$

where

$$\Upsilon_k(t) = \gamma_{k,A_k(t)}(\varsigma_k(t))\overline{\mathfrak{F}}(t),$$

and ς_k is defined in the same manner as ς_1^N with A_k instead of A_1^N. (This definition follows a similar idea from Lemma 6.1 as in [8].) In this definition we use the same $(\lambda_{k,i}, \gamma_{k,i}, Q_k)$ as in the definition of the model in section 2. Moreover, since $((\lambda_{k,i})_i, (\gamma_{k,i})_i, Q_k)_{k\geq 1}$ are i.i.d, $(A_k)_{k\geq 1}$ are also i.i.d.

Remark 6.1. *From Lemma 6.1 we have*

$$\overline{\mathfrak{F}}(t) = \mathbb{E}\left[\lambda_{1,A_1(t)}(\varsigma_1(t))\right] \ and \ \overline{\mathfrak{S}}(t) = \mathbb{E}\left[\gamma_{1,A_1(t)}(\varsigma_1(t))\right].$$

Now for each $k \geq 1$, we compare the process $A_k^N(t), t \geq 0$ with the process $A_k(t), t \geq 0$.

Lemma 6.2. *For $k \in \mathbb{N}$ and $T \geq 0$,*

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|A_k^N(t)-A_k(t)\right|\right] \leq \int_0^T \mathbb{E}\left[\left|\Upsilon_k^N(t)-\Upsilon_k(t)\right|\right]dt =: \delta^N(T) \tag{6.8}$$

and

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|\varsigma_k^N(t)-\varsigma_k(t)\right|\right] \leq T\delta^N(T).$$

Moreover,

$$\delta^N(T) \leq \frac{\lambda^*}{\sqrt{N}}T\exp(2\lambda^*T). \tag{6.9}$$

Proof. We adapt here the proof of Theorem IV.1 in [8] to our setting. Since

$$\left| A_k^N(t) - A_k(t) \right| \le \int_0^t \int_0^{+\infty} \mathbb{1}_{\min\left(\Upsilon_k^N(r-),\Upsilon_k(r-)\right) < u \le \max\left(\Upsilon_k^N(r-),\Upsilon_k(r-)\right)} Q_k(du, dr),$$

we have

$$\mathbb{E}\left[\sup_{t \in [0,T]} \left| A_k^N(t) - A_k(t) \right| \right] \le \int_0^T \mathbb{E}\left[\left| \Upsilon_k^N(t) - \Upsilon_k(t) \right| \right] dt = \delta^N(T).$$

We recall that

$$\Upsilon_k^N(t) = \gamma_{k,A_k^N(t)}(\varsigma_k^N(t))\overline{\mathfrak{F}}^N(t) \quad \text{and} \quad \Upsilon_k(t) = \gamma_{k,A_k(t)}(\varsigma_k(t))\overline{\mathfrak{F}}(t).$$

However, since $\gamma_{k,i} \le 1$ and $0 \le \overline{\mathfrak{F}}^N(t), \overline{\mathfrak{F}}(t) \le \lambda^*$, we obtain

$$\mathbb{E}\left[\left| \Upsilon_k^N(t) - \Upsilon_k(t) \right| \right] \le \mathbb{E}\left[\left| \Upsilon_k^N(t) - \Upsilon_k(t) \right| \mathbb{1}_{A_k^N(t)=A_k(t),\varsigma_k(t)=\varsigma_k^N(t)} \right] +$$
$$\lambda^* \mathbb{P}\left(A_k^N(t) \ne A_k(t) \text{ or } \varsigma_k(t) \ne \varsigma_k^N(t) \right). \quad (6.10)$$

On the other hand, using $\overline{\mathfrak{F}}(t) = \mathbb{E}\left[\lambda_{1,A_1(t)}(\varsigma_1(t)) \right]$, we have

$$\mathbb{E}\left[\left| \Upsilon_k^N(t) - \Upsilon_k(t) \right| \mathbb{1}_{A_k^N(t)=A_k(t),\varsigma_k(t)=\varsigma_k^N(t)} \right]$$
$$\le \mathbb{E}\left[\left| \overline{\mathfrak{F}}^N(t) - \overline{\mathfrak{F}}(t) \right| \right]$$
$$= \mathbb{E}\left[\left| \frac{1}{N} \sum_{j=1}^N \left(\lambda_{j,A_j^N(t)}(\varsigma_j^N(t)) - \mathbb{E}\left[\lambda_{1,A_1(t)}(\varsigma_1(t)) \right] \right) \right| \right]$$
$$\le \mathbb{E}\left[\left| \frac{1}{N} \sum_{j=1}^N \left(\lambda_{j,A_j^N(t)}(\varsigma_j^N(t)) - \lambda_{j,A_j(t)}(\varsigma_j(t)) \right) \right| \right]$$
$$+ \mathbb{E}\left[\left| \frac{1}{N} \sum_{j=1}^N \left(\lambda_{j,A_j(t)}(\varsigma_j(t)) - \mathbb{E}\left[\lambda_{1,A_1(t)}(\varsigma_1(t)) \right] \right) \right| \right]. \quad (6.11)$$

Since $((\lambda_{k,i})_i, A_k, \varsigma_k)_k$ are i.i.d, $(\lambda_{k,A_k(t)}(\varsigma_k(t))_k$ are i.i.d., hence, by Hölder's inequality we have

$$\mathbb{E}\left[\left| \frac{1}{N} \sum_{j=1}^N \left(\lambda_{j,A_j(t)}(\varsigma_j(t)) - \mathbb{E}\left[\lambda_{1,A_1(t)}(\varsigma_1(t)) \right] \right) \right| \right]$$
$$\le \frac{1}{N} \left(\mathbb{E}\left[\left(\sum_{j=1}^N \left(\lambda_{j,A_j(t)}(\varsigma_j(t)) - \mathbb{E}\left[\lambda_{j,A_j(t)}(\varsigma_j(t)) \right] \right) \right)^2 \right] \right)^{\frac{1}{2}}$$
$$= \frac{1}{N} \left(\sum_{j=1}^N \mathbb{E}\left[\left(\lambda_{j,A_j(t)}(\varsigma_j(t)) - \mathbb{E}\left[\lambda_{j,A_j(t)}(\varsigma_j(t)) \right] \right)^2 \right] \right)^{\frac{1}{2}}$$
$$\le \frac{\lambda^*}{\sqrt{N}}. \quad (6.12)$$

Here the equality holds because $(\lambda_{k,A_k(t)}(\varsigma_k(t)) - \mathbb{E}\left[\lambda_{k,A_k(t)}(\varsigma_k(t)) \right])_k$ are i.i.d. and the last inequality holds since $\lambda_{k,i}$ is bounded by λ^*.

