On Functional Limit Theorems for the Cumulative Times in Alternating Renewal Processes

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Abstract. We provide new proofs for two functional central limit theorems, and prove strong approximations for the cumulative "on" times in alternating renewal processes. The proofs rely on a first-passage-time representation of the cumulative "on" time process. As an application, we establish strong approximations for the queueing process in a single-server fluid queue with "on–off" sources.

1. Introduction

Consider an alternating renewal process $N = \{N(t) : t \geq 0\}$ with i.i.d. alternating "on-off" cycles $\{(U_i, V_i) : i \in \mathbb{N}\}$, where the U_i and V_i are "on" and "off" durations in the i^{th} cycle, $i \in \mathbb{N}$. Assume that the process starts at the beginning of an "on" period. Let $m_u = E[U_1] \in (0, \infty)$ and $m_v = E[V_1] \in (0, \infty)$, and $\sigma_u^2 = Var(U_1) < \infty$ and $\sigma_v^2 = Var(V_1) < \infty$. Let $T_i = \sum_{k=1}^{i}(U_k + V_k)$ for $i \in \mathbb{N}$ and $T_0 \equiv 0$. Then $N(t) = \max\{i \geq 0 : T_i \leq t\}$ for $t \geq 0$. Define the indicator process $\xi = \{\xi(t) : t \geq 0\}$ by

$$\xi(t) := \begin{cases} 1 & \text{if } T_i \leq t < T_i + U_{i+1}, \\ 0 & \text{if } T_i + U_{i+1} \leq t < T_{i+1}, \end{cases}$$

for each $i \in \mathbb{N}$. When $\xi(t) = 1$, the process is in the "on" period and otherwise the process is in the "off" period. Define the cumulative "on" and "off" processes $X = \{X(t) : t \geq 0\}$ and $Y = \{Y(t) : t \geq 0\}$, respectively, by

$$X(t) := \int_0^t \mathbf{1}(\xi(s) = 1)ds = \int_0^t \xi(s)ds, \tag{1.1}$$

$$Y(t) := \int_0^t \mathbf{1}(\xi(s) = 0)ds = \int_0^t (1 - \xi(s))ds = t - X(t).$$

We focus on the analysis of the cumulative "on" time process X. It is well known (see, e.g., Example 3.6(A) of Section 3.6 in [18]) that

$$\lim_{t\to\infty} \frac{E[X(t)]}{t} = \gamma_u, \quad \lim_{t\to\infty} \frac{E[Y(t)]}{t} = \gamma_v = 1 - \gamma_u,$$

where

$$\gamma_u := \frac{m_u}{m_u + m_v}. \tag{1.2}$$

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Representation of the Cumulative "On" Time as the First Passage Time. The process X can be represented as the first passage time for the random walk associated with the "on–off" cycle times, as observed in [23]. Let $N_u = \{N_u(t) : t \geq 0\}$ be defined by

$$N_u(t) := \max\{k \geq 0 : T_{u,k} \leq t\}, \quad T_{u,k} := \sum_{i=1}^{k} U_i, \ k \in \mathbb{N}, \ T_{u,0} := 0. \tag{1.3}$$

Define the compound processes $Z_u = \{Z_u(t) : t \geq 0\}$ by

$$Z_u(t) := \sum_{k=1}^{N_u(t)} V_k, \quad t \geq 0. \tag{1.4}$$

Then we can write $X(t)$ directly as

$$X(t) = \inf\{s > 0 : Z_u(s) > t - s\}, \quad t \geq 0.$$

Now define an auxiliary process $\check{Z}_u = \{\check{Z}_u(t) : t \geq 0\}$ by

$$\check{Z}_u(t) := Z_u(t) + t, \quad t \geq 0. \tag{1.5}$$

Thus, we obtain the following representation of the process X as the first passage time of the process $\check{Z}_u$:

$$X(t) = \inf\{s > 0 : \check{Z}_u(s) > t\}, \quad t \geq 0. \tag{1.6}$$

In this paper, we first review two functional central limit theorems (FCLTs) for the cumulative "on" time process X and provide new proofs for these FCLTs (Theorems 2.1 and 2.2 in Section 2). In Theorem 2.1, the "on" and "off" times are of the same order and the result is stated in [21, Theorem 8.3.1] (ours is a slight modifcation) and its proof is given in Section 5.3 of [22]. That proof applies Theorem 12.5.1 (iv) of [21] by controlling the oscillations of the cumulative "on" time process in the Skorohod M_1 topology. Our new proof takes advantage of the first passage time representation in (1.6) and thus applies the continuous mapping theorem for the inverse mapping with centering [21, Theorem 13.7.2]. This result has been used in establishing FCLTs for the queues with "on–off" sources (see, e.g., [20] and a good review in Section 8 of [21]).

In Theorem 2.2, the "on" and "off" times are of different orders, in particular, the "off" times are asymptotically negligible comparing with the "on" times. The result has been used in queueing systems with service interruptions and server vacations for single-server queues and networks [4, 11, 12, 21]. A similar result is also used for many-server queueing systems with service interruptions [14, 15, 16, 17]. This theorem can be proved with the argument as in the proof of Theorem 14.7.3 in [21]. The proof can also be done with an explicit construction of the parametric representations for the Skorohod M_1 topology (see Section 5.4 in [16]). Here we provide a new proof by applying the continuous mapping theorem to the inverse mapping with centering using the representation in (1.6) . The new proofs for these two FCLTs for the cumulative "on" time processes provide important insights on their understanding and future applications.

We prove the strong approximations for the cumulative "on" time processes (Theorem 3.1). Although strong approximations for renewal processes have been well studied and applied in queueing theory [3, 5, 6, 7, 8, 9, 10, 19], strong approximations for the cumulative "on" time processes in alternating renewal processes have remained open in the literature. The first-passage-time representation of the cumulative "on" time process in (1.6) plays a key role in establishing the strong approximations, since some existing results and proof techniques in [5, 6] on renewal processes and the inverse mapping can be applied and/or adapted for our purpose. In Theorem 3.1, we obtain the probability bounds and almost sure properties under the condition of either finite moment generating functions of the "on" and "off" times in a neighborhood of zero or their finite moments of order higher than two.

As an application, the strong approximations of the cumulative "on" time process are applied to a single-server fluid queue with "on–off" sources. Under the proper assumptions on the strong approximations of the input processes, we obtain the strong approximations of the queueing process by a reflected Brownian motion in the critically loaded regime or a Brownian motion in the overloaded regime (Theorem 3.2). Heavy-traffic approximations for fluid queues with "on–off" sources have been well studied in the literature (see a good review in Section 8 of [21]). However, strong approximations for fluid queues with "on–off" sources have remained open. To the best of our knowledge, Theorem 3.2 is the first result on this subject.

