CAAM 402/502 Spring 2013
Homework 1
Solutions

1. (a) Prove that there is no real valued function $f : V \to \mathbb{R}$, defined on a set $V \subseteq \mathbb{R}^n$, that has directional derivates at a give $\mathbf{c} \in V$, satisfying $D_{\mathbf{u}}f(\mathbf{c}) > 0$, for all $\mathbf{u} \in \mathbb{R}^n$, $\|\mathbf{u}\| = 1$.

Proof. Suppose that there exists a function f satisfying the given conditions. Consider a $\mathbf{u} \in \mathbb{R}^n$ and $\|\mathbf{u}\| = 1$, we have $D_{\mathbf{u}}f(\mathbf{c}) > 0$ and $D_{\text{-}\mathbf{u}}f(\mathbf{c}) > 0$ since $\|\text{-}\mathbf{u}\| = 1$

$$0 < D_{\mathbf{u}}f(\mathbf{c}) = \lim_{h\to 0} \frac{f(\mathbf{c}+h\mathbf{u}) - f(\mathbf{c})}{h} = -\lim_{-h\to 0} \frac{f(\mathbf{c}+(-h)(\text{-}\mathbf{u})) - f(\mathbf{c})}{-h} = -D_{\text{-}\mathbf{u}}f(\mathbf{c}) < 0$$

We have $0 < 0$ from the above relations, which is a contradiction. Hence proved. □

(b) Give an example of a function $f : \mathbb{R}^n \to \mathbb{R}$, satisfying $D_{\mathbf{u}}f(\mathbf{c}) > 0$, for all $\mathbf{c} \in \mathbb{R}^n$, and a given unit vector $\mathbf{u} \in \mathbb{R}^n$
Consider $f(\mathbf{x}) = \mathbf{x} \cdot \mathbf{u}$,

$$D_{\mathbf{u}}f(\mathbf{c}) = \lim_{t\to 0} \frac{f(\mathbf{c}+t\mathbf{u}) - f(\mathbf{c})}{t} = \lim_{t\to 0} \frac{(\mathbf{c}+t\mathbf{u})\cdot\mathbf{u} - \mathbf{c}\cdot\mathbf{u}}{t} = \lim_{t\to 0} \frac{t\mathbf{u}\cdot\mathbf{u}}{t} = \mathbf{u}\cdot\mathbf{u} = 1 > 0$$

Hence $D_{\mathbf{u}}f(\mathbf{c}) > 0$ for all $\mathbf{c} \in \mathbb{R}^n$ and a given unit vector $\mathbf{u} \in \mathbb{R}^n$

2. $f : V \to \mathbb{R}$, V is an open set in $\mathbb{R}^n$ and f is homogeneous of degree p over V. i.e.,

$$f(\lambda\mathbf{x}) = \lambda^p f(\mathbf{x})$$

for all $\lambda \in \mathbb{R}$ and $\mathbf{x} \in V$ such that $\lambda\mathbf{x} \in V$.

Show that if f is differentiable and homogenous of degree p, then

$$\mathbf{x} \cdot \text{grad} f(\mathbf{x}) = pf(\mathbf{x})$$

Proof. Define $g(\lambda) = f(\lambda\mathbf{x})$. Using the chain rule for derivates $g'(\lambda) = \mathbf{x} \cdot \text{grad} f(\lambda\mathbf{x}) \Rightarrow g'(1) = \mathbf{x}\cdot\text{grad} f(\mathbf{x})$. Similarly we have $g(\lambda) = \lambda^p f(\mathbf{x}) \Rightarrow g'(\lambda) = p\lambda^{p-1} f(\mathbf{x})$. For $\lambda = 1$, we have $g'(1) = pf(\mathbf{x})$. Thus we have $\mathbf{x} \cdot \text{grad} f(\mathbf{x}) = g'(1) = pf(\mathbf{x})$. □

3. (a) Let $f : \mathbb{R}^n \to \mathbb{R}$ and suppose that $D_u f(x) = 0$ for all x in the open ball $B(c) \subset \mathbb{R}^n$ and for all unit vectors $u \in \mathbb{R}^n$. Prove that f is constant on $B(c)$