In addition, as $(A_j^N)_j$ are exchangeable we have

$$\mathbb{E}\left[\left|\frac{1}{N}\sum_{j=1}^{N}\left(\lambda_{j,A_j^N(t)}(\varsigma_j^N(t)) - \lambda_{j,A_j(t)}(\varsigma_j(t))\right)\right|\right]$$

$$= \mathbb{E}\left[\left|\frac{1}{N}\sum_{j=1}^{N}\left(\lambda_{j,A_j^N(t)}(\varsigma_j^N(t)) - \lambda_{j,A_j(t)}(\varsigma_j(t))\right)\mathbb{1}_{A_j(t)\neq A_j^N(t)\text{ or }\varsigma_j(t)\neq\varsigma_j^N(t)}\right|\right]$$

$$\leq \frac{\lambda^*}{N}\sum_{j=1}^{N}\mathbb{P}\left(A_j(t)\neq A_j^N(t)\text{ or }\varsigma_j(t)\neq\varsigma_j^N(t)\right)$$

$$= \lambda^*\mathbb{P}\left(A_k(t)\neq A_k^N(t)\text{ or }\varsigma_k(t)\neq\varsigma_k^N(t)\right). \tag{6.13}$$

Hence from (6.11), (6.12) and (6.13) we have

$$\mathbb{E}\left[\left|\Upsilon_k^N(t) - \Upsilon_k(t)\right|\mathbb{1}_{A_k^N(t)=A_k(t),\varsigma_k(t)=\varsigma_k^N(t)}\right] \leq \frac{\lambda^*}{\sqrt{N}} + \lambda^*\mathbb{P}\left(A_k(t)\neq A_k^N(t)\text{ or }\varsigma_k(t)\neq\varsigma_k^N(t)\right).$$

On the other hand, since

$$\left\{A_k^N(t)\neq A_k(t)\text{ or }\varsigma_k(t)\neq\varsigma_k^N(t)\right\} \subset \left\{\sup_{r\in[0,t]}|A_k^N(r) - A_k(r)| \geq 1\right\},$$

we have

$$\mathbb{P}\left(A_k(t)\neq A_k^N(t)\text{ or }\varsigma_k(t)\neq\varsigma_k^N(t)\right) \leq \mathbb{E}\left[\sup_{r\in[0,t]}|A_k^N(r) - A_k(r)|\right] \leq \delta^N(t).$$

Thus, from (6.10), we have

$$\mathbb{E}\left[\left|\Upsilon_k^N(t) - \Upsilon_k(t)\right|\right] \leq \frac{\lambda^*}{\sqrt{N}} + 2\lambda^*\delta^N(t).$$

Combining this with (6.8), we deduce that for any $T \geq 0$,

$$\delta^N(T) \leq \frac{\lambda^*}{\sqrt{N}}T + 2\lambda^*\int_0^T \delta^N(t)dt,$$

hence by Gronwall's lemma, we have

$$\delta^N(T) \leq \frac{\lambda^*}{\sqrt{N}}T\exp(2\lambda^*T).$$

Moreover,

$$\mathbb{E}\left[\sup_{t\in[0,T]}|\varsigma_k^N(t) - \varsigma_k(t)|\right] = \mathbb{E}\left[\mathbb{1}_{\{\exists t\in[0,T],\varsigma_k^N(t)\neq\varsigma_k(t)\}}\sup_{t\in[0,T]}|\varsigma_k^N(t) - \varsigma_k(t)|\right]$$

$$\leq T\mathbb{P}\left(\exists t\in[0,T],\varsigma_k^N(t)\neq\varsigma_k(t)\right)$$

$$= T\mathbb{P}\left(\sup_{t\in[0,T]}|A_k^N(t) - A_k(t)|\neq 0\right)$$

$$\leq T\mathbb{E}\left[\sup_{t\in[0,T]}|A_k^N(t) - A_k(t)|\right]$$

$$\leq T\delta^N(T).$$

This concludes the proof of the lemma. $\qquad\square$

From the proof of Lemma 6.2, we deduce the following Remark.

Remark 6.2. *For $k \in \mathbb{N}$ and $t \geq 0$ we have*

$$\mathbb{E}\left[\left|\overline{\mathfrak{F}}^N(t) - \overline{\mathfrak{F}}(t)\right|\right] \leq \frac{\lambda^*}{\sqrt{N}}\left(1 + t\exp(2\lambda^* t)\right), \qquad \mathbb{E}\left[\left|\overline{\mathfrak{S}}^N(t) - \overline{\mathfrak{S}}(t)\right|\right] \leq \frac{1}{\sqrt{N}}\left(1 + \lambda^* t\exp(2\lambda^* t)\right)$$

and $\mathbb{E}\left[\left|\Upsilon_k^N(t) - \Upsilon_k(t)\right|\right] \leq \frac{\lambda^*}{\sqrt{N}}\left(1 + 2\lambda^* t\exp(2\lambda^* t)\right).$

From Lemma 6.2, we deduce the following Lemma.

Lemma 6.3. *For $k \in \mathbb{N}$ and $T \geq 0$ we have*

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|\gamma_{k,A_k^N(t)}(\varsigma_k^N(t)) - \gamma_{k,A_k(t)}(\varsigma_k(t))\right|\right] \leq \frac{\lambda^*}{\sqrt{N}}T\exp(2\lambda^* T), \tag{6.14}$$

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|\lambda_{k,A_k^N(t)}(\varsigma_k^N(t)) - \lambda_{k,A_k(t)}(\varsigma_k(t))\right|\right] \leq \frac{\lambda^{*2}}{\sqrt{N}}T\exp(2\lambda^* T), \tag{6.15}$$

and

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|\mathbb{1}_{\varsigma_k^N(t)<\eta_{k,A_k^N(t)}} - \mathbb{1}_{\varsigma_k(t)<\eta_{k,A_k(t)}}\right|\right] \leq \frac{\lambda^*}{\sqrt{N}}T\exp(2\lambda^* T). \tag{6.16}$$

Proof. From (6.8) and the fact that $\gamma_{k,i} \leq 1$, we have

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|\gamma_{k,A_k^N(t)}(\varsigma_k^N(t)) - \gamma_{k,A_k(t)}(\varsigma_k(t))\right|\right]$$

$$= \mathbb{E}\left[\sup_{t\in[0,T]}\left|\gamma_{k,A_k^N(t)}(\varsigma_k^N(t)) - \gamma_{k,A_k(t)}(\varsigma_k(t))\right|\mathbb{1}_{\sup_{t\in[0,T]}\left|A_k^N(t)-A_k(t)\right|\geq 1}\right]$$

$$\leq \mathbb{P}\left(\sup_{t\in[0,T]}\left|A_k^N(t) - A_k(t)\right| \geq 1\right)$$

$$\leq \delta^N(T).$$

Similarly we also have

$$\begin{cases} \mathbb{E}\left[\sup_{t\in[0,T]}\left|\lambda_{k,A_k^N(t)}(\varsigma_k^N(t)) - \lambda_{k,A_k(t)}(\varsigma_k(t))\right|\right] \leq \lambda^*\delta^N(T), \\ \mathbb{E}\left[\sup_{t\in[0,T]}\left|\mathbb{1}_{\varsigma_k^N(t)<\eta_{k,A_k^N(t)}} - \mathbb{1}_{\varsigma_k(t)<\eta_{k,A_k(t)}}\right|\right] \leq \delta^N(T). \end{cases}$$

Hence the claims follow from (6.9). $\qquad\square$

Completing the proof of Theorem 3.2. For $t \geq 0$, we have

$$\overline{\mathfrak{F}}^N(t) = \frac{1}{N}\sum_{k=1}^{N}\lambda_{k,A_k^N(t)}(\varsigma_k^N(t))$$

$$= \frac{1}{N}\sum_{k=1}^{N}\left(\lambda_{k,A_k^N(t)}(\varsigma_k^N(t)) - \lambda_{k,A_k(t)}(\varsigma_k(t))\right) + \frac{1}{N}\sum_{k=1}^{N}\lambda_{k,A_k(t)}(\varsigma_k(t)),$$

and

$$\overline{\mathfrak{S}}^N(t) = \frac{1}{N}\sum_{k=1}^{N}\gamma_{k,A_k^N(t)}(\varsigma_k^N(t))$$

$$= \frac{1}{N} \sum_{k=1}^{N} \left(\gamma_{k,A_k^N(t)}(\varsigma_k^N(t)) - \gamma_{k,A_k(t)}(\varsigma_k(t)) \right) + \frac{1}{N} \sum_{k=1}^{N} \gamma_{k,A_k(t)}(\varsigma_k(t)).$$

From (6.14) and (6.15), we have

$$\begin{cases} \mathbb{E}\left[\frac{1}{N} \sum_{k=1}^{N} \sup_{t \in [0,T]} \left| \lambda_{k,A_k^N(t)}(\varsigma_k^N(t)) - \lambda_{k,A_k(t)}(\varsigma_k(t)) \right| \right] \leq \frac{\lambda^{*2}}{\sqrt{N}} T \exp(2\lambda^* T), \\[2ex] \mathbb{E}\left[\frac{1}{N} \sum_{k=1}^{N} \sup_{t \in [0,T]} \left| \gamma_{k,A_k^N(t)}(\varsigma_k^N(t)) - \gamma_{k,A_k(t)}(\varsigma_k(t)) \right| \right] \leq \frac{\lambda^*}{\sqrt{N}} T \exp(2\lambda^* T). \end{cases}$$

Hence,

$$\left(\frac{1}{N} \sum_{k=1}^{N} \left(\gamma_{k,A_k^N(t)}(\varsigma_k^N(t)) - \gamma_{k,A_k(t)}(\varsigma_k(t)) \right), \frac{1}{N} \sum_{k=1}^{N} \left(\lambda_{k,A_k^N(t)}(\varsigma_k^N(t)) - \lambda_{k,A_k(t)}(\varsigma_k(t)) \right) \right) \xrightarrow[N \to +\infty]{} (0,0)$$

locally uniformly in t.