1.1. **Notation.** We use $\mathbb{R}^k$ (and $\mathbb{R}^k_+$), $k \geq 1$, to denote real-valued k-dimensional (nonnegative) vectors, and write $\mathbb{R}$ and $\mathbb{R}_+$ for $k = 1$. Let $\mathbb{N}$ denote the natural numbers. For $x, y \in \mathbb{R}$, $x \vee y = \max\{x, y\}$, $x \wedge y = \min\{x, y\}$, $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$. Let $\mathbb{D}^k = \mathbb{D}([0,\infty), \mathbb{R}^k)$ denote the $\mathbb{R}^k$-valued function space of all right continuous functions on $[0,\infty)$ with left limits everywhere in $(0,\infty)$. Denote $\mathbb{D} \equiv \mathbb{D}^1$. Let $(\mathbb{D}, J_1)$ and $(\mathbb{D}, M_1)$ denote the space $\mathbb{D}$ equipped with Skorohod J_1 and M_1 topology, respectively. Let $(\mathbb{D}_k, J_1) = (\mathbb{D}, J_1) \times \cdots \times (\mathbb{D}, J_1)$ be the k-fold product of $(\mathbb{D}, J_1)$ with the product topology. Similarly, let $(\mathbb{D}_k, M_1) = (\mathbb{D}, M_1) \times \cdots \times (\mathbb{D}, M_1)$ be the k-fold product of $(\mathbb{D}, M_1)$ with the product topology. Notations $\to$ and $\Rightarrow$ mean convergence of real numbers and convergence in distributions. Let "$\stackrel{\mathrm{d}}{=}$" denote "equal in distribution" and ":=" be "definition by equation". The abbreviation *a.s.* means *almost surely.* All random variables and processes are defined on a common probability space $(\Omega, \mathcal{F}, P)$.

2. Functional Central Limit Theorems

In this section we state the two FCLTs for the diffusion-scaled processes of X and provide new proofs for them. We index the quantities and processes with n and use n as a scaling parameter, and let $n \to \infty$.

2.1. **"On" and "off" times of the same order.** We assume that the "on" and "off" times $\{(U^n_k, V^n_k) : k \in \mathbb{N}\}$ are of the same order, and for the simplicity of exposition, we set them to be independent of n in the scaling below. Note that the random vectors $\{(U^n_k, V^n_k) : k \in \mathbb{N}\}$ are i.i.d. and each pair U^n_k and V^n_k, $k \in \mathbb{N}$, can be correlated.

Define the partial sums associated with the "on" and "off" times

$$S_{u,n} := \sum_{k=1}^{n} U_k, \quad S_{v,n} := \sum_{k=1}^{n} V_k, \quad n \in \mathbb{N},$$

and the corresponding processes: for each $n \in \mathbb{N}$,

$$S_{u,n}(t) := \sum_{k=1}^{\lfloor nt \rfloor} U_k, \quad S_{v,n}(t) := \sum_{k=1}^{\lfloor nt \rfloor} V_k, \quad t \geq 0.$$

We make the following assumption on these partial sum processes. Let $Disc(\mathcal{X})$ be the random set of discontinuities in $\mathbb{R}_+$ of any stochastic process $\mathcal{X}$.

Assumption 1. *There exist positive constants* $\alpha_u \in (1,2]$ *and* $\alpha_v \in (1,2]$ *and stochastic processes* $\tilde{S}_u$ *and* $\tilde{S}_v$ *such that*

$$P(Disc(\tilde{S}_u) \cap Disc(\tilde{S}_v) = \emptyset) = 1,$$

and

$$(\tilde{S}_{u,n}, \tilde{S}_{v,n}) \Rightarrow (\tilde{S}_u, \tilde{S}_v) \quad in \quad (\mathbb{D}_2,\ M_1) \quad as \quad n \to \infty, \tag{2.1}$$

where the processes $\tilde{S}_{u,n} = \{\tilde{S}_{u,n}(t) : t \geq 0\}$ *and* $\tilde{S}_{v,n} = \{\tilde{S}_{v,n}(t) : t \geq 0\}$ *are defined by*

$$\tilde{S}_{u,n}(t) := n^{-1/\alpha_u}(S_{u,n}(t) - m_u nt), \quad t \geq 0,$$

and

$$\tilde{S}_{v,n}(t) := n^{-1/\alpha_v}(S_{v,n}(t) - m_v nt), \quad t \geq 0.$$

Define the diffusion-scaled processes $\tilde{X}^n = \{\tilde{X}^n(t) : t \geq 0\}$,

$$\tilde{X}^n(t) := n^{-1/(\alpha_u \wedge \alpha_v)}(X(nt) - \gamma_u nt), \quad t \geq 0.$$

We prove the following FCLT for the processes $\tilde{X}_n$.

Theorem 2.1. *Under Assumption 1,*

$$\tilde{X}^n \Rightarrow \tilde{X} \quad in \quad (\mathbb{D},\ M_1) \quad as \quad n \to \infty, \tag{2.2}$$

where the limit process $\tilde{X} = \{\tilde{X}(t) : t \geq 0\}$ *is given by*

$$\tilde{X}(t) := \begin{cases} -\gamma_u \tilde{S}_v(m_u^{-1}\gamma_u t) & if \quad \alpha_u > \alpha_v, \\ \gamma_v \tilde{S}_u(m_u^{-1}\gamma_u t) & if \quad \alpha_u < \alpha_v, \\ -\gamma_u \tilde{S}_v(m_u^{-1}\gamma_u t) + \gamma_v \tilde{S}_u(m_u^{-1}\gamma_u t) & if \quad \alpha_u = \alpha_v. \end{cases} \tag{2.3}$$

Proof. Define the diffusion-scaled process $\check{Z}_u^n = \{\check{Z}_u^n(t) : t \geq 0\}$ by

$$\check{Z}_u^n(t) := n^{-1/(\alpha_u \wedge \alpha_v)}\big(\check{Z}_u(nt) - \gamma_u^{-1} nt\big), \quad t \geq 0.$$

It follows from simple algebra that for each $t \geq 0$,

$$\begin{aligned} \check{Z}_u^n(t) &= n^{-1/(\alpha_u \wedge \alpha_v)}\Big(Z_u(nt) - \frac{m_v}{m_u} nt\Big) \\ &= n^{-1/(\alpha_u \wedge \alpha_v)} \sum_{i=1}^{N_u(nt)} (V_i - m_v) + m_v n^{-1/(\alpha_u \wedge \alpha_v)}(N_u(nt) - m_u^{-1} nt). \end{aligned}$$

By the continuity of the inverse mapping with centering [21, Theorem 13.7.2] and the convergence of $\tilde{S}_{u,n} \Rightarrow \tilde{S}_u$ in (2.1), we obtain that

$$n^{-1/\alpha_u}(N_u(nt) - m_u^{-1} nt) \Rightarrow -m_u^{-1}\tilde{S}_u(m_u^{-1} t) \quad \text{in} \quad (\mathbb{D},\ M_1) \quad \text{as} \quad n \to \infty.$$

This also implies that

$$n^{-1} N_u(nt) \Rightarrow m_u^{-1} t \quad \text{in} \quad (\mathbb{D},\ M_1) \quad \text{as} \quad n \to \infty.$$

