CAAM 415, Theoretical Neuroscience

October 14, 2013

Midterm
October 11, 2010

Instructions

1. **Staple** this cover sheet to your exam and solutions and return it to Yuri Dabaghian in DH 2006 by 12 am on Thursday, October 19.

2. `Print your name on the line below`

____

3. (5) Indicate your compliance with the honor system by writing out in full and signing the traditional pledge, *"On my honor, I have neither given nor received any unauthorized aid on this exam,"* on the lines below

4. **Do any five problems out the proposed seven.**

5. Work each part in great detail.

Problem 1 Consider the passive isopotential cell, where the potential, $v = V - V_{Cl}$, is governed by

$$\tau v'(t) + v(t) = f(t), \qquad v(0) = 0, \tag{1}$$

for $\tau = C_m/g_{Cl}$ and $f(t) = I_{stim}(t)/(AC_m)$.

(1a) Use the identity

$$\frac{d}{dt}\left(v(t)e^{t/\tau}\right) = (v'(t) + \frac{1}{\tau}v(t))e^{\frac{t}{\tau}} \tag{2}$$

to show

$$v(t) = \frac{1}{\tau}\int_0^t e^{(s-t)/\tau} f(s)ds. \tag{3}$$

(1b) Suppose the cell is driven by the impulse $I_{stim}(t) = I_0\delta(t - t_1)$, where δ denotes the Dirac-delta function. Compute $v_{\max} = \max_t v(t)$. At what time does the potential attain its maximum value?

(1c) Since the passive cell cannot model the action potential, we often assume the cell spikes when its potential reaches a threshold v_{th}. Using the same stimulus as in (b), determine the threshold current $I_\theta = \min(I_0)$ such that $v_{\max} \geq v_{th}$. When $I_0 \geq I_\theta$, at what time will the cell spike?

(1d) Assume the cell receives a delta-impulse every t ms, i.e.,

$$I_{stim}(t) = I_0 \sum_{k=1}^{\infty} \delta(t - k\Delta t). \tag{4}$$

Determine the threshold potential I_θ as a function of the input frequency $\omega = 1/\Delta t$. Sketch a plot of this function.

Hint: Evaluate $v(t_n)$ and use the geometric series:

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \tag{5}$$

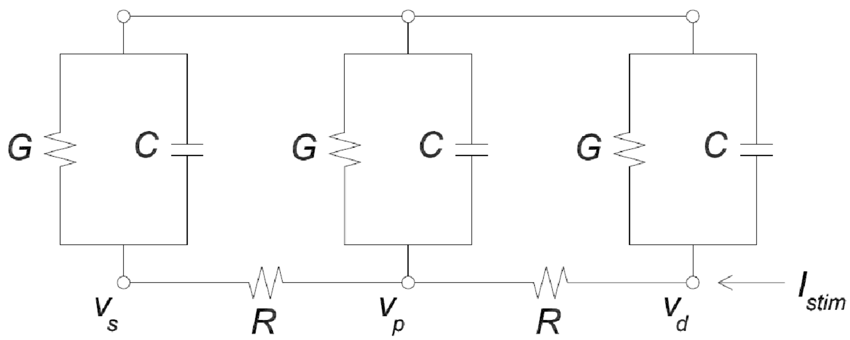
for $|x| < 1$.

Problem 2 Consider the isopotential cell with the active h-current:

$$C_m V'(t) = -g_{Cl}(V(t) - V_{Cl}) - \bar{g}_h q^2(t)(V(t) - V_h) + I_{stim}(t) \tag{6}$$

$$\tau_q(V)q'(t) = q_\infty(V) - q(t) \tag{7}$$

(2a) What equation must the resting potential, V_r, satisfy?



(2b) Assume that for some small ϵ, $I_{stim}(t) = \epsilon\tilde{I}(t)$, $V(t) = V_r + \epsilon\tilde{V}(t) + O(\epsilon^2)$, and $q(t) = q_\infty(V_r) + \epsilon\tilde{q}(t) + O(\epsilon^2)$. Expand each function, $q_\infty(V)$ and $\tau_q(V)$, as a Taylor series about V_r. Then, derive the quasi-active equations describing the linear perturbations from rest.