Moreover, as $\left(\gamma_{k,A_k(\cdot)}(\varsigma_k(\cdot)), \lambda_{k,A_k(\cdot)}(\varsigma_k(\cdot)) \right)_k$ is a collection of i.i.d. random variables in D^2, by the law of large numbers in D^2 [27, Theorem 1],

$$\left(\frac{1}{N} \sum_{k=1}^{N} \gamma_{k,A_k(\cdot)}(\varsigma_k(\cdot)), \frac{1}{N} \sum_{k=1}^{N} \lambda_{k,A_k(\cdot)}(\varsigma_k(\cdot)) \right) \xrightarrow[N \to +\infty]{\mathbb{P}} \left(\mathbb{E}\left[\gamma_{1,A_1(\cdot)}(\varsigma_1(\cdot)) \right], \mathbb{E}\left[\lambda_{1,A_1(\cdot)}(\varsigma_1(\cdot)) \right] \right) \text{ in } D^2.$$

We have shown in the proof on Lemma 6.1 that the pair $(\overline{\mathfrak{S}}, \overline{\mathfrak{F}})$ given in Remark 6.1 solves the set of equations (3.1)–(3.2). This proves the convergence (3.3).

For $(\overline{U}^N, \overline{I}^N)$, we have for $t \geq 0$,

$$\overline{I}^N(t) = \frac{1}{N} \sum_{k=1}^{N} \mathbb{1}_{\varsigma_k^N(t) < \eta_{k,A_k^N(t)}}$$

$$= \frac{1}{N} \sum_{k=1}^{N} \left(\mathbb{1}_{\varsigma_k^N(t) < \eta_{k,A_k^N(t)}} - \mathbb{1}_{\varsigma_k(t) < \eta_{k,A_k(t)}} \right) + \frac{1}{N} \sum_{k=1}^{N} \mathbb{1}_{\varsigma_k(t) < \eta_{k,A_k(t)}},$$

and

$$\overline{U}^N(t) = \frac{1}{N} \sum_{k=1}^{N} \mathbb{1}_{\varsigma_k^N(t) \geq \eta_{k,A_k^N(t)}}$$

$$= \frac{1}{N} \sum_{k=1}^{N} \left(\mathbb{1}_{\varsigma_k^N(t) \geq \eta_{k,A_k^N(t)}} - \mathbb{1}_{\varsigma_k(t) \geq \eta_{k,A_k(t)}} \right) + \frac{1}{N} \sum_{k=1}^{N} \mathbb{1}_{\varsigma_k(t) \geq \eta_{k,A_k(t)}}.$$

As above, from (6.16), we deduce that

$$\left(\frac{1}{N} \sum_{k=1}^{N} \left(\mathbb{1}_{\varsigma_k^N(t) \geq \eta_{k,A_k^N(t)}} - \mathbb{1}_{\varsigma_k(t) \geq \eta_{k,A_k(t)}} \right), \frac{1}{N} \sum_{k=1}^{N} \left(\mathbb{1}_{\varsigma_k^N(t) < \eta_{k,A_k^N(t)}} - \mathbb{1}_{\varsigma_k(t) < \eta_{k,A_k(t)}} \right) \right) \xrightarrow[N \to +\infty]{} (0,0)$$

locally uniformly in t.

Moreover, as $\left(\varsigma_k(t), \eta_{k,A_k(t)} \right)_k$ is a collection of i.i.d. random variables in D^2, by the law of large numbers in D^2 [27, Theorem 1],

$$\left(\frac{1}{N} \sum_{k=1}^{N} \mathbb{1}_{\varsigma_k(t) \geq \eta_{k,A_k(t)}}, \frac{1}{N} \sum_{k=1}^{N} \mathbb{1}_{\varsigma_k(t) < \eta_{k,A_k(t)}} \right) \xrightarrow[N \to +\infty]{\mathbb{P}} \left(\mathbb{E}\left[\mathbb{1}_{\varsigma_1(t) \geq \eta_{1,A_1(t)}} \right], \mathbb{E}\left[\mathbb{1}_{\varsigma_1(t) < \eta_{1,A_1(t)}} \right] \right) \text{ in } D^2.$$

Recall the formula (3.4) and (3.5). In order to complete the proof of Theorem 3.2, it remains to verify that:
$$\overline{I}(t) = \mathbb{E}\left[\mathbb{1}_{\varsigma_1(t)<\eta_{1,A_1(t)}}\right] \text{ and } \overline{U}(t) = \mathbb{E}\left[\mathbb{1}_{\varsigma_1(t)\geq\eta_{1,A_1(t)}}\right].$$
Denote the jump times of the process A_1 by $(\tau_i)_{i\geq1}$, if $t-\tau_i < \eta_{1,i}$, $\lambda_{1,i}(t-\tau_i) \neq 0$. From Assumption 2.1, we deduce that
$$\int_{\tau_i}^t \gamma_{1,i}(r-\tau_i)\overline{\mathfrak{F}}(r)dr = 0.$$
So $A_1(t) = A_1(\tau_i)$. Since $\mathbb{1}_{t<\eta_{1,0}} = \mathbb{1}_{t<\eta_{1,0}}\mathbb{1}_{A_1(t)=0}$ a.s., we have

$$\mathbb{E}\left[\mathbb{1}_{\varsigma_1(t)<\eta_{1,A_1(t)}}\right] = \mathbb{E}\left[\mathbb{1}_{t<\eta_{1,0}}\mathbb{1}_{A_1(t)=0} + \sum_{i\geq1}\mathbb{1}_{t-\tau_i<\eta_{1,i}}\mathbb{1}_{A_1(t)=i}\right]$$

$$= \mathbb{P}\left(\eta_{1,0}>t\big|\eta_{1,0}>0\right)\mathbb{P}\left(\eta_{1,0}>0\right) + \mathbb{E}\left[\sum_{i\geq1}\mathbb{1}_{t-\tau_i<\eta_{1,i}}\mathbb{1}_{\tau_i\leq t}\right]$$

$$= \overline{I}(0)F_0^c(t) + \sum_{i\geq1}\mathbb{E}\left[\mathbb{P}\left(t-\tau_i<\eta_{1,i}\big|\tau_i\right)\mathbb{1}_{\tau_i\leq t}\right].$$

Moreover, since $\eta_{1,i}$ and τ_i are independent, we obtain

$$\mathbb{E}\left[\mathbb{1}_{\varsigma_1(t)<\eta_{1,A_1(t)}}\right] = \overline{I}(0)F_0^c(t) + \sum_{i\geq1}\mathbb{E}\left[F^c\left(t-\tau_i<\eta_{1,1}\right)\mathbb{1}_{\tau_i\leq t}\right]$$

$$= \overline{I}(0)F_0^c(t) + \mathbb{E}\left[\int_0^t F^c(t-s)A_1(ds)\right]$$

$$= \overline{I}(0)F_0^c(t) + \int_0^t F^c(t-s)\overline{\mathfrak{F}}(s)\overline{\mathfrak{S}}(s)ds. \tag{6.17}$$