By the convergence of $\tilde{S}_{v,n} \Rightarrow \tilde{S}_v$ in (2.1) and the continuity of the composition mapping [21, Theorem 13.2.3], we obtain that

$$n^{-1/\alpha_v} \sum_{i=1}^{N_u(nt)} (V_i - m_v) \Rightarrow \tilde{S}_v(m_u^{-1} t) \quad \text{in} \quad (\mathbb{D},\ M_1) \quad \text{as} \quad n \to \infty.$$

Thus, by the continuity of addition for Skorohod M_1 topology ([21], Corollary 12.7.1), we have

$$\check{Z}_u^n(t) \Rightarrow \check{Z}_u(t) \quad \text{in} \quad (\mathbb{D},\ M_1) \quad \text{as} \quad n \to \infty,$$

where

$$\check{Z}_u(t) := \begin{cases} \tilde{S}_v(m_u^{-1}t) & \text{if} \quad \alpha_u > \alpha_v, \\ -m_v m_u^{-1} \tilde{S}_u(m_u^{-1}t) & \text{if} \quad \alpha_u < \alpha_v, \\ \tilde{S}_v(m_u^{-1}t) - m_v m_u^{-1} \tilde{S}_u(m_u^{-1}t) & \text{if} \quad \alpha_v = \alpha_u. \end{cases}$$

We next apply Theorem 13.7.2 in [21] to $\check{Z}_u^n$ and $\tilde{X}^n$. Note that we use the centering function of $\check{Z}_u^n$, i.e., $\gamma_u^{-1} t = (m_v/m_u + 1)t$ for $t \geq 0$. Thus we obtain that

$$\tilde{X}^n(t) \Rightarrow -\gamma_u \check{Z}_u(\gamma_u t) \quad \text{in} \quad (\mathbb{D},\ M_1) \quad \text{as} \quad n \to \infty.$$

It is evident that $\tilde{X}(t) = -\gamma_u \check{Z}_u(\gamma_u t)$ as given in (2.3). This completes the proof of the theorem. □

2.2. **Asymptotically negligible "off" times.** We now focus on the case where the "off" times are asymptotically negligible comparing with the "on" times (the opposite can be analyzed analogously). We make the following assumption.

Assumption 2. *There exist a positive constant $\kappa_v \in [1/2, 1)$ and a sequence of i.i.d. positive random vectors $\{(\hat{U}_k, \hat{V}_k) : k \in \mathbb{N}\}$ such that*

$$\{(n^{-1} U_k^n, n^{-\kappa_v} V_k^n) : k \in \mathbb{N}\} \to \{(\hat{U}_k, \hat{V}_k) : k \in \mathbb{N}\} \quad \textit{in} \quad (\mathbb{R}^2)^\infty \quad \textit{as} \quad n \to \infty,$$

and $\sum_{k=1}^\infty \hat{U}_k = \infty$ with probability one.

Note that $\hat{U}_k$ and $\hat{V}_k$ can be correlated for each $k \in \mathbb{N}$. Let the counting process $\hat{N}_u = \{\hat{N}_u(t) : t \geq 0\}$ associated with limiting "on" times $\{\hat{U}_k : k \in \mathbb{N}\}$ be defined by

$$\hat{N}_u(t) := \max\{k \geq 0 : \hat{T}_{u,k} \leq t\}, \quad \hat{T}_{u,k} := \sum_{i=1}^k \hat{U}_i, \quad \hat{T}_{u,0} := 0. \tag{2.4}$$

Let X^n be the cumulative "on" time process degenerated by the sequence $\{(U_k^n, V_k^n) : k \in \mathbb{N}\}$, as defined in (1.1). Let N_u^n, Z_u^n and $\check{Z}_u^n$ be the corresponding processes as defined in (1.3), (1.4) and (1.5), respectively. Define the diffusion-scaled processes $\tilde{X}_v^n = \{\tilde{X}_v^n(t) : t \geq 0\}$ by

$$\tilde{X}_v^n(t) := n^{-\kappa_v}(X^n(nt) - nt), \quad t \geq 0.$$

Theorem 2.2. *Under Assumption 2,*

$$\tilde{X}_v^n \Rightarrow \tilde{X}_v \quad \textit{in} \quad (\mathbb{D},\ M_1) \quad \textit{as} \quad n \to \infty,$$

where the limit process $\tilde{X}_v = \{\tilde{X}_v(t) : t \geq 0\}$ is given by

$$\tilde{X}_v(t) := -\sum_{k=1}^{\hat{N}_u(t)} \hat{V}_k, \quad t \geq 0.$$

Proof. Define the diffusion-scaled process $\breve{Z}_u^n = \{\breve{Z}_u^n(t) : t \geq 0\}$ by

$$\breve{Z}_u^n(t) := n^{-\kappa_v}(\check{Z}_u^n(nt) - nt), \quad t \geq 0,$$

where the process $\check{Z}_u^n$ is as defined in (1.5). It is evident that

$$\breve{Z}_u^n(t) = n^{-\kappa_v} Z_u^n(nt), \quad t \geq 0.$$

By Assumption 2, we have

$$\hat{T}^n_{u,k} := n^{-1}\sum_{i=1}^{k} U^n_i \Rightarrow \hat{T}_{u,k} := \sum_{i=1}^{k}\hat{U}_i \quad \text{as} \quad n \to \infty,$$

for each $k \in \mathbb{N}$. Thus,

$$\hat{T}^n_u(t) := n^{-1}\sum_{i=1}^{[t]} U^n_i \Rightarrow \hat{T}_u(t) := \sum_{i=1}^{[t]}\hat{U}_i \quad \text{as} \quad n \to \infty,$$

for each $t \geq 0$. It is easy to see that $\hat{T}^n_u(t)$ and $\hat{T}_u(t)$ are nondecreasing functions in $\mathbb{D}$. Recall that the convergence in M_1 topology reduces to pointwise convergence on a dense subset including 0 for nondecreasing functions in $\mathbb{D}$ ([21], Corollary 12.5.1). Thus, we have

$$\hat{T}^n_u(t) \Rightarrow \hat{T}_u(t) \quad \text{in} \quad (\mathbb{D},\ M_1) \quad \text{as} \quad n \to \infty.$$

Then by the definitions of $N^n_u(nt)$ and $\hat{N}_u(t)$, and the continuous mapping theorem applying to the inverse mapping ([21], Theorem 13.6.1), we obtain

$$N^n_u(nt) \Rightarrow \hat{N}_u(t) \quad \text{in} \quad (\mathbb{D},\ M_1) \quad \text{as} \quad n \to \infty,$$

where $\hat{N}_u(t)$ is defined in (2.4). By the continuity of the composition mapping [21, Theorem 13.2.3], we obtain

$$\breve{Z}^n_u \Rightarrow \hat{Z}_u = -\tilde{X}_v \quad \text{in} \quad (\mathbb{D},\ M_1) \quad \text{as} \quad n \to \infty.$$

We apply Theorem 13.7.2 in [21] to $\breve{Z}^n_u$ and $\tilde{X}^n_v$ in the M_1 topology. Note that the centering function for both processes is $t \to nt$ for $t \geq 0$. Thus we obtain

$$\tilde{X}^n_v(t) \Rightarrow -\hat{Z}_u = \tilde{X}_v \quad \text{in} \quad (\mathbb{D},\ M_1) \quad \text{as} \quad n \to \infty.$$

This completes the proof of the theorem. □

3. Strong Approximations

In this section we prove strong approximations for the cumulative "on" time process X by applying the first-passage-time representation in (1.6), and then apply them to a single-server fluid queue with "on–off" sources. We make the following assumptions on the "on" and "off" times.