Proof. Since the open ball $B(c)$ is convex, then given arbitrary two points $x_1, x_2 \in B(c)$, they can be joined by a straight line segment contained in $B(c)$. This line segment can be expressed as $\alpha(t) = tx_2 + (1-t)x_1$ where $t \in [0,1]$. From the chain rule (Theorem 2.2 page 382 in Lang) and the fundamental theorem of calculus, we have

$$\begin{aligned} f(x_2) - f(x_1) &= f(\alpha(1)) - f(\alpha(0)) = \int_0^1 f'(\alpha(t))dt = \int_0^1 (\text{grad} f)(\alpha(t)) \cdot \alpha'(t)dt \\ &= \|x_2 - x_1\| \int_0^1 (\text{grad} f)(\alpha(t)) \cdot u \, dt = \|x_2 - x_1\| \int_0^1 D_u f(\alpha(t))dt = 0, \end{aligned}$$

where $u = (x_2 - x_1)/\|x_2 - x_1\|$. Hence $f(x_2) = f(x_1)$ for any two points $x_1, x_2 \in B(c)$. Thus f is contant in $B(c)$. □

(b) What can you conclude about f, if you know that $D_u f(x) = 0$ for all $x \in B(c)$ and a single unit vector u ? We conclude that f is contant along lines parallel to u.

Proof. This proof is very similar to that of part (a). Let $x_1, x_2 \in B(c)$ such that $u = (x_2 - x_1)/\|x_2 - x_1\|$. Let $\alpha(t) = tx_2 + (1-t)x_1$ where $t \in [0,1]$. From the chain rule (Theorem 2.2 page 382 in Lang) and the fundamental theorem of calculus, we have

$$\begin{aligned} f(x_2) - f(x_1) &= f(\alpha(1)) - f(\alpha(0)) = \int_0^1 f'(\alpha(t))dt = \int_0^1 (\mathrm{grad} f)(\alpha(t)) \cdot \alpha'(t)dt \\ &= \|x_2 - x_1\| \int_0^1 (\mathrm{grad} f)(\alpha(t)) \cdot u \, dt = \|x_2 - x_1\| \int_0^1 D_u f(\alpha(t))dt = 0, \end{aligned}$$

Hence $f(x_2) = f(x_1)$ for two points $x_1, x_2 \in B(c)$ whose difference points in the direction of u. □

4. Problem XV.1.10 in Lang.

(a) In $\mathbb{R}^3$, suppose that $f(x,y,z) = g(r)$ where $r = \sqrt{x^2+y^2+z^2}$. Show that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{d^2 g}{\partial r^2} + \frac{2}{r}\frac{\partial g}{\partial r}$$

Proof. We simply use the chain rule to obtain

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= \frac{x^2}{r^2}\frac{d^2 g}{\partial r^2} + \frac{1}{r}\frac{\partial g}{\partial r} - \frac{x^2}{r^3}\frac{\partial g}{\partial r} \\ \frac{\partial^2 f}{\partial y^2} &= \frac{y^2}{r^2}\frac{d^2 g}{\partial r^2} + \frac{1}{r}\frac{\partial g}{\partial r} - \frac{y^2}{r^3}\frac{\partial g}{\partial r} \\ \frac{\partial^2 f}{\partial z^2} &= \frac{z^2}{r^2}\frac{d^2 g}{\partial r^2} + \frac{1}{r}\frac{\partial g}{\partial r} - \frac{z^2}{r^3}\frac{\partial g}{\partial r} \end{aligned}$$

Adding the above equations, we obtain

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} &= \frac{x^2+y^2+z^2}{r^2}\frac{d^2 g}{\partial r^2} + \frac{3}{r}\frac{\partial g}{\partial r} - \frac{x^2+y^2+z^2}{r^3}\frac{\partial g}{\partial r} \\ &= \frac{d^2 g}{\partial r^2} + \frac{2}{r}\frac{\partial g}{\partial r} \end{aligned}$$

□

(b) Assume that f is harmonic except possibly at the origin on $\mathbb{R}^n$, and that there is a C^2 function g such that $f(X) = g(r)$ where $r = \|X\|$. Let $n \geq 3$. Show that there exist constants C and K such that $g(r) = Kr^{2-n} + C$. What if $n = 2$?