If $\eta_{1,0}>t$, $\lambda_{1,0}(t)>0$ ($\eta_{1,1}>t$, $\lambda_{1,1}(t)>0$), from Assumption 2.1 by using the fact that $\lambda_{1,1}(t)>0$ ($\lambda_{1,0}(t)>0$) implies that $\gamma_{1,1}(s)=0$ ($\gamma_{1,0}(s)=0$) for $0\leq s\leq t$, from (6.17) we also have,

$$\mathbb{E}\left[\mathbb{1}_{\varsigma_1(t)<\eta_{1,A_1(t)}}\right] = \mathbb{E}\left[\mathbb{1}_{\eta_{1,0}>t}\exp\left(-\int_0^t\gamma_0(r)\overline{\mathfrak{F}}(r)dr\right)\right]$$

$$+ \int_0^t\mathbb{E}\left[\mathbb{1}_{\eta_{1,1}>t-s}\exp\left(-\int_s^t\gamma(r-s)\overline{\mathfrak{F}}(r)dr\right)\right]\overline{\mathfrak{F}}(s)\overline{\mathfrak{S}}(s)ds,$$

which combined with (3.8) yields

$$\mathbb{E}\left[\mathbb{1}_{\varsigma_1(t)\geq\eta_{1,A_1(t)}}\right] = 1 - \mathbb{E}\left[\mathbb{1}_{\varsigma_1(t)<\eta_{1,A_1(t)}}\right]$$

$$= \mathbb{E}\left[\mathbb{1}_{t\geq\eta_{1,0}}\exp\left(-\int_0^t\gamma_0(r)\overline{\mathfrak{F}}(r)dr\right)\right]$$

$$+ \int_0^t\mathbb{E}\left[\mathbb{1}_{t-s\geq\eta_{1,1}}\exp\left(-\int_s^t\gamma(r-s)\overline{\mathfrak{F}}(r)dr\right)\right]\overline{\mathfrak{F}}(s)\overline{\mathfrak{S}}(s)ds.$$

This completes the proof of Theorem 3.2. □

We also have the following convergence result on the empirical measure of the processes $(A_k^N(t), \varsigma_k^N(t))_{t\geq0}$. It is not used in our analysis, but a worth to be established.

Theorem 6.1.

$$\frac{1}{N} \sum_{k=1}^{N} \delta_{(A_k^N(t), \varsigma_k^N(t))_{t \geq 0}} \xrightarrow[N \to +\infty]{\mathbb{P}} \mathcal{L}\left((A_1(t), \varsigma_1(t))_{t \geq 0}\right) \ in \ \mathcal{P}(D^2). \tag{6.18}$$

Proof. By Lemma 6.2, we have

$$\mathbb{E}\left[\sup_{t \in [0,T]} \left(\left|A_1^N(t) - A_1(t)\right| + \left|A_2^N(t) - A_2(t)\right| + \left|\varsigma_1^N(t) - \varsigma_1(t)\right| + \left|\varsigma_2^N(t) - \varsigma_2(t)\right|\right)\right] \xrightarrow[N \to +\infty]{} 0.$$

Hence

$$\left(\left\{(A_1^N(t), \varsigma_1^N(t))\right\}_{t \geq 0}, \left\{(A_2^N(t), \varsigma_2^N(t))\right\}_{t \geq 0}\right) \xrightarrow[N \to +\infty]{\mathbb{P}} \left(\left\{(A_1(t), \varsigma_1(t))\right\}_{t \geq 0}, \left\{(A_2(t), \varsigma_2(t))\right\}_{t \geq 0}\right)$$

in $D^2 \times D^2$. Thus since the processes are exchangeable and D is a separable metric space, by [29, Proposition 2.2 i), page 177], the convergence in (6.18) holds. $\square$

7. PROOFS FOR THE ENDEMIC EQUILIBRIUM

In this section, we prove the results on the endemic equilibrium behaviors. We proceed in two subsections to prove the results in the scenarios $R_0 < \mathbb{E}\left[\frac{1}{\gamma_*}\right]$ and $R_0 \geq \mathbb{E}\left[\frac{1}{\gamma_*}\right]$. We have a complete theory in the first scenario as stated in Theorem 4.1, which we prove first. We then establish some of the partial results in the second scenario.

7.1. **Proof of Theorem 4.1.**

Proof of Theorem 4.1. This theorem is proved in two cases: $\mathbb{P}(\gamma_* = 0) > 0$ and $\mathbb{P}(\gamma_* = 0) = 0$.

 Case 1: $\mathbb{P}(\gamma_* = 0) > 0$. Recall (3.8). Note that, since $\gamma_* = 0$ implies $\gamma(t) = 0$ for all t,

$$\mathbb{E}\left[\exp\left(-\int_s^t \gamma(r-s)\overline{\mathfrak{F}}(r)dr\right)\right] \geq \mathbb{P}(\gamma_* = 0).$$

As a result, from (3.8), for all $t \geq 0$,

$$\int_0^t \overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)ds \leq \frac{1}{\mathbb{P}(\gamma_* = 0)}. \tag{7.1}$$

Consequently,

$$\int_0^{+\infty} \overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)ds < +\infty. \tag{7.2}$$

Since $\lambda(t) \leq \lambda_*$ a.s and $\lambda(t) = 0$ for all $t \geq \eta$, from Assumption 4.1 and the dominated convergence theorem $\overline{\lambda}_0(t) \to 0$, $\overline{\lambda}(t) \to 0$ as $t \to +\infty$. By the dominated convergence theorem applied to (3.7), using (7.2), we obtain

$$\lim_{t \to +\infty} \overline{\mathfrak{F}}(t) = \lim_{t \to +\infty} \int_0^t \overline{\lambda}(t-s)\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)ds = 0.$$

Next by the dominated convergence theorem applied to (3.6) using (7.2) again, $\overline{\mathfrak{S}}_* = \lim_{t \to +\infty} \overline{\mathfrak{S}}(t)$ as given in (4.2). This concludes the proof of the first case of Theorem 4.2.

 Case 2: $\mathbb{P}(\gamma_* = 0) = 0$. We first note that

$$\int_0^{+\infty} \overline{\mathfrak{F}}(u)du < +\infty \Leftrightarrow \int_0^{+\infty} \overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)ds < +\infty. \tag{7.3}$$

Indeed, from (3.7), and from Fubuni's theorem

$$\int_0^{+\infty} \overline{\mathfrak{F}}(u)du = \overline{I}(0)\int_0^{+\infty} \overline{\lambda}_0(u)du + R_0\int_0^{+\infty} \overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)ds. \tag{7.4}$$

Next from Assumption 4.2, we have

$$\int_0^{+\infty} \overline{\lambda}_0(u)du \leq R_0 < +\infty. \tag{7.5}$$

This combined with (7.4) implies (7.3).