Assumption 3. *The "on" and "off" times U_k and V_k are independent for each k. In addition, either of the following conditions holds:*

(a) *The moment generating functions of U_1 and V_1 satisfy*

$$E\big[e^{\vartheta U_1}\big] < \infty \quad \textit{and} \quad E\big[e^{\vartheta V_1}\big] < \infty, \tag{3.1}$$

in a neighborhood of 0.

(b) *The moments of U_1 and V_1 satisfy*

$$E\big[U_1^{\beta}\big] < \infty \quad \textit{and} \quad E\big[V_1^{\beta}\big] < \infty, \quad \textit{for} \quad \beta > 2. \tag{3.2}$$

These assumptions are in the same fashion as those used in [5, 6, 7, 9, 10] to prove strong approximations for random walks/partial sums and renewal processes. The representation of renewal processes via the inverse mapping of the corresponding partial sum processes is critical in proving their strong approximations; see [6, 9, 10]. Strong approximations for the cumulative "on" time processes seems difficult to establish directly. The representation via the inverse mapping of the stopped partial sum process in (1.6) makes it convenient to applying the existing results, as in [5, 6]. We prove the following theorem.

Theorem 3.1. *There exists a standard Wiener process* $W = \{W(t) : t \geq 0\}$ *such that*

(i) *under Assumption 3(a),*

$$P\left(\sup_{0\leq t\leq T} \left|\sigma_X^{-1}\left(X(t) - \gamma_u t\right) - W(t)\right| > A_1 \log T + x\right) \leq B_1 e^{-C_1 x}, \tag{3.3}$$

and as $T \to \infty$,

$$\sup_{0\leq t\leq T} \left|\sigma_X^{-1}\left(X(t) - \gamma_u t\right) - W(t)\right| = O(\log(T)) \quad a.s., \tag{3.4}$$

where $\sigma_X^2 = (\sigma_v^2/m_u + m_v^2\sigma_u^2/m_u^3)\gamma_u^{3/2}$, *and* A_1, B_1 *and* C_1 *are positive constants;*

(ii) *under Assumption 3(b),*

$$P\left(\sup_{0\leq t\leq T} \left|\sigma_X^{-1}\left(X(t) - \gamma_u t\right) - W(t)\right| > x\right) \leq D_1(T)(Tx^{-\beta} + T^{-\kappa}), \tag{3.5}$$

for every $\kappa > 0$ *and for all* $D_2T^{1/\beta} + \sigma_u^{-1}m_u^{3/2} \leq x \leq D_3(T\log T)^{1/2} + \sigma_u^{-1}m_u^{3/2}$, *and as* $T \to \infty$,

$$\sup_{0\leq t\leq T} \left|\sigma_X^{-1}\left(X(t) - \gamma_u t\right) - W(t)\right| = o(T^{1/\beta}) \quad a.s., \tag{3.6}$$

where $D_1(T) \to 0$ *as* $T \to \infty$ *and* D_2 *and* D_3 *are positive constants.*

3.1. **Application to a single-server fluid queue with "on–off" sources.** We now apply the above results to obtain strong approximations for a single-server fluid queue with "on–off" sources. Suppose the arrival process is generated by a stochastic process $R = \{R(t) : t \geq 0\}$ with nondecreasing sample paths and $R(0) = 0$, whenever the underlying alternating renewal process is in the "on" periods. We use the same notation for the alternating renewal process as above. Note that the process R and the alternating renewal process are assumed to be independent. Then the cumulative arrival process $A = \{A(t) : t \geq 0\}$ is given by

$$A(t) := R(X(t)), \quad t \geq 0. \tag{3.7}$$

We assume that the process R satisfies the following strong approximation properties.

Assumption 4. *There exists a standard Wiener process* $W_R = \{W_R(t) : t \geq 0\}$ *such that either of the following properties holds:*

(a)

$$P\left(\sup_{0\leq t\leq T} \left|\sigma_R^{-1}\left(R(t) - rt\right) - W_R(t)\right| > A_1' \log T + x\right) \leq B_1' e^{-C_1' x}, \tag{3.8}$$

for all $x > 0$ *and as* $T \to \infty$,

$$\sup_{0\leq t\leq T} \left|\sigma_R^{-1}\left(R(t) - rt\right) - W_R(t)\right| = O(\log(T)) \quad a.s., \tag{3.9}$$

where r, σ_R, A_1', B_1' *and* C_1' *are positive constants.*

(b)

$$P\left(\sup_{0\le t\le T}\left|\sigma_R^{-1}\left(R(t)-rt\right)-W_R(t)\right|>x\right)\le D_1'(T)Tx^{-\varrho}, \tag{3.10}$$

for all $D_2'T^{1/\varrho}\le x\le D_3'(T\log T)^{1/2}$ *and as* $T\to\infty$,

$$\sup_{0\le t\le T}\left|\sigma_R^{-1}\left(R(t)-rt\right)-W_R(t)\right|=o(T^{1/\varrho})\quad a.s., \tag{3.11}$$

where ϱ, D_2', D_3' *are positive constants and* $D_1'(T)\to 0$ *as* $T\to\infty$.

Consider a single-server fluid queueing model with this arrival process A in (3.7) and a constant service rate μ. Let $Q=\{Q(t):t\ge 0\}$ be the queueing process. Then we can write

$$Q(t)=Q(0)+A(t)-\mu B(t),\quad t\ge 0, \tag{3.12}$$

where $B(t)$ is the cumulative busy time of the server by time t. We assume that the system is work-conserving under the first-come first-served (FCFS) service discipline. For any real-valued function x, let ϕ be the Skorohod mapping of x defined by

$$\phi(x)(t):=x(t)+\sup_{0\le s\le t}(-x(s))^+=x(t)-\inf_{0\le s\le t}\{x(s)\wedge 0\},\quad t\ge 0. \tag{3.13}$$

Theorem 3.2. *There exists a standard Wiener process* $W_Q=\{W_Q(t):t\ge 0\}$ *such that*