Proof. For $n \geq 3$ we have that

$$\Delta f = \frac{d^2 g}{\partial r^2} + \frac{n-1}{r}\frac{\partial g}{\partial r}$$

Since f is harmonic (except possibly at the origin) then

$$\frac{d^2 g}{\partial r^2} = \frac{1-n}{r}\frac{\partial g}{\partial r}, \quad r \neq 0.$$

Let $\phi(r) = dg(r)/dr$ and we get that $r\phi'(r) = (1-n)\phi(r)$, a differential equation whose solution has the form $\phi(r) = \tilde{K} r^{1-n}$. It follows that

$$g(r) = \frac{\tilde{K}}{2-n} r^{2-n} + C,$$

and the result is obtained by setting $K = \tilde{K}/(2-n)$.
For the case $n = 2$, we repeat the same steps to obtain $\phi(r) = Kr^{-1}$ which implies that $g(r) = K\ln(r) + C$. □

5. Show that any two points on a sphere of radius 1 in n-space centered at the origin can be joined by a differentiable curve that lies on the sphere.

Proof. Let $x_1, x_2 \in S^{n-1}$ be two arbitrary points. Following the hint, let P be any plane passing through 0, x_1 and x_2 (not necessarily unique since the three points may be co-linear). Let $A, B \in P \cap S^{n-1}$ be perpendicular. Consider the curve $\alpha(t) = \cos(t)A + \sin(t)B$. Since $A \perp B$ then $\|\alpha(t)\|^2 = \cos^2(t)\|A\|^2 + \sin^2(t)\|B\|^2 = 1$ which means that the curve $\alpha(t) \in S^{n-1}$. In addition, $\alpha(t)$ is clearly differentiable. Finally, we have that $P \cap S^{n-1} = \{\alpha(t) : 0 \leq t \leq 2\pi\}$ which means that $\alpha(t)$ joins x_1 and x_2. □

6. Let f be a differentiable function on $\mathbb{R}^n$, and assume that there is a differentiable function h such that $(\text{grad}f)(x) = h(x)x$. Show that f is constant on the sphere of radius r centered at the origin in $\mathbb{R}^n$.

Proof. Let x_1 and x_2 be any two points on the sphere of radius r. From Problem 5, we know that the two points can be joined by a differentiable curve $\alpha(t)$ that lies on the sphere. Hence, there exists t_1 and t_2 such that $x_1 = \alpha(t_1)$ and $x_2 = \alpha(t_2)$. Now define the following function $\psi : [t_1, t_2] \to \mathbb{R}^n$ by $\psi(t) = f(\alpha(t))$. From Theorem 2.2 (page 382 in Lang) then

$$\psi'(t) = (\text{grad}f)(\alpha(t)) \cdot \alpha'(t) = h(\alpha(t))\alpha(t) \cdot \alpha'(t).$$

An application of the fundamental theorem of calculus implies that

$$\begin{aligned} f(x_2) - f(x_1) &= \psi(t_2) - \psi(t_1) = \int_{t_1}^{t_2} \psi'(t)dt \\ &= \int_{t_1}^{t_2} h(\alpha(t))\alpha(t) \cdot \alpha'(t)dt \\ &= \frac{1}{2}\int_{t_1}^{t_2} h(\alpha(t))(\alpha(t) \cdot \alpha(t))'dt \end{aligned}$$

where $(\alpha(t) \cdot \alpha(t))' = (\|\alpha(t)\|^2)' = (r^2)' = 0$ since $\alpha(t)$ lies on the sphere of radius r. It follows that $f(x_2) = f(x_1)$ for all points x_1 and x_2 on the sphere. Therefore, f is constant on this sphere. □

ADDITIONAL PROBLEMS FOR CAAM 502

1. Prove the converse of Euler's theorem for homogeneous functions. If

$$x \cdot (\text{grad}f)(x) = pf(x), \quad \forall\ x \in V, \tag{1}$$

then f must be homogeneous of degree p over V.