Thus from the proof of Case 1 and (7.3), it suffices to show that $\int_0^{+\infty}\overline{\mathfrak{F}}(s)ds < +\infty$. We prove this claim by contradiction. Suppose that

$$\int_0^{+\infty} \overline{\mathfrak{F}}(u)du = +\infty. \tag{7.6}$$

By (3.7), using Fubuni's theorem, we obtain

$$\int_0^t \overline{\mathfrak{F}}(u)du = \overline{I}(0)\int_0^t \overline{\lambda}_0(u)du + \int_0^t \left(\int_0^{t-u} \overline{\lambda}(s)ds\right)\overline{\mathfrak{S}}(u)\overline{\mathfrak{F}}(u)du$$

$$= \overline{I}(0)\int_0^t \overline{\lambda}_0(u)du + R_0\int_0^t \overline{\mathfrak{S}}(u)\overline{\mathfrak{F}}(u)du - \int_0^t \left(\int_{t-u}^{+\infty} \overline{\lambda}(s)ds\right)\overline{\mathfrak{S}}(u)\overline{\mathfrak{F}}(u)du.$$

Consequently,

$$\frac{\int_0^t \overline{\mathfrak{S}}(u)\overline{\mathfrak{F}}(u)du}{\int_0^t \overline{\mathfrak{F}}(u)du} = \frac{1}{R_0} + \frac{\int_0^t \left(\int_{t-u}^{+\infty}\overline{\lambda}(s)ds\right)\overline{\mathfrak{S}}(u)\overline{\mathfrak{F}}(u)du}{R_0\int_0^t \overline{\mathfrak{F}}(u)du} - \frac{\overline{I}(0)\int_0^t \overline{\lambda}_0(u)du}{R_0\int_0^t \overline{\mathfrak{F}}(u)du}. \tag{7.7}$$

Thus by (7.5) and (7.6), we have

$$\frac{\int_0^t \overline{\lambda}_0(u)du}{\int_0^t \overline{\mathfrak{F}}(u)du} \xrightarrow[t\to+\infty]{} 0. \tag{7.8}$$

In addition, since $\int_t^{+\infty} \overline{\lambda}(s)ds \to 0$ as $t \to +\infty$, for $\epsilon > 0$ there exists $T_\epsilon > 0$ such that $\int_{T_\epsilon}^{+\infty}\overline{\lambda}(s)ds < \epsilon$. Hence, for $t \geq T_\epsilon$,

$$\int_0^t \left(\int_{t-u}^{+\infty}\overline{\lambda}(s)ds\right)\overline{\mathfrak{S}}(u)\overline{\mathfrak{F}}(u)du$$

$$= \int_0^t \left(\int_u^{+\infty}\overline{\lambda}(s)ds\right)\overline{\mathfrak{S}}(t-u)\overline{\mathfrak{F}}(t-u)du$$

$$\leq \int_0^{T_\epsilon} \left(\int_u^{+\infty}\overline{\lambda}(s)ds\right)\overline{\mathfrak{S}}(t-u)\overline{\mathfrak{F}}(t-u)du + \epsilon\int_{T_\epsilon}^t \overline{\mathfrak{S}}(t-u)\overline{\mathfrak{F}}(t-u)du$$

$$\leq R_0\lambda_* T_\epsilon + \epsilon\int_0^t \overline{\mathfrak{F}}(u)du.$$

Thus by (7.6), we have

$$\frac{\int_0^t \left(\int_{t-u}^{+\infty}\overline{\lambda}(s)ds\right)\overline{\mathfrak{S}}(u)\overline{\mathfrak{F}}(u)du}{\int_0^t \overline{\mathfrak{F}}(u)du} \xrightarrow[t\to+\infty]{} 0. \tag{7.9}$$

Hence under the assumption (7.6), from (7.7), (7.8) and (7.9), we obtain

$$\frac{\int_0^t \overline{\mathfrak{S}}(u)\overline{\mathfrak{F}}(u)du}{\int_0^t \overline{\mathfrak{F}}(u)du} \xrightarrow[t\to+\infty]{} \frac{1}{R_0}. \tag{7.10}$$

On the other hand, from (3.8) and the fact that $\gamma \leq \gamma_*$ we have

$$\int_0^u \mathbb{E}\left[\exp\left(-\gamma_* \int_s^u \overline{\mathfrak{F}}(r)dr\right)\right]\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)ds < 1. \tag{7.11}$$

Next, multiplying by $\overline{\mathfrak{F}}(u)$ and integrating from 0 to t both sides of (7.11), we have

$$\int_0^t \left(\int_0^u \mathbb{E}\left[\overline{\mathfrak{F}}(u)\exp\left(-\gamma_* \int_s^u \overline{\mathfrak{F}}(r)dr\right)\right]\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)ds\right)du < \int_0^t \overline{\mathfrak{F}}(u)du,$$

and by Fubuni's theorem,

$$\int_0^t \mathbb{E}\left[\int_s^t \overline{\mathfrak{F}}(u)\exp\left(-\gamma_* \int_s^u \overline{\mathfrak{F}}(r)dr\right)du\right]\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)ds < \int_0^t \overline{\mathfrak{F}}(u)du,$$

from which we obtain

$$\int_0^t \mathbb{E}\left[\frac{1-\exp\left(-\gamma_* \int_s^t \overline{\mathfrak{F}}(r)dr\right)}{\gamma_*}\right]\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)ds < \int_0^t \overline{\mathfrak{F}}(u)du.$$

Hence for $0 < \epsilon \leq 1$, since $\mathbb{1}_{\gamma_* \geq \epsilon} \leq 1$,

$$\int_0^t \mathbb{E}\left[\frac{1-\exp\left(-\gamma_* \int_s^t \overline{\mathfrak{F}}(r)dr\right)}{\gamma_*}\mathbb{1}_{\gamma_* \geq \epsilon}\right]\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)ds < \int_0^t \overline{\mathfrak{F}}(u)du.$$

Thus,

$$\mathbb{E}\left[\frac{1}{\gamma_*}\mathbb{1}_{\gamma_* \geq \epsilon}\right]\int_0^t \overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)ds < \int_0^t \overline{\mathfrak{F}}(u)du + \int_0^t \mathbb{E}\left[\frac{1}{\gamma_*}\mathbb{1}_{\gamma_* \geq \epsilon}\exp\left(-\gamma_* \int_s^t \overline{\mathfrak{F}}(r)dr\right)\right]\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)ds. \tag{7.12}$$

Moreover, from (7.11),

$$\int_0^t \mathbb{E}\left[\frac{1}{\gamma_*}\mathbb{1}_{\gamma_* \geq \epsilon}\exp\left(-\gamma_* \int_s^t \overline{\mathfrak{F}}(r)dr\right)\right]\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)ds \leq \frac{1}{\epsilon}.$$

Consequently, under the assumption (7.6), we have

$$\frac{\int_0^t \mathbb{E}\left[\frac{1}{\gamma_*}\mathbb{1}_{\gamma_* \geq \epsilon}\exp\left(-\gamma_* \int_s^t \overline{\mathfrak{F}}(r)dr\right)\right]\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)ds}{\int_0^t \overline{\mathfrak{F}}(u)du} \xrightarrow[t\to+\infty]{} 0.$$

This implies that, by (7.12), for all $0 < \epsilon \leq 1$,

$$\limsup_{t\to+\infty}\frac{\int_0^t \overline{\mathfrak{S}}(u)\overline{\mathfrak{F}}(u)du}{\int_0^t \overline{\mathfrak{F}}(u)du} \leq \left(\mathbb{E}\left[\frac{1}{\gamma_*}\mathbb{1}_{\gamma_* \geq \epsilon}\right]\right)^{-1}.$$

Since $\mathbb{P}(\gamma_* = 0) = 0$, we deduce by the monotone convergence theorem that

$$\limsup_{t\to+\infty}\frac{\int_0^t \overline{\mathfrak{S}}(u)\overline{\mathfrak{F}}(u)du}{\int_0^t \overline{\mathfrak{F}}(u)du} \leq \left(\mathbb{E}\left[\frac{1}{\gamma_*}\right]\right)^{-1},$$

However, this contradicts (7.10) since $R_0 < \mathbb{E}\left[\frac{1}{\gamma_*}\right]$ by the assumption of Theorem 4.1.

This completes the proof of the second case. $\qquad\square$

7.2. **Proofs in the case $R_0 \geq \mathbb{E}\left[\frac{1}{\gamma_*}\right]$.** In this subsection, we prove Theorem 4.2.