(i) *under Assumptions 3(a) and 4(a), if* $r\gamma_u=\mu$,

$$P\left(\sup_{0\le t\le T}\left|Q(t)-\tilde{Q}(t)\right|>A_2'\log T+x\right)\le B_2'e^{-C_2'x}, \tag{3.14}$$

for all $x>0$ *and as* $T\to\infty$,

$$\sup_{0\le t\le T}\left|Q(t)-\tilde{Q}(t)\right|=O(\log(T))\quad a.s., \tag{3.15}$$

and if $r\gamma_u>\mu$,

$$P\left(\sup_{0\le t\le T}\left|Q(t)-(r\gamma_u-\mu)t-\tilde{W}(t)\right|>A_2''\log T+x\right)\le B_2''e^{-C_2''x}, \tag{3.16}$$

for all $x>0$ *and as* $T\to\infty$,

$$\sup_{0\le t\le T}\left|Q(t)-(r\gamma_u-\mu)t-\tilde{W}(t)\right|=O(\log(T))\quad a.s., \tag{3.17}$$

where the process $\tilde{Q}=\{\tilde{Q}(t):t\ge 0\}$ *is a reflected Brownian motion, defined by*

$$\tilde{Q}(t):=\phi(\tilde{W})(t),\quad t\ge 0, \tag{3.18}$$

with $\tilde{W}=\{\tilde{W}(t):t\ge 0\}$ *being a Brownian motion defined by*

$$\tilde{W}(t):=Q(0)+\theta W_Q(t),\quad t\ge 0, \tag{3.19}$$

$$\theta^2:=r^2\sigma_X^2+\gamma_u\sigma_R^2, \tag{3.20}$$

and A_2', A_2'', B_2', B_2'', C_2' *and* C_2'' *are positive constants;*

(ii) *under Assumptions 3(b) and 4(b), if* $r\gamma_u=\mu$,

$$P\left(\sup_{0\le t\le T}\left|Q(t)-\tilde{Q}(t)\right|>x\right)\le D_4'(T)(Tx^{-\varrho\wedge 1}+1), \tag{3.21}$$

for all $D_5' T^{1/(\beta\wedge\varrho)} + \sigma_u^{-1} m_u^{3/2} \le x \le D_6'(T\log T)^{1/2} + \sigma_u^{-1} m_u^{3/2}$ *and as* $T\to\infty$,

$$\sup_{0\le t\le T} \big|Q(t) - \tilde{Q}(t)\big| = o(T^{1/\beta}) \quad a.s., \tag{3.22}$$

where $\tilde{Q}$ *is defined in* (3.18), $D_4'(T)\to 0$ *as* $T\to\infty$, D_5', D_6' *are positive constants, and if* $r\gamma_u > \mu$,

$$P\Big(\sup_{0\le t\le T} \big|Q(t) - (r\gamma_u - \mu)t - \tilde{W}(t)\big| > x\Big) \le D_4''(T)(Tx^{-\varrho\wedge 1} + 1), \tag{3.23}$$

for all $D_5'' T^{1/(\beta\wedge\varrho)} + \sigma_u^{-1} m_u^{3/2} \le x \le D_6''(T\log T)^{1/2} + \sigma_u^{-1} m_u^{3/2}$ *and as* $T\to\infty$,

$$\sup_{0\le t\le T} \big|Q(t) - (r\gamma_u - \mu)t - \tilde{W}(t)\big| = o(T^{1/\beta}) \quad a.s., \tag{3.24}$$

where $\tilde{W}$ *is defined in* (3.19), *and* $D_4''(T)\to 0$ *as* $T\to\infty$, D_5'', D_6'' *are positive constants.*

Remark 3.1. The results in Theorem 3.2 can be easily extended to single-server fluid queues with a finite number of "on–off" sources, as shown in the FCLTs for such models in Section 5 of [20] and Section 8.3.2 of [21].

4. Proofs of Theorems 3.1 and 3.2

Proof of Theorem 3.1. Let $\tilde{N}_u = \{\tilde{N}_u(t) : t\ge 0\}$ be the "extended" renewal process defined by

$$\tilde{N}_u(t) := \min\{k > 0 : T_{u,k} > t\}, \quad t\ge 0.$$

Then by Theorem B in [5], there exists a standard Wiener process $W_{\tilde{N}} = \{W_{\tilde{N}}(t) : t\ge 0\}$ such that under Assumption 3(a),

$$P\Big(\sup_{0\le t\le T} \big|\sigma_{\tilde{N}}^{-1}\big(\tilde{N}_u(t) - m_u^{-1}t\big) - W_{\tilde{N}}(t)\big| > A_2\log T + x\Big) \le B_2 e^{-C_2 x}, \tag{4.1}$$

for all $x\ge 0$ and as $T\to\infty$,

$$\sup_{0\le t\le T} \big|\sigma_{\tilde{N}}^{-1}\big(\tilde{N}_u(t) - m_u^{-1}t\big) - W_{\tilde{N}}(t)\big| = O(\log(T)) \quad a.s., \tag{4.2}$$

and under Assumption 3(b),

$$P\Big(\sup_{0\le t\le T} \big|\sigma_{\tilde{N}}^{-1}\big(\tilde{N}_u(t) - m_u^{-1}t\big) - W_{\tilde{N}}(t)\big| > x\Big) \le D_4(T)Tx^{-\beta}, \tag{4.3}$$

for all $D_5 T^{1/\beta} \le x \le D_6(T\log T)^{1/2}$ and as $T\to\infty$,

$$\sup_{0\le t\le T} \big|\sigma_{\tilde{N}}^{-1}\big(\tilde{N}_u(t) - m_u^{-1}t\big) - W_{\tilde{N}}(t)\big| = o(T^{1/\beta}) \quad a.s., \tag{4.4}$$

where A_2, B_2, C_2, D_5, D_6 are positive constants, $D_4(T)\to 0$ as $T\to\infty$, and $\sigma_{\tilde{N}}^2 := \sigma_u^2 m_u^{-3}$. It is clear that $N_u(t) = \tilde{N}_u(t) - 1$, for $t\ge 0$. Thus, (4.1)–(4.4) hold with $\tilde{N}_u$ replaced by N_u, only with changes in constants and the range for x. In particular, B_2 becomes $B_2\exp(C_2\sigma_{\tilde{N}}^{-1})$ in (4.1) and the range for x above becomes $D_5 T^{1/\beta} + \sigma_{\tilde{N}}^{-1} \le x \le D_6(T\log T)^{1/2} + \sigma_{\tilde{N}}^{-1}$.