(2c) Construct the quasi-active system

$$y'(t) = By(t) + f(t), \tag{8}$$

where $y = [\tilde{q} \quad \tilde{V}]^\top$ ("$\top$" means "transposed"). Identify each element of B and f.

Problem 3 Consider the simplified cable with three compartments, where v_s is the soma potential, v_p is the potential for the proximal compartment, and v_d is the potential for the distal compartment. Let C be the membrane capacitance, G be the membrane conductance, and R be the axial resistance.

(3a) Write the governing equations for v_s, v_p, and v_d, where $v_s(0) = v_p(0) = v_d(0) = 0$. Show that if $v \equiv [vs\ vp\ vd]^\top$, then v can be written as

$$v'(t) = Bv(t) + f(t). \tag{9}$$

Write the matrix B and driving term f.

(3b) Assume $Bq_n = z_n q_n$ with orthonormal eigenvectors:

$$q_i^\top q_j = \begin{cases} 1, & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

Let

$$f(t) = \sum_{i=1}^{3} c_i(t) q_i \tag{11}$$

$$v(t) \quad = \quad \sum_{i=1} a_i(t) q_i. \tag{12}$$

Write the soma potential v_s in terms of the eigenvalues and eigenvectors of B. You may find Equation (1) from Problem 1 helpful in solving for the coefficients a_i.

(3c) Let $R = C = G = 1$. Then B has the orthonormal eigenvectors

$$q_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad q_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad q_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \tag{13}$$

Compute the corresponding eigenvalues. Is the resting potential stable?

(3d) Assume the stimulus is constant, $I_{stim}(t) = I_0$. Solve for the steady-state soma potential,

$$\lim_{t \to \infty} v_s(t) \tag{14}$$

in terms of I_0.

(3e) Assume the proximal compartment receives the stimulus $I_{stim} = I_0$ instead of the distal compartment. What is the new steady-state potential at the soma?

Problem 4. Together, we shall study, by hand, the passive sealed fiber undergoing a known distributed current stimulus, I. That is,

$$\frac{a}{2R_2} v_{xx} = C_m v_t + G_m v + I(x,t), \quad 0 < x < l \tag{15}$$

$$v_x(0,t) = v_x(l,t) = 0, \quad v(x,0) = 0. \tag{16}$$

We will solve this at first for a general I and then finish up with a particular case.

(4a) We shall expand everything in terms of the solutions to

$$-q''(x) = zq(x), \quad q'(0) = q'(l) = 0. \tag{17}$$

Solve for the eigenpairs $\{q_n, z_n\}_{n=0}^{\infty}$ of (2) and normalize the eigenfunctions in order that $\int_0^l q_n^2(x)\,dx = 1$.

(4b) Now write

$$v(x,t) = \sum_{n=0}^{\infty} q_n(x) T_n(t) I(x,t) = \sum_{n=0}^{\infty} q_n(x) i_n(t) \tag{18}$$

and plug these into (1) and derive a sequence of ordinary differential equations (and initial conditions) for the unknown T_n in terms of the known i_n.

(4c) Solve these ordinary differential equations.

(4d) Explicitly compute each i_n in the case that $I(x,t) = t\exp(-t)\cos(\pi x/l)$. What are the corresponding T_n? And finally, what is v?

Problem 5. If the stimulus above was a conductance change, rather than a direct current stimulus we would instead be faced with

$$\begin{aligned} \frac{a}{2R_2}v_{xx} &= C_m v_t + G_m v + G(x,t)(v-E), \quad 0 < x < l \\ v_x(0,t) &= v_x(l,t) = 0 \\ v(x,0) &= 0. \end{aligned} \tag{19}$$

you might think of G as some effective synaptic conductance and E as the (constant) synaptic reversal potential. As our eigentechnique of problem 1 does not directly apply we turn to a strictly numerical attack.

(5a) Equipartition the fiber into N equipotential compartments and show how (3) becomes a system of N ordinary differential equations for the N compartment potentials. Write this in matrix terms.

(5b) Setup both forward and backward Euler time discretizations of this system of ordinary differential equations.

Problems 6 and 7. Do any two problems from the "Kicking passive Neurons" handout, http://www.caam.rice.edu/ caam415/syl13.html