Proof of Theorem 4.2 (i). As $\overline{\mathfrak{S}}(t) \to \overline{\mathfrak{S}}_*$ and $\overline{\mathfrak{F}}(t) \to \overline{\mathfrak{F}}_*$ as $t \to \infty$, $R_0 < +\infty$ and $\overline{\mathfrak{F}}(t)\overline{\mathfrak{S}}(t) \leq \lambda_*$, by the dominated convergence theorem, we have

$$\lim_{t\to+\infty} \int_0^t \overline{\lambda}(t-s)\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)ds = \lim_{t\to+\infty} \int_0^{+\infty} \overline{\lambda}(s)\overline{\mathfrak{S}}(t-s)\overline{\mathfrak{F}}(t-s)\mathbb{1}_{[0,t]}(s)ds = \overline{\mathfrak{S}}_*\overline{\mathfrak{F}}_* R_0. \quad (7.13)$$

Thus by (7.13) and (3.7), using the fact that $\overline{\lambda}_0(t) \to 0$ as $t \to \infty$,

$$\overline{\mathfrak{F}}_* = \overline{\mathfrak{S}}_*\overline{\mathfrak{F}}_* R_0.$$

As a result, either $\overline{\mathfrak{F}}_* = 0$ or else $\overline{\mathfrak{S}}_* = \frac{1}{R_0}$.

In the following, we assume that $\overline{\mathfrak{F}}_* > 0$, then $\overline{\mathfrak{S}}_* = \frac{1}{R_0}$.

Recall (3.8). Since $\mathbb{E}\left[\frac{1}{\gamma_*}\right] \leq R_0 < +\infty$ and $\mathbb{P}(\gamma_* = 0) = 0$, $\gamma(t+\xi)\overline{\mathfrak{F}}(t) \to \gamma_*\overline{\mathfrak{F}}_* > 0$ with probability one when $t \to +\infty$ and hence

$$\lim_{t\to+\infty} \int_0^t \gamma(r+\xi)\overline{\mathfrak{F}}(r)dr = +\infty \text{ and } \lim_{t\to+\infty} \int_0^t \overline{\mathfrak{F}}(r)dr = +\infty \text{ almost surely.}$$

It follows that

$$\lim_{t\to+\infty} \mathbb{E}\left[\exp\left(-\int_0^t \gamma_0(r)\overline{\mathfrak{F}}(r)dr\right)\right] = 0.$$

Hence from (3.8) we have

$$\lim_{t\to+\infty} \int_0^t \mathbb{E}\left[\exp\left(-\int_s^t \gamma(r-s)\overline{\mathfrak{F}}(r)dr\right)\right] \overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)ds = 1. \quad (7.14)$$

Since $\overline{\mathfrak{F}}(t) \to \overline{\mathfrak{F}}_*$ when $t \to +\infty$, there exists $t_0 > 0$ such that for all $t \geq t_0$, $\overline{\mathfrak{F}}(t) \geq \frac{\overline{\mathfrak{F}}_*}{2}$. Then,

$$\int_0^t \mathbb{E}\left[\exp\left(-\int_s^t \gamma(r-s)\overline{\mathfrak{F}}(r)dr\right)\right] \overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)ds$$

$$= \int_0^{t_0} \mathbb{E}\left[\exp\left(-\int_0^{t-s} \gamma(r)\overline{\mathfrak{F}}(s+r)dr\right)\right] \overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)ds$$

$$+ \int_0^{t-t_0} \mathbb{E}\left[\exp\left(-\int_0^s \gamma(r)\overline{\mathfrak{F}}(t+r-s)dr\right)\right] \overline{\mathfrak{S}}(t-s)\overline{\mathfrak{F}}(t-s)ds. \quad (7.15)$$

The first term on the right hand side converges to zero as $t \to \infty$ by the dominated convergence theorem. On the other hand, for $0 < s < t - t_0$,

$$\exp\left(-\int_0^s \gamma(r)\overline{\mathfrak{F}}(r+t-s)dr\right) \leq \exp\left(-\frac{\overline{\mathfrak{F}}_*}{2} \int_0^s \gamma(r)dr\right), \quad (7.16)$$

and by Assumption 4.3, we deduce that

$$\int_0^{+\infty} \mathbb{E}\left[\exp\left(-\frac{\overline{\mathfrak{F}}_*}{2} \int_0^s \gamma(r)dr\right)\right] ds \leq \mathbb{E}\left[t_*\right] + \mathbb{E}\left[\int_0^{+\infty} \exp\left(-\frac{\overline{\mathfrak{F}}_*\gamma_*}{4}s\right) ds\right]$$

$$= \mathbb{E}\left[t_*\right] + \frac{4}{\overline{\mathfrak{F}}_*}\mathbb{E}\left[\frac{1}{\gamma_*}\right] < +\infty \quad (7.17)$$

Thus, applying the dominated convergence theorem to the second term on the right-hand-side of (7.15) and using (7.14), we obtain

$$\int_0^{+\infty} \mathbb{E}\left[\exp\left(-\overline{\mathfrak{F}}_* \int_0^s \gamma(r)dr\right)\right] \overline{\mathfrak{S}}_*\overline{\mathfrak{F}}_* ds = 1. \quad (7.18)$$

Next by a change of variables in (7.18) and the fact that $\overline{\mathfrak{S}}_* = \frac{1}{R_0}$, we obtain (4.4).

To conclude, Lemma 7.1 below implies that the equation $H(x) = R_0$ has a unique positive solution if and only if $R_0 > \mathbb{E}\left[\frac{1}{\gamma_*}\right]$, which yields the result.

On the other hand, as $F_0^c(t) \to 0$ as $t \to +\infty$ and $(\overline{\mathfrak{S}}(t), \overline{\mathfrak{F}}(t)) \to (\overline{\mathfrak{S}}_*, \overline{\mathfrak{F}}_*)$ as $t \to +\infty$, applying the dominated convergence theorem to (3.5) we obtain

$$\overline{I}_* = \lim_{t \to +\infty} \int_0^t F^c(t-s)\overline{\mathfrak{F}}(s)\overline{\mathfrak{S}}(s)ds = \overline{\mathfrak{S}}_*\overline{\mathfrak{F}}_*\mathbb{E}[\eta] = \frac{\overline{\mathfrak{F}}_*}{R_0}\mathbb{E}[\eta].$$

This proves the claims of Theorem 4.2 (i). $\qquad\square$

Let $H : \mathbb{R}_+ \to \mathbb{R}_+$ as follows,

$$\text{for } x > 0,\ H(x) := \int_0^{+\infty} \mathbb{E}\left[\exp\left(-\int_0^s \gamma\left(\frac{r}{x}\right)dr\right)\right]ds;\ H(0) = \mathbb{E}\left[\frac{1}{\gamma_*}\right].$$

Lemma 7.1. *If γ is nondecreasing and $\mathbb{E}\left[\frac{1}{\gamma_*}\right] < +\infty$, then the function $H : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and strictly increasing. Moreover,*

$$H(0) = \mathbb{E}\left[\frac{1}{\gamma_*}\right], \quad and \quad \lim_{x \to +\infty} H(x) = +\infty.$$

Proof. Note that $H(x)$ is also given by the following equivalent formula

$$H(x) = x\int_0^{\infty} \mathbb{E}\left[\exp\left(-x\int_0^s \gamma(r)dr\right)\right]ds.$$

This formula and monotone convergence imply the continuity of H. The fact that H is nondecreasing follows readily from the first formula and the nondecreasing property of γ. Moreover we have

$$H(0) = \mathbb{E}\left[\frac{1}{\gamma_*}\right], \quad and \quad \lim_{x \to +\infty} H(x) = +\infty.$$

To prove that H is strictly increasing, assume that $H(x) = H(y)$ for some $y > x$. Then, for a.e. r, $\gamma(r/x) = \gamma(r/y)$ almost surely, which implies that γ is constant.