We next follow the similar steps in the proof of Theorem 1.1 in [5] (given by Lemma 1, 2 and 3 there) to obtain the strong approximations for $\check{Z}_u(t)$. Note that Theorem 1.1 in [5] cannot be applied directly, since it provides probability bounds and almost sure statements for $S(N(t))$ given those of $S(t)$ and $N(t)$, with $S(t)$ being a partial sum process and $N(t)$ an "extended" renewal process. However, in our setting, given the strong approximations for the processes $\sum_{1\le k\le t} V_k$ (by Theorem A in [5]) and the renewal process $N_u(t)$ (in the above paragraph), by a slight modification of the proof of Theorem 1.1 in [5], we obtain the probability bounds and almost sure statements as (4.1)–(4.4) for $Z_u(t)$. Since $\check{Z}_u(t) = Z_u(t) + t$ for each $t \ge 0$, we conclude that there exists a standard Wiener process $W_{\check{Z}_u} = \{W_{\check{Z}_u}(t) : t \ge 0\}$ such that under Assumption 3(a),

$$P\bigg(\sup_{0\le t\le T}\big|\sigma_{\check{Z}_u}^{-1}\big(\check{Z}_u(t)-\gamma_u^{-1}t\big)-W_{\check{Z}_u}(t)\big|>A_3\log T+x\bigg)\le B_3e^{-C_3x}, \tag{4.5}$$

for all $x \ge 0$ and as $T \to \infty$,

$$\sup_{0\le t\le T}\big|\sigma_{\check{Z}_u}^{-1}\big(\check{Z}_u(t)-\gamma_u^{-1}t\big)-W_{\check{Z}_u}(t)\big|=O(\log(T))\quad a.s., \tag{4.6}$$

and under Assumption 3(b),

$$P\bigg(\sup_{0\le t\le T}\big|\sigma_{\check{Z}_u}^{-1}\big(\check{Z}_u(t)-\gamma_u^{-1}t\big)-W_{\check{Z}_u}(t)\big|>x\bigg)\le D_7(T)Tx^{-\beta}, \tag{4.7}$$

for all $D_8T^{1/\beta}+\sigma_{\tilde{N}}^{-1}\le x\le D_9(T\log T)^{1/2}+\sigma_{\tilde{N}}^{-1}$ and as $T\to\infty$,

$$\sup_{0\le t\le T}\big|\sigma_{\check{Z}_u}^{-1}\big(\check{Z}_u(t)-\gamma_u^{-1}t\big)-W_{\check{Z}_u}(t)\big|=o(T^{1/\beta})\quad a.s., \tag{4.8}$$

where A_3, B_3, C_3, D_8, D_9 are positive constants, $D_7(T) \to 0$ as $T \to \infty$, and

$$\sigma_{\check{Z}_u}^2 := \sigma_v^2/m_u + m_v^2\sigma_u^2/m_u^3.$$

Now we turn to the process $X(t)$, which is the "inverse" process of $\check{Z}_u$ as defined in (1.6). By applying Theorem 3.1 in [6], (4.6) and (4.8) imply (3.4) and (3.6) directly.

To prove (3.3), we modify the proof of Corollary 4.2 in [6]. By (4.5) (counterpart of (4.14) in [6]) and a similar proof of (4.4) in [6], we obtain

$$P(X(T) > 2T\gamma_u + A_4x) \le B_4e^{-C_4x},$$

for some positive constants A_4, B_4 and C_4. Define $M(t) := \inf\{x \ge 0 : \gamma_u\sigma_{\check{Z}_u}W_{\check{Z}_u}(x) = t-x\}$ if $t \ge 0$ and $M(t) \equiv 0$ otherwise. Then by a similar argument of (4.5) in [6], we obtain

$$\begin{aligned}P\big(M(\gamma_ut-(A_5\log T+A_6x))\le X(t)\le M(\gamma_ut+(A_7\log T+A_8x)),0\le t\le T\big)\\ \ge 1-B_5e^{-C_5x},\end{aligned}$$

for some positive constants A_5, A_6, A_7, A_8, B_5 and C_5. This inequality combined with Theorem 2.1 in [6] and Lemma 1.2.1 in [7] completes the proof of (3.3).

Finally, given (4.7), we apply Theorem 4.1 in [6] to obtain (3.5). This completes the proof. □

We next prove Theorem 3.2. We need the following lemmas. Recall θ defined in (3.20).

Lemma 4.1 (Lemma 3 in [5]). *There exists a standard Wiener process $W_Q = \{W_Q(t) : t \ge 0\}$ such that*

$$P\bigg(\sup_{0\le t\le T}\big|\big(\sigma_RW_R(\gamma_ut+\sigma_XW(t))+r\sigma_XW(t)\big)-\theta W_Q(t)\big|>\tilde{A}\log T+x\bigg)\le\tilde{B}e^{-\tilde{C}x}, \tag{4.9}$$

for any $x \ge 0$, where $\tilde{A}, \tilde{B}$ and $\tilde{C}$ are positive constants.

Lemma 4.2. *Under Assumptions 3(a) and 4(a), there exists a standard Wiener process $W_Q = \{W_Q(t) : t \geq 0\}$ such that*

$$P\bigg(\sup_{0\leq t\leq T} \big|R(X(t)) - r\gamma_u t - \theta W_Q(t)\big| > A_2' \log T + x\bigg) \leq B_3' e^{-C_3' x} \tag{4.10}$$

for any $x \geq 0$, where A_2' is as given in (3.14)*, and B_3', C_3' are positive constants.*

Proof. By Lemma 4.1, there exists a standard Wiener process W_Q such that (4.9) holds. By adding and subtracting terms, we obtain that

$$R(X(t)) - r\gamma_u t - \theta W_Q(t) = \sum_{i=1}^{3} X_i(t) + \big(r\sigma_X W(t) + \sigma_R W_R(\gamma_u t + \sigma_X W(t)) - \theta W_Q(t)\big) \tag{4.11}$$

for $t \geq 0$, where

$$\begin{aligned} X_1(t) &:= R(X(t)) - rX(t) - \sigma_R W_R(X(t)), \\ X_2(t) &:= r\big(X(t) - \gamma_u t - \sigma_X W(t)\big), \\ X_3(t) &:= \sigma_R\big(W_R(X(t)) - W_R(\gamma_u t + \sigma_X W(t))\big). \end{aligned}$$

Notice that $0 \leq X(t) \leq t$, by Assumption 4 (a), we have

$$P\bigg(\sup_{0\leq t\leq T} |X_1(t)| > A_2' \log T + x\bigg) \leq B_{3,1}' e^{-C_{3,1}' x}, \tag{4.12}$$

for positive constants $B_{3,1}'$ and $C_{3,1}'$. By Theorem 3.1 (i), we obtain

$$P\bigg(\sup_{0\leq t\leq T} |X_2(t)| > A_2' \log T + x\bigg) \leq B_{3,2}' e^{-C_{3,2}' x}, \tag{4.13}$$

for postive constants $B_{3,2}'$ and $C_{3,2}'$. Further, by (4.13) and Lemma 1.2.1 in [7]

$$\begin{aligned} &P\bigg(\sup_{0\leq t\leq T} |X_3(t)| \geq A_2' \log T + x\bigg) \\ &\leq P\bigg(\sup_{0\leq t\leq T}\ \sup_{0\leq s\leq A_3' \log T + x} |W_R(t+s) - W_R(t)| \geq \frac{1}{\sigma_R} A_2' \log T + \frac{1}{\sigma_R} x\bigg) \\ &\quad + B_{3,2}' e^{-C_{3,2}' x} \\ &\leq B_{3,3}' e^{-C_{3,3}' x}, \end{aligned} \tag{4.14}$$

for positive constants $B_{3,3}'$ and $C_{3,3}'$. Now, (4.10) is simply implied by (4.9) and (4.11)–(4.14) and the proof is complete. □

Lemma 4.3. *There exists a standard Wiener process $W_Q' = \{W_Q'(t) : t \geq 0\}$ such that for any $\alpha > 3$ and all $\hat{A} T^{1/\alpha} \leq x \leq \hat{B}(T \log T)^{1/2}$,*

$$P\bigg(\sup_{0\leq t\leq T} \big|\big(\sigma_R W_R(\gamma_u t + \sigma_X W(t)) + r\sigma_X W(t)\big) - \theta W_Q'(t)\big| > x\bigg) \leq \hat{C}(T) T x^{-\alpha}, \tag{4.15}$$

where $\hat{A}, \hat{B}$ are positive constants and $\hat{C}(T) \to 0$ as $T \to \infty$.