Proof. Consider the function $g(\lambda) = f(\lambda x)$. The combination of the chain rule Theorem 2.2 (page 382 in Lang) and (**??**) yields,

$$g'(\lambda) = (\text{grad}f)(\lambda x) \cdot x = \frac{1}{\lambda}(\text{grad}f)(\lambda x) \cdot (\lambda x) = \frac{p}{\lambda}f(\lambda x) = \frac{p}{\lambda}g(\lambda).$$

Therefore, g satisfies the following initial value problem

$$\begin{aligned} &\lambda g'(\lambda) = pg(\lambda), \\ &g(1) = f(x). \end{aligned}$$

The unique solution is given by $g(\lambda) = \lambda^p f(x)$, which implies that $f(\lambda x) = \lambda^p f(x)$, ie., f is homogeneous of degree p over V. □

2. Let $p : (a, b) \to \mathbb{R}$ be a smooth function and M be the differential operator

$$(Mf)(y) = -f''(y) + p(y)f(y), \quad y \in (a, b) \tag{2}$$

acting on a sufficiently smooth function f. A Green's function for the differential operator M is a function $g : (a, b) \times (a, b) \to \mathbb{R}$ such that

$$M \int_a^b g(y, z)f(z)dz = f(y), \quad y \in (a, b) \tag{3}$$

for all $f \in C^\infty(a, b)$ with compact support. Consider the following conditions:

GF 1. g is infinitely differentiable in each variable except on the diagonal $y = z$, with the derivative having one-sided limits at the diagonal.

GF 2. If $y \neq z$ then $M_y g(y, z) = 0$.

GF 3. $D_1 g(y, y+) - D_1 g(y, y-) = 1$.

Prove: Let g be a continuous function satisfying **GF 1** and **GF 2**. Then g is a Green's function for the operator M if and only if g also satisfies the jump condition **GF 3**.

Proof. Let g be a continuous function satisfying conditions **GF 1** and **GF 2**. We follow the hint, and the use of the Lemma below with $G = g$ and $G = D_1 g$ together with the continuity of g yields

$$\begin{aligned}
\frac{d^2}{dy^2} \int_a^b g(y, z)f(z)dz &= \frac{d}{dy}\left(\int_a^y D_1 g(y, z)f(z)dz + \int_y^b D_1 g(y, z)f(z)dz \right) \\
&= \int_a^y D_1^2 g(y, z)f(z)dz + \int_y^b D_1^2 g(y, z)f(z)dz \\
&\quad + D_1 g(y, y-)f(y) - D_1 g(y, y+)f(y),
\end{aligned}$$

for all $f \in C^\infty(a, b)$. From the definition of the differential operator M, we have that

$$M \int_a^b g(y, z)f(z)dz = \int_a^y Mg(y, z)f(z)dz + \int_y^b Mg(y, z)f(z)dz + D_1 g(y, y+)f(y) - D_1 g(y, y-)f(y).$$

Now from condition **GF 2** and the above result, it follows that

$$M \int_a^b g(y, z)f(z)dz = D_1 g(y, y+)f(y) - D_1 g(y, y-)f(y),$$

for all $f \in C^\infty(a, b)$. Hence, condition **GF 3** is satisfied if and only if g is a Green's function for M. □

Lemma: Let G satisfy **GF 1** and $f \in C^\infty(a, b)$. Then,

$$\frac{d}{dy} \int_a^y G(y, z)f(z)dz = \int_a^y D_1 G(y, z)f(z)dz + G(y, y-)f(y) \tag{4}$$

$$\frac{d}{dy} \int_y^b G(y, z)f(z)dz = \int_y^b D_1 G(y, z)f(z)dz - G(y, y+)f(y) \tag{5}$$

Proof. We proceed to prove (**??**) and the proof for (**??**) is similar. Let $0 < h$ and consider,

$$\begin{aligned}
\frac{d}{dy} \int_a^y G(y, z)f(z)dz &= \lim_{h \to 0} \frac{1}{h}\left(\int_a^y G(y, z)f(z)dz - \int_a^{y-h} G(y - h, z)f(z)dz \right) \\
&= \lim_{h \to 0} \int_a^y \frac{(G(y, z) - G(y - h, z))}{h} f(z)dz + \int_{y-h}^y \frac{G(y - h, z)}{h} f(z)dz \\
&= \int_a^y D_1 G(y, z)f(z)dz + \lim_{h \to 0} G(y - h, c(h))f(c(h)),
\end{aligned}$$

where $y - h < c(h) < y$. We have used Theorem 7.1 (page 276 in Lang) and the mean value theorem for integrals. The result (**??**) follows from the one-sided continuity of G and continuity of f. □