Moreover, from the second formula for H, it follows readily that $x \mapsto H(x)/x$ is non-increasing. From the continuity at 0, we deduce that there exists $x > 0$ such that $H(x) < \infty$. Now for any $y > x$, $H(y) \leq y\frac{H(x)}{x} < \infty$. Hence H takes values in $\mathbb{R}_+$. The rest of the statement is easy to verify. $\qquad\square$

Proof of Theorem 4.2 (ii). The goal of this Lemma is to prove that, if γ_* is deterministic, $R_0 > \frac{1}{\gamma_*}$, for all $\delta > 0$ there exists t_δ such that $\gamma(t_\delta) \geq (1-\delta)\gamma_*$ and if there exists a positive decreasing function h such that for all $0 \leq s, t$, $\overline{\lambda}(t+s) \geq h(t)\overline{\lambda}(s)$, and $\overline{\mathfrak{F}}(0) > 0$, then there exists $c > 0$ such that for all $t \geq 0$, $\overline{\mathfrak{F}}(t) \geq c$.

Let $\delta > 0$ be such that $(1-\delta)\gamma_* > \frac{1}{R_0}$. From Assumption 4.4, there exists $s_1 \geq 0$ deterministic such that $\gamma_0(s_1) \wedge \gamma(s_1) \geq (1-\delta)\gamma_*$ a.s. Let $\epsilon > 0$ be such that $(1-\delta)\gamma_* > \frac{1+\epsilon}{R_0}$.

Let x be the solution of the following Volterra equation

$$x(t) = h(s_1 + t) + (1+\epsilon)\int_0^t p(t-s)x(s)ds,\ t \geq 0, \tag{7.19}$$

with

$$p(t) = \frac{\overline{\lambda}(t)}{R_0}.$$

As

$$(1+\epsilon)\int_0^{+\infty} p(t)dt = 1 + \epsilon > 1, \qquad \int_0^{+\infty} h(t)dt < +\infty$$

where the integrability of h results from Assumption 4.5 and the integrability of $\overline{\lambda}$. Moreover, since h and p are bounded and non-negative, by [10, Remark following Theorem 4, page 253], $x(t) \to +\infty$ as $t \to +\infty$, hence there exists $s_2 \geq 0$ such that $x(s_2) > 2$.

Let $c_2 > 0$ be such that

$$(1 - \delta)\gamma_* \exp(-c_2(s_1 + s_2)) \geq \frac{1 + \epsilon}{R_0}. \tag{7.20}$$

Let

$$c_1 = \frac{1}{2}\min(c_2, \overline{\overline{\mathfrak{F}}}(0)), \quad \text{and} \quad c_0 = \frac{c_1}{2}h(s_1 + s_2),$$

$$t_0 = \inf\{t \geq 0, \overline{\overline{\mathfrak{F}}}(t) \leq c_0\}, \quad \text{and} \quad t_1 = \sup\{t \leq t_0, \overline{\overline{\mathfrak{F}}}(t) \geq c_1\}.$$

Since h is decreasing and $h(0) = 1$ we have $c_0 < c_1$. We want to show that $t_0 = \infty$, and will prove by contradict. We suppose that $t_0 < +\infty$, then $t_1 < +\infty$. From Assumption 4.5, by the continuity of $\overline{\overline{\mathfrak{F}}}$ and the definition of t_1, for all $t \geq t_1$, we obtain

$$\overline{\overline{\mathfrak{F}}}(t) = \overline{\lambda}_0(t)\overline{I}(0) + \int_0^t \overline{\lambda}(t - s)\overline{\mathfrak{F}}(s)\overline{\mathfrak{S}}(s)ds$$

$$\geq \overline{\lambda}_0(t - t_1 + t_1)\overline{I}(0) + \int_0^{t_1} \overline{\lambda}(t - t_1 + t_1 - s)\overline{\mathfrak{F}}(s)\overline{\mathfrak{S}}(s)ds$$

$$\geq h(t - t_1)\left(\overline{\lambda}_0(t_1)\overline{I}(0) + \int_0^{t_1} \overline{\lambda}(t_1 - s)\overline{\mathfrak{F}}(s)\overline{\mathfrak{S}}(s)ds\right)$$

$$= h(t - t_1)\overline{\overline{\mathfrak{F}}}(t_1)$$

$$\geq c_1 h(t - t_1).$$

The definition of t_0 and the continuity of $\overline{\overline{\mathfrak{F}}}$, implies that $\overline{\overline{\mathfrak{F}}}(t_0) \leq c_0$. Combining with the last inequality evaluated at $t = t_0$, we have $c_0 \geq c_1 h(t_0 - t_1)$. Hence, by the definition of c_0 and the fact that h is decreasing, we deduce that, $t_0 - t_1 > s_1 + s_2$. So $t_0 > t_1 + s_1 + s_2$ and for all $t \in [t_1, t_0]$

$$\overline{\overline{\mathfrak{F}}}(t) \leq c_1 < c_2. \tag{7.21}$$

On the other hand, as $\gamma_0(t) \leq 1$ and $\gamma(t) \leq 1$, we have, for all $t \geq t_1$,

$$\overline{\mathfrak{S}}(t) = \mathbb{E}\left[\gamma_0(t)\exp\left(-\int_0^t \gamma_0(r)\overline{\mathfrak{F}}(r)dr\right)\right]$$

$$+ \int_0^t \mathbb{E}\left[\gamma(t - s)\exp\left(-\int_s^t \gamma(r - s)\overline{\mathfrak{F}}(r)dr\right)\right]\overline{\mathfrak{F}}(s)\overline{\mathfrak{S}}(s)ds$$

$$\geq \exp\left(-\int_{t_1}^t \overline{\mathfrak{F}}(r)dr\right)\left(\mathbb{E}\left[\gamma_0(t)\exp\left(-\int_0^{t_1} \gamma_0(r)\overline{\mathfrak{F}}(r)dr\right)\right]\right.$$

$$\left. + \int_0^{t_1} \mathbb{E}\left[\gamma(t - s)\exp\left(-\int_s^{t_1} \gamma(r - s)\overline{\mathfrak{F}}(r)dr\right)\right]\overline{\mathfrak{F}}(s)\overline{\mathfrak{S}}(s)ds\right).$$

But, as for $t \in [t_1 + s_1, t_1 + s_1 + s_2]$ and $s \in [0, t_1]$, $\gamma_0(t) \wedge \gamma(t - s) \geq \gamma_0(s_1) \wedge \gamma(s_1) \geq (1 - \delta)\gamma_*$, and using (3.8) at time t_1 we deduce that, for all $t \in [t_1 + s_1, t_1 + s_1 + s_2]$,

$$\overline{\mathfrak{S}}(t) \geq (1 - \delta)\gamma_* \exp\left(-\int_{t_1}^t \overline{\mathfrak{F}}(r)dr\right).$$

Moreover, since from (7.21) for $t \in [t_1 + s_1, t_1 + s_1 + s_2]$, $\overline{\overline{\mathfrak{F}}}(t) \leq c_2$,

$$\overline{\mathfrak{S}}(t) \geq (1 - \delta)\gamma_* \exp(-c_2(s_2 + s_1)).$$

Then from (7.20)

$$\forall t \in [t_1 + s_1, t_1 + s_1 + s_2], \quad \overline{\mathfrak{S}}(t) \geq \frac{1 + \epsilon}{R_0}. \tag{7.22}$$