Proof. We follow similar steps in the proof of Lemma 3 in [5].

Let $\{Z_i :\ i \geq 1\}$ be a sequence of i.i.d random variables with $E[Z_1] = r\gamma_u^{-1}\sigma_X^2$ and $Var(Z_1) = \gamma_u^{-1}\sigma_R^2\sigma_X^2$. In addition, we also assume that the c.d.f. of Z_1 satisfies conditions

(i) and (ii) in equation (1.2) of [13], as required in Theorem 4 in [13]. Also let $\nu(t)$ be a unit-rate Poisson process independent of $\{Z_i\}$.

Instead of (A.7) in [5], we use the approximation method in [13] (see Theorem 4 and its proof in [13]) to obtain that there exists a Wiener process $W_1' = \{W_1'(t) : t \geq 0\}$ such that for $\alpha > 3$ and all $T^{1/\alpha} \leq x \leq \hat{B}_1(T\log T)^{1/2}$,

$$P\bigg(\sup_{0\leq t\leq T}\bigg|\bigg(\sum_{0\leq i\leq \nu(t)} Z_i - r\gamma_u^{-1}\sigma_X^2 t\bigg)/\hat{\theta} - W_1'(t)\bigg| > x\bigg) \leq \hat{C}_1(T)Tx^{-\alpha}, \tag{4.16}$$

where

$$\hat{\theta}^2 := \gamma_u^{-2}\sigma_X^2\big(\gamma_u\sigma_R^2 + r^2\sigma_X^2\big) = \gamma_u^{-2}\sigma_X^2\theta^2,$$

$\hat{B}_1$ is a positive constant and $\hat{C}_1(T) \to 0$ as $T \to \infty$.

By Theorems A and B and Lemma 2 in [5], there exist two independent Wiener processes W_R' and W' such that for all $\hat{A}_2T^{1/\alpha} \leq x \leq \hat{B}_2(T\log T)^{1/2}$,

$$P\bigg(\sup_{0\leq t\leq T}\bigg|\bigg(\sum_{0\leq i\leq \nu(t)} Z_i - r\gamma_u^{-1}\sigma_X^2 t\bigg) - \Big(\gamma_u^{-1/2}\sigma_R\sigma_X W_R'(t + W(t)) + r\gamma_u^{-1}\sigma_X^2 W'(t)\Big)\bigg| > x\bigg) \leq \hat{C}_2(T)Tx^{-\alpha}, \tag{4.17}$$

where $\hat{A}_2$, $\hat{B}_2$ are positive constants, and $\hat{C}_2(T) \to 0$ as $T \to \infty$.

Now by (4.16)-(4.17), we obtain for all $\hat{A}_3T^{1/\alpha} \leq x \leq \hat{B}_3(T\log T)^{1/2}$,

$$P\bigg(\sup_{0\leq t\leq T}\Big|\hat{\theta}W_1' - \Big(\gamma_u^{-1/2}\sigma_R\sigma_X W_R'(t+W(t)) + r\gamma_u^{-1}\sigma_X^2 W'(t)\Big)\Big| > x\bigg) \leq \hat{C}_3(T)Tx^{-\alpha}, \tag{4.18}$$

where $\hat{A}_3$, $\hat{B}_3$ are positive constants, and $\hat{C}_3(T) \to 0$ as $T \to \infty$.

Note that

$$\begin{aligned} &\big\{\gamma_u^{-1/2}\sigma_R\sigma_X W_R'(t + W(t)) + r\gamma_u^{-1}\sigma_X^2 W'(t) : t \geq 0\big\} \\ \overset{\mathrm{d}}{=}\ &\big\{\sigma_R W_R'(\gamma_u^{-1}\sigma_X^2 t + \sigma_X W(\gamma_u^{-2}\sigma_X^2 t)) + r\sigma_X W'(\gamma_u^{-2}\sigma_X^2 t) : t \geq 0\big\}. \end{aligned} \tag{4.19}$$

By the time transformation $\gamma_u^{-2}\sigma_X^2 t \to t$ for each $t \geq 0$, the latter becomes

$$\big\{\sigma_R W_R'(\gamma_u t + \sigma_X W(t)) + r\sigma_X W'(t) : t \geq 0\big\}.$$

Thus, by Theorem A.1 in [1], from this observation and the bound in (4.18), we complete the proof of the lemma. □

Lemma 4.4. *Under Assumptions 3(b) and 4(b), there exists a standard Wiener process* $W_Q' = \{W_Q'(t) : t \geq 0\}$ *such that*

$$P\bigg(\sup_{0\leq t\leq T}\big|R(X(t)) - r\gamma_u t - \theta W_Q'(t)\big| > x\bigg) \leq D_7'(T)(Tx^{-\varrho\wedge 1} + 1) \tag{4.20}$$

for all $D_8'T^{1/(\beta\wedge\varrho)} + \sigma_u^{-1}m_u^{3/2} \leq x \leq D_9'(T\log T)^{1/2} + \sigma_u^{-1}m_u^{3/2}$*, where* $D_7'(T) \to 0$ *as* $T \to \infty$ *and* D_8', D_9' *are positive constants.*

Proof. By applying Lemma 4.3, there exists a standard Wiener process W_Q' such that (4.15) holds. Thus, similar to (4.11), we obtain

$$R(X(t)) - r\gamma_u t - \theta W_Q'(t) = \sum_{i=1}^{3} X_i'(t) + \big(r\sigma_X W(t) + \sigma_R W_R(\gamma_u t + \sigma_X W(t)) - \theta W_Q'(t)\big), \tag{4.21}$$

where X_i', $i = 1,2,3$, are as defined in (4.11) (the primes are used to avoid confusion).

For $X_1'(t)$, by Assumption 4 (b), we have

$$P\biggl(\sup_{0\le t\le T}\bigl|X_1'(t)\bigr| > x\biggr) \le D_1'(T)Tx^{-\varrho}, \tag{4.22}$$

for all $D_2'T^{1/\varrho} \le x \le D_3'(T\log T)^{1/2}$, where $D_1'(T) \to 0$ as $T \to \infty$, and D_2', D_3' are positive constants.