Let $y(t) = \overline{\overline{\mathfrak{F}}}(t + t_1 + s_1)$ and define g as follows:

$$g(t) = \overline{I}(0)\overline{\lambda}_0(t_1 + s_1 + t) + \int_0^{t_1+s_1} \overline{\lambda}(t_1 + s_1 + t - s)\overline{\overline{\mathfrak{F}}}(s)\overline{\mathfrak{S}}(s)ds,$$

where we recall that

$$p(t) = \frac{\overline{\lambda}(t)}{R_0}.$$

Then using (7.22) for any $t \geq 0$,

$$y(t) \geq g(t) + (1 + \epsilon) \int_{t_1+s_1}^{t_1+s_1+t} p(t_1 + s_1 + t - s)y(s - t_1 - s_1)ds$$

$$= g(t) + (1 + \epsilon) \int_0^t p(t - s)y(s)ds.$$

However, from Assumption 4.5 we deduce that

$$g(t) \geq \overline{\overline{\mathfrak{F}}}(t_1)h(s_1 + t)$$

and as $\overline{\overline{\mathfrak{F}}}(t_1) = c_1$ by continuity, we deduce that

$$y(t) \geq c_1 h(s_1 + t) + (1 + \epsilon) \int_0^t p(t - s)y(s)ds. \tag{7.23}$$

Thus by Theorem 1.2.19 in [5] applied to (7.23), we have

$$y(t) \geq c_1 x(t)$$

where x is given by (7.19). However $x(s_2) > 2$. Hence $\overline{\overline{\mathfrak{F}}}(t_1 + s_1 + s_2) > 2c_1 > c_1$ and $t_0 \geq t_1 + s_1 + s_2$, this contradicts the definition of t_1. Hence $t_0 = +\infty$. This concludes the proof. $\square$

ACKNOWLEDGEMENT

REFERENCES

[1] M. V. Barbarossa and G. Röst. Immuno-epidemiology of a population structured by immune status: a mathematical study of waning immunity and immune system boosting. *Journal of Mathematical Biology*, 71:1737–1770, 2015.

[2] P. Billingsley. *Convergence of probability measures*. John Wiley & Sons, 1999.

[3] D. Breda, O. Diekmann, W. De Graaf, A. Pugliese, and R. Vermiglio. On the formulation of epidemic models (an appraisal of kermack and mckendrick). *Journal of Biological Dynamics*, 6(sup2):103–117, 2012.

[4] T. Britton and E. Pardoux. *Stochastic epidemic models with inference*. Springer, 2019.

[5] H. Brunner. *Volterra Integral Equations: An Introduction to Theory and Applications*. Cambridge University Press, Jan. 2017.

[6] R.-M. Carlsson, L. M. Childs, Z. Feng, J. W. Glasser, J. M. Heffernan, J. Li, and G. Röst. Modeling the waning and boosting of immunity from infection or vaccination. *Journal of Theoretical Biology*, 497:110265, 2020.

[7] J. Chevallier. Fluctuations for mean-field interacting age-dependent Hawkes processes. *Electronic Journal of Probability*, 22, 2017.

[8] J. Chevallier. Mean-field limit of generalized Hawkes processes. *Stochastic Processes and their Applications*, 127(12):3870–3912, 2017.

[9] J. Y. Fan, K. Hamza, P. Jagers, and F. C. Klebaner. Limit theorems for multi-type general branching processes with population dependence. *Advances in Applied Probability*, 52(4):1127–1163, 2020.

[10] W. Feller. On the integral equation of renewal theory. In *Selected Papers I*, pages 567–591. Springer, 2015.

[11] R. Forien, G. Pang, and É. Pardoux. Epidemic models with varying infectivity. *SIAM Journal on Applied Mathematics*, 81(5):1893–1930, 2021.

[12] R. Forien, G. Pang, and É. Pardoux. Estimating the state of the COVID-19 epidemic in rance using a model with memory. *Royal Society Open Science*, 8(3):202327, 2021.

[13] R. Forien, G. Pang, and É. Pardoux. Multi-patch multi-group epidemic model with varying infectivity. *Probab. Uncertain. Quant. Risk*, 7:333–364, 2022.

[14] K. Hamza, P. Jagers, and F. C. Klebaner. The age structure of population-dependent general branching processes in environments with a high carrying capacity. *Proceedings of the Steklov Institute of Mathematics*, 282(1):90–105, 2013.

[15] K. Hamza, P. Jagers, and F. C. Klebaner. On the establishment, persistence, and inevitable extinction of populations. *Journal of Mathematical Biology*, 72(4):797–820, 2016.

[16] H. Inaba. Kermack and McKendrick revisited: the variable susceptibility model for infectious diseases. *Japan Journal of Industrial and Applied Mathematics*, 18(2):273–292, 2001.

[17] H. Inaba. Endemic threshold analysis for the Kermack–McKendrick reinfection model. *Josai Math. Monogr*, 9:105–133, 2016.

[18] H. Inaba. Variable Susceptibility, Reinfection, and Immunity. In *Age-Structured Population Dynamics in Demography and Epidemiology*, pages 379–442. Springer, 2017.

[19] H. Inaba and H. Sekine. A mathematical model for chagas disease with infection-age-dependent infectivity. *Mathematical Biosciences*, 190(1):39–69, 2004.

[20] P. Jagers and F. C. Klebaner. Population-size-dependent and age-dependent branching processes. *Stochastic Processes and their Applications*, 87(2):235–254, 2000.

[21] W. O. Kermack and A. G. McKendrick. A contribution to the mathematical theory of epidemics. *Proceedings of the Royal Society of London. Series A, Containing papers of a mathematical and physical character*, 115(772):700–721, 1927.

[22] W. O. Kermack and A. G. McKendrick. Contributions to the mathematical theory of epidemics. II.—The problem of endemicity. *Proceedings of the Royal Society of London. Series A, containing papers of a mathematical and physical character*, 138(834):55–83, 1932.

[23] W. O. Kermack and A. G. McKendrick. Contributions to the mathematical theory of epidemics–III. Further studies of the problem of endemicity. 1933. *Proceedings of the Royal Society of London. Series A, containing papers of a mathematical and physical character*, 141(843):89–118, 1933.

[24] M. E. Khalifi and T. Britton. Extending sirs epidemics to allow for gradual waning of immunity. *Journal of Royal Society Interface*, 20:20230042, 2023.

[25] K. Oelschlager. Limit theorems for age-structured populations. *Annals of Probability*, 18(1):290–318, 1990.

[26] G. Pang and É. Pardoux. Functional limit theorems for non-Markovian epidemic models. *Annals of Applied Probability*, 32(3):1615–1665, 2022.

[27] R. R. Rao. The law of large numbers for $D[0,1]$-valued random variables. *Theory of Probability & Its Applications*, 8(1):70–74, 1963.

[28] M. T. Sofonea, B. Reyné, B. Elie, R. Djidjou-Demasse, C. Selinger, Y. Michalakis, and S. Alizon. Memory is key in capturing COVID-19 epidemiological dynamics. *Epidemics*, 35:100459, 2021.

[29] A.-S. Sznitman. Topics in propagation of chaos. In *Ecole d'été de probabilités de Saint-Flour XIX—1989*, pages 165–251. Springer, 1991.

[30] H. R. Thieme and J. Yang. An endemic model with variable re-infection rate and applications to influenza. *Mathematical Biosciences*, 180(1-2):207–235, 2002.

[31] V. C. Tran. Large population limit and time behaviour of a stochastic particle model describing an age-structured population. *ESAIM: Probability and Statistics*, 12:345–386, 2008.

INRAE, Centre INRAE PACA, Domaine St-Paul - Site Agroparc 84914 Avignon Cedex FRANCE
Email address: raphael.forien@inrae.fr

Department of Computational Applied Mathematics and Operations Research, George R. Brown School of Engineering, Rice University, Houston, TX 77005
Email address: gdpang@rice.edu

Aix Marseille Univ, CNRS, I2M, Marseille, France
Email address: etienne.pardoux@univ.amu.fr

Aix Marseille Univ, CNRS, I2M, Marseille, France
Email address: arsene-brice.zotsa-ngoufack@univ-amu.fr