By Theorem 3.1 (ii), we obtain that for every $\kappa > 0$,

$$P\biggl(\sup_{0\le t\le T}\bigl|X_2'(t)\bigr| > x\biggr) \le D_1(T)(Tx^{-\beta} + T^{-\kappa}), \tag{4.23}$$

for all $D_2T^{1/\beta} + \sigma_u^{-1}m_u^{3/2} \le x \le D_3(T\log T)^{1/2} + \sigma_u^{-1}m_u^{3/2}$ where $D_1(T) \to 0$ as $T \to \infty$, and D_2, D_3 are positive constants.

Note that there exists a positive constant $\hat{K}_1$ such that $\hat{K}_1\log T \le D_2T^{1/\beta}$. Thus, by (4.23) and Lemma 1.2.1 in [7], we obtain that for all $D_2T^{1/\beta} + \sigma_u^{-1}m_u^{3/2} \le x \le D_3(T\log T)^{1/2} + \sigma_u^{-1}m_u^{3/2}$,

$$\begin{aligned} & P\biggl(\sup_{0\le t\le T}\bigl|X_3'(t)\bigr| > x\biggr) \\ \le\ & P\biggl(\sup_{0\le t\le T}\sup_{0\le s\le x}|W_R(t+s) - W_R(t)| > x\biggr) + D_1(Tx^{-\beta} + T^{-\kappa}) \\ \le\ & P\biggl(\sup_{0\le t\le T}\sup_{0\le s\le x}|W_R(t+s) - W_R(t)| > (x\hat{K}_1\log T)^{1/2}\biggr) + D_1(T)(Tx^{-\beta} + T^{-\kappa}) \\ \le\ & \bigl(\hat{K}_2T^{-\hat{K}_1/3}\bigr)Tx^{-1} + D_1(T)\bigl(Tx^{-\beta} + T^{-\kappa}\bigr), \end{aligned} \tag{4.24}$$

where $\hat{K}_2 > 0$ is a constant. Therefore, (4.20) follows from Lemma 4.3 and (4.22)–(4.24). □

Proof of Theorem 3.2. We first prove case (i). Recall the strong approximations of X in Theorem 3.1 and R in Assumption 4, and the associated Brownian motions W and W_R. Note that W and W_R are independent by assumption. Also recall Lemma 4.2, the existence of the Brownian motion W_Q satisfying (4.10).

By the complementarity condition under the work-conserving service policy, we can represent the process Q in (3.12) as (see, e.g., Section 9.2 in [21])

$$Q(t) = \phi(\breve{Q})(t), \quad t \ge 0, \tag{4.25}$$

where the process $\breve{Q} := \{\breve{Q}(t) : t \ge 0\}$ is defined by

$$\begin{aligned} \breve{Q}(t) &:= Q(0) + R(X(t)) - \mu t \\ &= \bigl(R(X(t)) - r\gamma_u t - \theta W_Q(t)\bigr) + (r\gamma_u - \mu)t + \tilde{W}(t), \quad t \ge 0, \end{aligned} \tag{4.26}$$

with $\tilde{W}(t)$ defined in (3.19).

Let L be the Lipschitz constant of the Skorohod mapping ϕ. We first consider the case $r\gamma_u = \mu$. By (3.18) and (4.25)–(4.26), we obtain

$$\begin{aligned} & P\biggl(\sup_{0\le t\le T}\bigl|Q(t) - \tilde{Q}(t)\bigr| > A_2'\log T + x\biggr) \\ \le\ & P\biggl(\sup_{0\le t\le T}|R(X(t)) - r\gamma_u t - \theta W_Q(t)| > \frac{1}{L}A_2'\log T + \frac{1}{L}x\biggr). \end{aligned} \tag{4.27}$$

Now, the inequality above with Lemma 4.2 implies (3.14). (3.15) is immediate from (3.14).

We next consider the case $r\gamma_u > \mu$. By (4.25) and (4.26), we obtain

$$\begin{aligned}
& |Q(t) - (r\gamma_u - \mu)t - \tilde{W}(t)| \\
= \ & \big|\big(R(X(t)) - r\gamma_u t - \theta W_Q(t)\big) - \inf_{0\le s\le t}\{(Q(0) + R(X(s)) - \mu s) \wedge 0\}\big| \\
\le \ & |R(X(t)) - r\gamma_u t - \theta W_Q(t)| \\
& \quad + \sup_{0\le s\le t}\big[\big(Q(0) + R(X(s) - r\gamma_u s - \theta W_Q(s)\big) + \big(\theta W_Q(s) + (r\gamma_u - \mu)s\big)\big]^- \\
\le \ & |R(X(t)) - r\gamma_u t - \theta W_Q(t)| \\
& \quad + \sup_{0\le s\le t}|Q(0) + R(X(s) - r\gamma_u s - \theta W_Q(s)| + \sup_{0\le s\le t}[\theta W_Q(s) + (r\gamma_u - \mu)s]^- .
\end{aligned}$$

Thus, by Lemma 4.2, (3.16) follows from

$$P\bigg(\sup_{0\le t\le T}[\theta W_Q(t) + (r\gamma_u - \mu)t]^- \ge \theta A_2'' \log T + x\bigg) \le B_{2,1}'' e^{-C_{2,1}'' x}. \tag{4.28}$$

for some positive constants $B_{2,1}''$ and $C_{2,1}''$. Observe that

$$\sup_{0\le t\le T}[\theta W_Q(t) + (r\gamma_u - \mu)t]^- = 0 \vee \sup_{0\le t\le T}[-\theta W_Q(t) - (r\gamma_u - \mu)t] .$$

Thus, by the symmetry property of Brownian motions, to prove (4.28), it suffices to prove

$$P\bigg(\sup_{0\le t\le T}[\theta W_Q(t) - (r\gamma_u - \mu)t] \ge \theta A_2'' \log T + x\bigg) \le B_{2,1}'' e^{-C_{2,1}'' x}.$$

By the Darling and Siegert formula for the distribution of the supremum of drifted Brownian motion (see, e.g., Theorem 2 in [2]), we obtain that

$$\begin{aligned}
& P\bigg(\sup_{0\le t\le T}[W_Q(t) - (r\gamma_u - \mu)t/\theta] \ge A_2'' \log T + x/\theta\bigg) \\
= \ & 1 - \Phi\big(T^{-1/2}(A_2'' \log T + x/\theta) + (r\gamma_u - \mu)T^{1/2}/\theta\big) \\
& \quad + e^{2(r\gamma_u - \mu)\theta^{-1}(A_2'' \log T + x/\theta)}\big(1 - \Phi\big(T^{-1/2}(A_2'' \log T + x/\theta) - (r\gamma_u - \mu)T^{1/2}/\theta\big)\big) \\
\le \ & B_2'' e^{-C_2'' x},
\end{aligned}$$

where Φ is the c.d.f of standard normal, and the inequality follows from applying $1 - \Phi(x) \le e^{-x^2}/(x\sqrt{2\pi})$ for $x > 0$ and some simple algebra. This completes the proof of (3.16) and also (3.17).

The proof for case (ii) follows from the similar arguments as in case (i), by Lemmas 4.3–4.4. This completes the proof of the theorem. □

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